COMPLEX ANALYSIS

M.A. (Previous)

Directorate of Distance Education
Maharshi Dayanand University
ROHTAK – 124 001

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Contents

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M.Marks: 100
Time: 3 Hours

Note: *Question paper will consist of three sections. Section I consisting of one question with ten parts of 2 marks each covering whole of the syllabus shall be compulsory. From Section-II, 10 questions to be set selecting two questions from each unit. The candidate will be required to attempt any seven questions each of five marks. Section-III, five questions to be set, one from each unit. The candidate will be required to attempt any three questions each of fifteen marks.*

Unit I: Analysis functions, Cauchy-Riemann equation in cartesian and polar coordinates . Complex integration. Cauchy-Goursat Theorem. Cauchy's integral formula. Higher order derivatives. Morera's Theorem. Cauchy's inequality and Liouville's theorem. The fundamental theorem of algebra. Taylor's theorem.

Unit-II: Isolated singularities. Meromorphic functions. Maximum modulus principle. Schwarz lemma. Laurent's series. The argument principle. Rouche's theorem. Inverse function theorem.

Residues. Cauchy's residue theorem. Evaluation of integrals. Branches of many valued functions with special reference to arg z, log z and z^a.

Unit-III: Bilinear transformations, their properties and classifications. Definitions and examples of Conformal mappings.

Space of analytic functions. Hurwitz's theorem. Montel's theorem. Riemann mapping theorem.

Weierstrass' factorisation theorem. Gammar function and its properties. Riemann Zeta function. Riemann's functional equation. Runge's theorem. Mittag-Leffler's theorem.

Unit IV: Analytic Continuation. Uniqueness of direct analytic continuation. Uniqueness of analytic continuation along a curve. Power series method of analytic continuation. Schwarz Reflection principle. Monodromy theorem and its consequences. Harmonic functions on a disk. Harnack's inequality and theorem. Dirichlet problem. Green's function.

Canonical products. jensen's formula. Poisson-Jensen formula. Hadamard's three circles theorem.

Unit V: Order of an entire function. Exponent of Convergence. Borel's theorem. Hadamard's factorization theorem.

The range of an analytic function. Bloch's theorem. The Little Picard theorem. Schottky's theorem. Montel Caratheodory and the Great picard theorem.

Univalent functions. Bieberbach's conjecture (Statement only) and the "1/4 theorem.

UNIT – I

1. Analytic Functions

We denote the set of complex numbers by ∀. Unless stated to the contrary, all functions will be assumed to take their values in ∀. It has been observed that the definitions of limit and continuity of functions in ∀ are analogous to those in real analysis. Continuous functions play only an ancillary and technical role in the subject of complex analysis. Much more important are the analytic functions which we discuss here. Loosely, analytic means differentiable. Differentiation in ∀ is set against the background of limits, continuity etc. To some extent the rules for differentiation of a function of complex variable are similar to those of differentiation of a function of real variable. Since ∀ is merely R^2 with the additional structure of addition and multiplication of complex numbers, we can immediately transform most of the concepts of R^2 into those for the complex field ∀.

Let us consider the complex function w = f(z) of a complex variable z. If z and w be separated into their real and imaginary parts and written as z = x + iy, w = u + iv, then the relation w = f(z) becomes

$$u + iv = f(x + iy)$$

From here, it is clear that u and v, in general, depend upon x and y in a certain definite manner so that the function w = f(z) is nothing but the ordered pair of two real functions u and v of two real variables x and y so that we may write

$$w = u\,(x, y) + i\,v(x, y)$$

If we use the polar form, then f can be written as

$$w = f(z) = u\,(r, \theta) + iv\,(r, \theta)$$

1.1. Definition. A function f defined on an open set G of ∀ is differentiable at an interior point z_0 of G if the limit

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} \qquad (1)$$

exists. If $z = z_0 + h$, h being complex, then (1) is equivalently written as

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} \qquad (2)$$

When the limit (1) (or (2)) exists, it must be the same regardless of the way in which z approaches z_0 (or h approaches zero). The value of the limit, denoted by $f'(z_0)$, is called the derivative of f at z_0. In $\in - \delta$ language, the above definition of derivative is the statement that for every positive number $\in$ there exist a positive number δ such that

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| < \in \qquad (3)$$

whenever $0 < |z - z_0| < \delta$.

For the general point z, we have

$$f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

which may also be expressed as

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

Suppose w = f(z). We sometimes define

$$\Delta w = f(z + \Delta z) - f(z)$$

and write the derivative as

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

If f is differentiable at each point of G, we say that f is differentiable on G. We observe that if f is differentiable on G, then $f'(z)$ defines a function $f' : G \to \forall$. If f' is continuous, then we say that f is continuously differentiable. If f' is differentiable, then f is said to be twice differentiable. Continuing in this manner, a differentiable function such that each successive derivative is again differentiable, is called infinitely differentiable. It is immediate that the derivative of a constant function is zero.

If f is differentiable at a point z_0 in G, then f is continuous at z_0, since

$$\lim_{z \to z_0} [f(z) - f(z_0)] = \lim_{z \to z_0} \left[\frac{f(z) - f(z_0)}{z - z_0}\right] \lim_{z \to z_0} (z - z_0)$$

$$= f'(z_0).\ 0 = 0$$

i.e. $$\lim_{z \to z_0} f(z) = f(z_0)$$

A continuous function is not necessarily differentiable. In fact differentiable functions possess many special properties. For example, $f(z) = \bar{z}$ is obviously continuous but does not possess derivative, since, by definition

$$f'(z) = \lim_{h \to 0} \frac{\overline{z+h} - \bar{z}}{h} = \lim_{h \to 0} \frac{\bar{h}}{h}$$

If we write $h = re^{i\theta}$, then

$$f'(z) = \lim_{h \to 0} e^{-2i\theta}$$

So, if $h \to 0$ along the positive real axis ($\theta = 0$), then $f'(z) = 1$ and if $h \to 0$ along the positive imaginary axis ($\theta = \pi/2$), then $f'(z) = -1$. Hence $f'(z)$ is not unique and it depends on how h approaches zero. Thus, we find the surprising result that the function $f(z) = \bar{z}$ is not differentiable anywhere, even though it is continuous everywhere. In fact, this situation will be seen for general complex functions unless the real and imaginary parts satisfy certain compatibility conditions.

Similarly, $f(z) = |z|^2$ is continuous everywhere but is differentiable only at z = 0 and the functions |z|, Re z, Im z are all nowhere differentiable in ∀ .

1.2. Definition. Let G be an open set in ∀. A function $f : G \to \forall$ is analytic (holomorphic) in G if f(z) is differentiable at each point of G. Here, it is important to stress that the open set G is a part of the definition.

Equivalently, a function f(z) is said to be analytic at $z = z_0$ if f(z) is differentiable at every point of some neighbourhood of z_0. We observe that $f(z) = |z - a|^2$ is differentiable at z = a but it is not analytic at z = a because there does not exist a neighbourhood of a in which $|z-a|^2$ is differentiable at each point of the neighbourhood.

If in a domain D of the complex plane, f(z) is analytic throughout, we sometimes say that f(z) is regular in D to emphasize that every point of D is a point at which f(z) is analytic. Further, if f(z) is analytic at each point of the entire finite plane, then $f(z)$ is called an entire function. A point where the function fails to be analytic, is called a singular point or singularity of the function.

The set (class) of functions holomorphic in G is denoted by H(G). The usual differentiation rules apply for analytic functions. Thus, if f, $g \varepsilon H(G)$, then $f + g \varepsilon H(G)$ and $fg \varepsilon H(G)$, so that H(G) is a ring. Further, superpositions of analytic functions are analytic, chain rule of differentiation applies. Thus, if f and g are analytic on G and G_1 respectively and $f(G) \subset G_1$, then $g o f$ is analytic on G and

$$(g o f)'(z) = g'(f(z))\, f'(z) \text{ for all z in G.}$$

1.3. Remark. The theory of analytic functions cannot be considered as a simple generalization of calculus. To point out how vastly different the two subjects are, we shall show that every analytic function is infinitely differentiable and also has a power series expansion about each point of its domain. These results have no analogue in the theory of functions of real variables. Further, in the complex variable case, there are an infinity of directions in which a variable z can approach a point z_0, at which differentiability is considered. In the real case, however, there are only two avenues of approach (e.g. continuity of a function in real case, can be discussed in terms of left and right continuity).

Thus, we notice that the statement that a function of a complex variable has a derivative is stronger than the same statement about a function of a real variable.

1.4. Cauchy-Riemann Equations. Now we come to the earlier mentioned compatibility relationship between the real and imaginary parts of a complex function which are necessarily satisfied if the function is differentiable. These relations are known as Cauchy-Riemann equations (C–R equations). We have seen that every complex function can be expressed as

$$f(z) = u(x, y) + iv(x, y), \text{ where } u(x, y) \equiv u \text{ and } v(x, y) \equiv v$$

are real functions of two real variables x and y. We shall denote the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x \partial y}$ by u_x, u_y, u_{xx}, u_{yy}, u_{xy} respectively.

1.5. Theorem. (Necessary condition for $f(z)$ to be analytic). Let $f(z) = u(x, y) + iv(x, y)$ be defined on an open set G and be differentiable at $z = x + iy \,\varepsilon\, G$, then the four first order partial derivatives u_x, u_y, v_x, v_y exist and satisfy the Cauchy-Riemann Equations $u_x = v_y$, $u_y = -v_x$ **.**

Proof. By definition, we have

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

We evaluate this limit in two different ways. Let $h = h_1 + i\, h_2 \neq 0$, $h_1, h_2 \,\varepsilon\, R$ First let $h \to 0$ through real values of h i.e. $h = h_1$, $h_2 = 0$. Thus, we get

$$\frac{f(z+h) - f(z)}{h} = \frac{f(x+iy+h_1) - f(x+iy)}{h_1}$$

$$= \frac{u(x+h_1, y) - u(x, y)}{h_1} + i\frac{v(x+h_1, y) - v(x, y)}{h_1}$$

Letting $h_1 \to 0$, we obtain

$$f'(z) = \frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x}(x, y) \qquad (1)$$

Secondly, let $h \to 0$ through purely imaginary $h = ih_2$, $h_1 = 0$. Then, we have

$$\frac{f(z+h)-f(z)}{h} = \frac{f(x+iy+ih_2)-f(x+iy)}{h_2}$$
$$= -i\,\frac{u(x,y+h_2)-u(x,y)}{h_2} + \frac{v(x,y+h_2)-v(x,y)}{h_2}$$

Letting $h_2 \to 0$, we obtain

$$f'(z) = -i\,\frac{\partial u}{\partial y}(x,y) + i\frac{\partial v}{\partial y}(x,y) \qquad (2)$$

Since $f'(z)$ exists, i.e., $f(z)$ has unique derivative, so from (1) and (2), equating real and imaginary parts, we get the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad (3)$$

i.e. $u_x = v_y$ and $u_y = -v_x$

1.6. Remarks. (i) We have $f(z) = u + iv$ which gives

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

From these two results, C−R equations, in complex form, can be put as

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}.$$

(ii) We note that unless the differential equations (3) i.e. C−R equations are satisfied, $f(z) = u + iv$ cannot be differentiable at any point even if the four first order partial derivatives exist.

For example, let us take

$$f(z) = \text{Re } z = x,\ z = x + iy$$

Then
$$\frac{\partial u}{\partial x} = 1,\ \frac{\partial u}{\partial y} = 0,\ \frac{\partial v}{\partial x} = 0,\ \frac{\partial v}{\partial y} = 0$$

Thus, although the partial derivatives exist every where, C−R equations are not satisfied at any point of the complex plane. Hence the function $f(z) = \text{Re } z$ is not differentiable at any point.

(iii) The condition of the above theorem is not sufficient. Actually, C−R equations are useful for proving non-differentiability. They are not, on their own, a sufficient condition for differentiability. For this, as an example, we consider the function

$$f(z) = \begin{cases} (\bar{z})^2/z, z \neq 0 \\ 0 \quad , \quad z = 0 \end{cases} \quad , z = x + iy$$

and show that $f(z)$ is not differentiable at the origin, although C−R equations are satisfied at that point. By definition, we have

$$f'(0) = \lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{z\to 0}\frac{(\bar{z})^2}{z^2}$$
$$= \lim_{(x,y)\to(0,0)}\left(\frac{x-iy}{x+iy}\right)^2$$

$$= \begin{cases} 1 \text{ if } z \to 0 \text{ along real axis} \\ 1 \text{ if } z \to 0 \text{ along imaginary axis} \\ -1 \text{ if } z \to 0 \text{ along the line } y = x \end{cases}$$

Thus $f'(0)$ is not unique and hence $f(z)$ is not differentiable at the origin.

Now, to verify C–R equations, we have

$$f(0) = 0 \Rightarrow u(0, 0) = 0,\ v(0, 0) = 0$$

Also $$f(z) = \frac{(\bar{z})^2}{z} = \frac{(\bar{z})^3}{z\bar{z}} = \frac{(x-iy)^3}{x^2+y^2}$$

From here, $$u(x, y) = \frac{x^3-3xy^2}{x^2+y^2},\ v(x,y) = \frac{y^3-3x^2y}{x^2+y^2}$$

Therefore, at (0, 0)

$$u_x = 1,\ u_y = 0,\ v_x = 0,\ v_y = 1$$

Thus C–R equations are satisfied at the origin.

To make C–R equations as sufficient an additional condition of continuity on partial derivatives is imposed.

1.7. Theorem. (Sufficient condition for *f*(z) to be analytic). Suppose that $f(z) = u(x, y) + i\,v(x, y)$ for $z = x + iy$ in an open set G, where u and v have continuous first order partial derivatives and satisfy Cauchy–Riemann equations in G. Then f is analytic in G.

Proof. For $z = x + iy$, let $h = h_1 + ih_2 \in G$. We have

$$u = u(x, y)$$
$$u + \Delta u = u(x + h_1, y + h_2)$$

so that

$$\Delta u = u(x + h_1, y + h_2) - u(x, y) \tag{1}$$

Similarly,

$$\Delta v = v(x + h_1, y + h_2) - v(x, y) \tag{2}$$

Since u_x, u_y, v_x, v_y are continuous at the point (x, y), applying mean value theorem for functions of two real variables, we get

$$\Delta u = (u_x + \epsilon_1) h_1 + (u_y + \epsilon_2) h_2$$
$$= (u_x h_1 + u_y h_2) + (\epsilon_1 h_1 + \epsilon_2 h_2) \tag{3}$$
$$\Delta v = (v_x h_1 + v_y h_2) + (\eta_1 h_1 + \eta_2 h_2) \tag{4}$$

where $\epsilon_1 = \epsilon_1(h_1, h_2),\ \epsilon_2 = \epsilon_2(h_1, h_2),\ \eta_1 = \eta_1(h_1, h_2)$

and $\eta_2 = \eta_2(h_1, h_2) \to 0$ as $h_1, h_2 \to 0$

Thus $$\Delta u + i\,\Delta v = (u_x + iv_x) h_1 + (u_y + iv_y) h_2 + (\epsilon_1 + i\eta_1) h_1 + (\epsilon_2 + i\eta_2) h_2$$

Making use of C–R equations, we obtain

$$\begin{aligned} \Delta u + i\Delta v &= (u_x + iv_x) h_1 + (-v_x + iu_x) h_2 + (\epsilon_1 + i\eta_1) h_1 + (\epsilon_2 + i\eta_2) h_2 \\ &= (u_x + iv_x)(h_1 + ih_2) + (\epsilon_1 + i\eta_1) h_1 + (\epsilon_2 + i\eta_2) h_2 \\ &= (u_x + iv_x) h + (\epsilon_1 + i\eta_1) h_1 + (\epsilon_2 + i\eta_2) h_2. \end{aligned}$$

Therefore,

$$\frac{f(z+h) - f(z)}{h} = \frac{\Delta u + i\Delta v}{h}$$

$$= u_x + iv_x + \left[\frac{(\in_1 + i\eta_1)h_1 + (\in_2 + i\eta_2)h_2}{h}\right] \quad (5)$$

Now, we note that $|h_1| \le |h|$, $|h_2| \le |h|$ so that

$$\left|\frac{(\in_1 + i\eta_1)h_1 + (\in_2 + i\eta_2)h_2}{h}\right| \le |\in_1 + i\eta_1| \left|\frac{h_1}{h}\right| + |\in_2 + i\eta_2| \left|\frac{h_2}{h}\right|$$

$$\le |\in_1 + i\eta_1| + |\in_2 + i\eta_2|$$

Therefore, as $h = h_1 + ih_2 \to 0$, the expression in the square bracket of (5) approaches zero. Consequently, taking limit as $h \to 0$ in (5), we obtain

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = u_x + iv_x$$

which shows that $f(z)$ is analytic at every point of G.

The two results, those of necessary and sufficient conditions for $f(z)$ to be analytic, can be combined in the form of the following theorem.

1.8. Theorem. Let u and v be real-valued functions defined on a region G and suppose that u and v have continuous first order partial derivatives. Then $f : G \to \forall$ defined by
$f(z) = u(x, y) + iv(x, y)$ is analytic iff u and v satisfy the Cauchy-Riemann equations.

1.9. C–R Equations in Polar Co-ordinates : We know that in polar co-ords. (r, θ),

$$x = r\cos\theta,\ y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2},\ \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Now,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$= \frac{\partial u}{\partial r}\left(\frac{x}{\sqrt{x^2+y^2}}\right) + \frac{\partial u}{\partial \theta}\left(-\frac{y}{x^2+y^2}\right)$$

$$= \frac{\partial u}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta \quad (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$= \frac{\partial u}{\partial r}\left(\frac{y}{\sqrt{x^2+y^2}}\right) + \frac{\partial u}{\partial \theta}\left(\frac{x}{x^2+y^2}\right)$$

$$= \frac{\partial u}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial u}{\partial \theta}\cos\theta \quad (2)$$

Similarly,

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial v}{\partial \theta}\sin\theta \quad (3)$$

and

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial v}{\partial \theta}\cos\theta \quad (4)$$

Using C–R equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ with (1) and (4), $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ with (2) and (3), we get

$$\left(\frac{\partial u}{\partial r}-\frac{1}{r}\frac{\partial v}{\partial \theta}\right)\cos\theta-\left(\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial \theta}\right)\sin\theta = 0 \qquad (5)$$

$$\left(\frac{\partial u}{\partial r}-\frac{1}{r}\frac{\partial v}{\partial \theta}\right)\sin\theta+\left(\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial \theta}\right)\cos\theta = 0 \qquad (6)$$

Multiplying (5) by cos θ and (6) by sin θ and then adding, we find

$$\frac{\partial u}{\partial r}-\frac{1}{r}\frac{\partial v}{\partial \theta}=0$$

i.e. $$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta} \qquad (7)$$

Again, multiplying (5) by sin θ and (6) by cos θ and then subtracting, we have

$$\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial \theta}=0$$

i.e. $$\frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta} \qquad (8)$$

Equations (7) and (8) are the required C−R equations in polar co-ordinates.

1.10. Remark. We can express $f'(z)$ in polar co-ords. as

$$\begin{aligned}
f'(z) &= \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\\
&= \frac{\partial u}{\partial r}\cos\theta-\frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta+i\frac{\partial v}{\partial r}\cos\theta\\
&\qquad -i\,\frac{1}{r}\frac{\partial v}{\partial \theta}\sin\theta\\
&= \frac{\partial u}{\partial r}\cos\theta+\frac{\partial v}{\partial r}\sin\theta+i\frac{\partial v}{\partial r}\cos\theta-i\frac{\partial u}{\partial r}\sin\theta\\
&= \left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right)\cos\theta-\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right)i\sin\theta\\
&= (\cos\theta - i\sin\theta)\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right)\\
&= e^{-i\theta}\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right)
\end{aligned}$$

i.e., $$\frac{dw}{dz}=e^{-i\theta}\frac{\partial w}{\partial r}$$

Similarly, we get $$\frac{dw}{dz}=-\frac{i}{r}e^{-i\theta}\frac{\partial w}{\partial \theta}$$

1.11. Theorem. A real function of a complex variables either has derivative zero or the derivative does not exist.

Proof. Suppose that $f(z)$ is a real function of complex variable whose derivative exists at z_0. Then, by definitions

$$f'(z_0)=\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}$$

let $h = h_1 + ih_2$.

If we take the limit $h \to 0$ along the real axis, $h = h_1 \to 0$, then $f'(z_0)$ is real (since f is real). If we take the limit $h \to 0$ along the imaginary axis, $h = ih_2 \to 0$, then $f'(z_0)$ becomes purely imaginary number, where f is real. So we must have $f'(z_0) = 0$.

Further, in this case we also observe that if $f(z)$ is analytic then, using C–R equations, we conclude that $f(z)$ is a constant function.

1.12. Example. Show that the function $f(0) = 0$,

$$f(z) = u + iv = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$$

$$= \frac{x^3 - y^3}{x^2+y^2} + i\frac{x^3+y^3}{x^2+y^2}$$

is continuous and that the C–R equations are satisfied at the origin, yet $f'(0)$ does not exist

Solution. We have

$$u = \frac{x^3 - y^3}{x^2+y^2}, \quad v = \frac{x^3+y^3}{x^2+y^2}$$

When $z \neq 0$, u and v are rational functions of x and y with non zero denominators. It follows that they are continuous when $z \neq 0$. To test them for continuity at $z = 0$, we change to polars and get

$$u = r(\cos^3\theta - \sin^3\theta), \; u = r(\cos^3\theta + \sin^3\theta)$$

each of which tends to zero as $r \to 0$, whatever value θ may have. Now, the actual values of u and v at origin are zero since $f(0) = 0$. So the actual and the limiting values of u and v at the origin are equal, they are continuous there. Hence $f(z)$ is a continuous function for all values of z. Now, at the origin

$$\frac{\partial u}{\partial x} = \lim_{x\to 0}\frac{u(x,0) - (0,0)}{x}$$

$$= \lim_{x\to 0}\frac{x^3/x^2}{x} = 1$$

Similarly, $\quad \dfrac{\partial u}{\partial y} = -1, \; \dfrac{\partial v}{\partial x} = 1, \dfrac{\partial v}{\partial y} = 1$

Hence C–R equations are satisfied at the origin.

Again, $\quad f'(0) = \lim_{z\to 0}\dfrac{f(z) - f(0)}{z}$

$$= \lim_{z\to 0}\frac{(x^3-y^3) + i(x^3+y^3)}{x^2+y^2}.\frac{1}{x+iy}$$

If we let $z \to 0$ along real axis ($y = 0$), then $f'(0) = 1 + i$ and if $z \to 0$ along $y = x$, then $f'(0) = \dfrac{i}{1+i}$

Thus $f'(0)$ is not unique and hence $f(z)$ is not differentiable at the origin.

Similar conclusion (as for example 1.12) holds for the following two functions

$$\text{(i) } f(z) = u + iv = \begin{cases} \dfrac{I_m(z^2)}{|z|^2}, & z \neq 0 \\ 0 \quad , & z = 0 \end{cases}$$

(ii) $f(z) = u + iv = \begin{cases} \dfrac{z^5}{|z|^4}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

1.13. Example. Real and imaginary parts of an analytic function satisfy Laplace equation.

Solution. Let $f(z) = u + iv$ be an analytic function so that C–R equations $u_x = v_y$, $u_y = -v_x$ are satisfied. Differentiating first equation w.r.t. x and second w.r.t. y and adding, we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x}$$

where continuity of partial derivatives implies that the mixed derivatives are equal i.e. $v_{xy} = v_{yx}$. Hence, we get

$$\Delta^2 u = 0$$

Similarly, differentiating first equation w.r.t y and second w.r.t x and then subtracting, we find

$$\frac{\partial^2 u}{\partial y \partial x} - \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$$

i.e. $\Delta^2 v = 0$

1.14. Definition. A real valued function $\phi(x, y)$ of real variables x and y is said to be harmonic on a domain $D \subset \forall$, if for all points (x, y) in D, it satisfies the Laplace equation in two variables. Thus, from the above example 1.13, we observe that u and v are harmonic functions. In such a case, u and v are called conjugate harmonic functions i.e. u is referred to as the harmonic conjugate of v and vice-versa where $f(z) = u + iv$ is analytic. Harmonic functions play a part in both physics and mathematics.

1.15. Remarks. (i) C–R equations, in polar form, are

$$u_r = \frac{1}{r} v_\theta,\ v_r = -\frac{1}{r} u_\theta$$

Differentiating first equation w.r.t r and second w.r.t θ, we get

$$v_{\theta r} = u_r + r\, u_{rr},\ v_{r\theta} = -\frac{1}{r} u_{\theta\theta}$$

Thus, using the continuity of second order partial derivatives, we get

$$u_r + r\, u_{rr} = -\frac{1}{r} u_{\theta\theta}$$

i.e. $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$ which is the polar form of Laplace equation.

(ii) The function u (or v) can be obtained from v (or u) via C–R equations. Thus, we can obtain an analytic function $f(z) = u + iv$ if either u or v is given. For this we use

$$x = \frac{z + \bar{z}}{2},\ y = \frac{z - \bar{z}}{2i} \text{ where } z = x + iy$$

Suppose u is given.

We denote $\frac{\partial u}{\partial x}$ by $\phi(x, y)$, $\frac{\partial u}{\partial y}$ by $\psi(x, y)$

Therefore, $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

$= \phi(x, y) - i\psi(x, y)$

$$= \phi(z, 0) - i\psi\,(z, 0)$$

Where we have set x = z, y = 0

Then, $f(z) = \int [\phi(z, 0) - i\psi(z, 0)]\, dz + c$, c being a constant.

Similarly, if v is given, we can find $f(z)$.

1.16. Power series. An infinite series of the form

$$\text{(i)} \sum_{n=0}^{\infty} a_n z^n \qquad \text{or} \qquad (2) \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

where a_n, z, z_0 are in general complex, is called a power series. Since the series (2) can be transformed into the series (1) by means of change of origin, it is sufficient to consider only the series of type (1).

The circle |z| = R which includes all the values of z for which the power series $\sum_{n=0}^{\infty} a_n z^n$ converges, is called the circle of convergence and the radius R of this circle is called the radius of convergence of the series. Thus, the series converges for |z| < R and diverges for |z| > R, nothing is claimed about the convergence on the circle.

The radius of convergence R of a power series, using ratio test or Cauchy's root test, is given by the formula

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} |a_n|^{-\frac{1}{n}}$$

or

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$

The number R is unique and R = ∞ is allowed, in that case the series converges for arbitrarily large |z|.

The given power series $\sum_{n=0}^{\infty} a_n z^n$ and the derived series $\sum_{n=1}^{\infty} n\, a_n z^{n-1}$ (obtained by differentiating the given series) have the same radius of convergence due to the fact that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

1.17. Remark. Our interest in power series is in their behaviour as functions. The power series can be used to give examples of analytic functions. A power series $\sum_{n=0}^{\infty} a_n z^n$ with non-zero radius of convergence R, converges for |z|< R, and so we can define a function f by $f(z) = \sum_{n=0}^{\infty} a_n z^n$ (|z| < R). The function $f(z)$ is called the sum function of the power series.

1.18. Theorem. A power series represents an analytic function inside its circle of convergence.

Proof. Let the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ be R and let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,\ \phi(z) = \sum_{n=1}^{\infty} n\, a_n z^{n-1}.$$

The radius of convergence of the second series is also R. Suppose that z is any point within the circle of convergence so that |z| < R. Then there exists a positive number r such that |z| < r < R. For convenience, we write |z| = ρ , |h| = ∈. Then ρ < R. Also h may be so chosen that ρ + ∈ < r.

Since $\Sigma\, a_n z^n$ is convergent in $|z| < R$, $\Sigma\, a_n r^n$ is bounded for $0 < r < R$ so that $|a_n r^n| < M$ where M is finite positive constant. Thus we have

$$\left|\frac{f(z+h)-f(z)}{h}-\phi(z)\right|$$
$$= \left|\sum_{n=0}^{\infty} a_n\left[\frac{(z+h)^n - z^n}{h} - n z^{n-1}\right]\right|$$
$$= \left|\sum_{n=0}^{\infty} a_n\left[\frac{n(n-1)}{2} z^{n-2}h + \ldots + h^{n-1}\right]\right|$$
$$\le \sum_{n=0}^{\infty} |a_n| \left[\frac{n(n-1)}{2} |z|^{n-2} |h| + \ldots + |h|^{n-1}\right]$$
$$< \sum_{n=0}^{\infty} \frac{M}{r^n}\left[\frac{n(n-1)}{2}\rho^{n-2} \in + \ldots + \in^{n-1}\right]$$
$$= \sum_{n=0}^{\infty} \frac{M}{r^n}\frac{1}{\in}\left[\frac{n(n-1)}{2}\rho^{n-2} \in^2 + \ldots + \in^{n}\right]$$
$$= \sum_{n=0}^{\infty} \frac{M}{r^n \in}\left[(\rho+\in)^n - \rho^n - n\rho^{n-1}\in\right]$$
$$= \frac{M}{\in}\sum_{n=0}^{\infty}\left[\left(\frac{\rho+\in}{r}\right)^n - \left(\frac{\rho}{r}\right)^n + \frac{\in}{\rho} n\left(\frac{\rho}{r}\right)^n\right] \qquad (2)$$

Now, $$\sum_{n=0}^{\infty}\left(\frac{\rho+\in}{r}\right)^n = 1 + \frac{\rho+\in}{r} + \left(\frac{\rho+\in}{r}\right)^2 + \ldots..$$
$$= \frac{1}{1-\frac{\rho+\in}{r}}$$
$$= \frac{r}{r-\rho-\in}$$

and $$\sum_{n=0}^{\infty}\left(\frac{\rho}{r}\right)^n = 1 + \frac{\rho}{r} + \left(\frac{\rho}{r}\right)^2 + \ldots...$$
$$= \frac{1}{1-\frac{\rho}{r}} = \frac{r}{r-\rho}$$

Let us write $S = \Sigma\, n\left(\frac{\rho}{r}\right)^n = 1.\frac{\rho}{r} + 2.\left(\frac{\rho}{r}\right)^2 + 3\left(\frac{\rho}{r}\right)^3 + \ldots$

Then $$S\frac{\rho}{r} = \left(\frac{\rho}{r}\right)^2 + 2\left(\frac{\rho}{r}\right)^3 + \ldots....$$

Subtracting, we get

$$S\left(1-\frac{\rho}{r}\right) = \frac{\rho}{r} + \left(\frac{\rho}{r}\right)^2 + \ldots$$
$$= \frac{\rho/r}{1-\rho/r} = \frac{\rho}{r-\rho}$$

or $$S = \frac{\rho r}{(r-\rho)^2}$$

Using the values of these sums, (2) becomes

$$\left|\frac{f(z+h)-f(z)}{h} - \phi(z)\right| < \frac{M}{\in}\left[\frac{r}{r-\rho-\in} - \frac{r}{r-\rho} + \frac{\in r}{(r-\rho)^2}\right]$$

$$= \frac{Mr\in}{(r-\rho-\in)(r-\rho)^2}$$

which tends to zero as $\in \to 0$

Hence $$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \phi(z)$$

It follows that $f(z)$ has the derivative $\phi(z)$. Thus $f(z)$ is differentiable so that $f(z)$ is analytic for $|z| < R$.

Again, since the radius of convergence of the derived series is also R, so $\phi(z)$ is also analytic in $|z|<R$. Successively differentiating and applying the theorem, we see that the sum function $f(z)$ of a power series possesses derivatives of all orders within its circle of convergence and all these derivatives are obtained by term by term differentiation of the series. In other words, a power series represents an analytic function inside its circle of convergence.

2. Complex Integration

Let [a, b] be a closed interval, where a, b are real numbers. Divide [a, b] into subintervals

$$[a = t_0, t_1], [t_1, t_2], \ldots, [t_{n-1}, t_n = b] \qquad (1)$$

by inserting n−1 points $t_1, t_2, \ldots, t_{n-1}$ satisfying the inequalities

$$a = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = b$$

Then the set $P = \{t_0, t_1, \ldots, t_n\}$ is called the partition of the interval [a, b] and the greatest of the numbers $t_1\ t_0,\ t_2 - t_1, \ldots, t_n - t_{n-1}$ is called the norm of the partition P. Thus the norm of the partition P is the maximum length of the subintervals in (1).

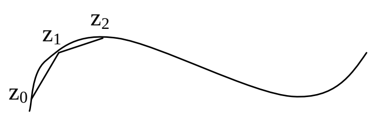
2.1. Arcs and Curves in the Complex Plane. An arc (path) L in a region $G \subset \forall$ is a continuous function $z(t) : [a, b] \to G$ for $t\ \varepsilon\ [a, b]$ in R. The arc L, given by $z(t) = x(t) + iy(t)$, $t\ \varepsilon\ [a, b]$, where $x(t)$ and $y(t)$ are continuous functions of t, is therefore a set of all image points of a closed interval under a continuous mapping. The arc L is said to be differentiable if $z'(t)$ exists for all t in [a, b]. In addition to the existence of $z'(t)$, if $z'(t) : [a, b] \to \forall$ is continuous, then $z(t)$ is a smooth arc. In such case, we may say that L is regular and smooth. Thus a regular arc is characterized by the property that $\dot{x}(t)$ and $\dot{y}(t)$ exist and are continuous over the whole range of values of t.

We say that an arc is simple or Jordan arc if $z(t_1) = z(t_2)$ only when $t_1 = t_2$ i.e. the arc does not intersect itself. If the points corresponding to the values a and b coincide, the arc is said to be a closed arc (closed curve). An arc is said to be piecewise continuous in [a,b] if it is continuous in every subinterval of [a, b].

2.2. Rectifiable Arcs. Let $z = x(t) + iy(t)$ be the equation of the Jordan arc L, the range for the parameter t being $t_0 \le t \le T$.

Let $z_0, z_1, \ldots, z_n$ be the points of this arc corresponding to the values $t_0, t_1, \ldots, t_n$ of t, where $t_0 < t_1 < t_2 < \ldots < t_n = T$. Evidently, the length of the polygonal arc obtained by joining successively z_0 and z_1, z_1 and z_2 etc by st. line segments is given by

$$\Sigma_n = \sum_{r=1}^{n} |z_r - z_{r-1}|$$
$$= \sum_{r=1}^{n} |(x_r + iy_r) - (x_{r-1} + iy_{r-1})|$$
$$= \sum_{r=1}^{n} |(x_r - x_{r-1}) + i\,(y_r - y_{r-1})|$$
$$= \sum_{r=1}^{n} [(x_r - x_{r-1})^2 + (y_r - y_{r-1})^2]^{1/2}$$



If this sum Σ_n tends to a unique limit $l<\infty$, as $n\to\infty$ and the maximum of the differences $t_r - t_{r-1}$ tends to zero, we say that the arc L defined by z = x(t) + iy(t) is rectifiable and that its length is *l*. In this connection, we have the following result.

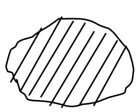
"A regular arc z = x(t) + iy(t), $t_0 \le t \le T$ is rectifiable and its length is

$$\int_{t_0}^{T} [(\dot{x}(t))^2 + (\dot{y}(t))^2]^{1/2}\, dt".$$

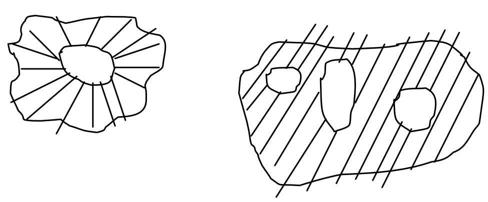
2.3. Contours. Let PQ and QR to be two rectifiable arcs with only Q as common point, then the arc PR is evidently rectifiable and its length is the sum of lengths of PQ and QR. Thus it follows that Jordan arc which consists of a finite number of regular arcs is rectifiable, its length being the sum of lengths of regular arcs of which it is composed. Such an arc is called contour. Thus a contour C is continuous chain of finite number of regular arcs. i.e. a contour is a piecewise smooth arc.

By a closed contour we shall mean a simple closed Jordan arc consisting of a finite number of regular arcs. Clearly, every closed contour is rectifiable. Circle rectangle, ellipse etc. are examples of closed contour.

2.4. Simply Connected Region A region D is said to be simply connected if every simple closed contour within it encloses only points of D. In such a region every closed curve can be shrunk (contracted) to a point without passing out of the region(Fig.1). If the region is not simply connected, then it is called multiply connected(Fig. 2).



Simply connected region
Fig. 1



Multiply connected regions
Fig. 2

2.5. Riemann's Definition of Complex Integration

First, we define the integral as the limit of a sum and later on, deduce it as the operation inverse to that of differentiation.

Let us consider a function $f(z)$ of the complex variable z. We assume that $f(z)$ has a definite value at each point of a rectifiable arc L having equation

$$z(t) = x(t) + iy(t),\ t_0 \le t \le T.$$

We divide this arc into n smaller arcs by points $z_0, z_1, z_2, \ldots, z_{n-1}, z_n$ (= Z, say) which correspond to the values

$t_0 < t_1 < t_2, \ldots, < t_{n-1} < t_n$ (= T) of the parameter t and then form the sum

$$\Sigma = \sum_{r=1}^{n} f(\xi_r)\,(z_r - z_{r-1})$$

where ξ_r is a point of L between z_{r-1} and z_r. If this sum Σ tends to a unique limit I as $n\to\infty$ and the maximum of the differences $t_r - t_{r-1}$ tends to zero, we say that $f(z)$ is integrable from z_0 to Z along the arc L, and we write

$$I = \int_L f(z)\, dz$$

The direction of integration is from z_0 to Z, since the points on x(t) + iy(t) describe the arc L in this sense when t increases.

2.6. Remarks. (i) Some of the most obvious properties of real integrals extend at once to complex integrals, for example,

$$\int_L [f(z) + g(z)]\, dz = \int_L f(z)\, dz + \int_L g(z)\, dz,$$

$$\int_L K\, f(z)\, dz = K \int_L f(z)\, dz, \text{ K being constant}$$

and

$$\int_{L'} f(z)\, dz = -\int_L f(z)\, dz,$$

where L′ denotes the arc L described in opposite direction.

(ii) In the above definition of the complex integral, although z_0, Z play much the same parts as the lower and upper limits in the definite integral of a function of a real variable, we do not write

$$I = \int_{z_0}^{Z} f(z)\, dz$$

This is dictated essentially by the fact that the value of I depends, in general, not only on the initial and final points of the arc L but also on its actual form.

In special circumstances, the integral may be independent of path from z_0 to Z as shown in the following example.

2.7. Example. Using the definition of an integral as the limit of a sum, evaluate the integrals

(i) $\int_L dz$ (ii) $\int_L |dz|$ (iii) $\int_L z\, dz$

where L is a rectifiable arc joining the points $z = \alpha$ and $z = \beta$.

Solution. We first observe that the integrals exist since the integrand is continuous on L in each case.

(i) By definition we have.

$$\int_L dz = \lim_{n\to\infty} \sum_{r=1}^{n} (z_r - z_{r-1})\, 1$$
$$= \lim_{n\to\infty} [z_1 - z_0 + z_2 - z_1 + \ldots + z_n - z_{n-1}]$$
$$= \lim_{n\to\infty} (z_n - z_0) = \beta - \alpha$$

(ii) $$\int_L |dz| = \lim_{n\to\infty} \sum_{r=1}^{n} |z_r - z_{r-1}|$$
$$= \lim_{n\to\infty} [|z_1 - z_0| + |z_2 - z_1| + \ldots + |z_n - z_{n-1}|]$$

= Arc length of L
= l (say)

(iii) Let $$I = \int_L z\, dz = \lim_{n\to\infty} \sum_{r=1}^{n} (z_r - z_{r-1})\, \xi_r \qquad (1)$$

where ξ_r is any point on the sub arc joining z_{r-1} and z_r.
Since ξ_r is arbitrary, we set $\xi_r = z_r$ and $\xi_{r-1} = z_{r-1}$ successively in (1) to find

$$I = \lim_{n\to\infty} \sum_{r=1}^{n} z_r (z_r - z_{r-1})$$
$$I = \lim_{n\to\infty} \sum_{r=1}^{n} z_{r-1} (z_r - z_{r-1})$$

Adding these two results, we get

$$2I = \lim_{n\to\infty} \sum_{r=1}^{n} (z_r + z_{r-1})(z_r - z_{r-1})$$
$$= \lim_{n\to\infty} \sum_{r=1}^{n} (z_r^2 - z_{r-1}^2) = \lim_{n\to\infty} (z_n^2 - z_0^2) = \beta^2 - \alpha^2$$
$$\therefore \qquad I = \frac{1}{2}(\beta^2 - \alpha^2)$$

In particular, if L is closed, then $\beta = \alpha$ and thus
$$\int_L dz = 0, \int_L z\, dz = 0.$$

2.8. Theorem (Integration along a regular arc). Let $f(z)$ be continuous on the regular arc L whose equation is $z(t) = x(t) + iy(t)$, $t_0 \le t \le T$. Prove that $f(z)$ is integrable along L and that

$$\int_L f(z)\, dz = \int_{t_0}^{T} F(t)\, [\dot{x}(t) + i\, \dot{y}(t)]dt\,,$$

where F(t) denotes the value of $f(z)$ at the point of L corresponding to the parametric value t.

Proof. Let us consider the sum

$$\Sigma = \sum_{r=1}^{n} f(\xi_r)\, (z_r - z_{r-1})$$

where ξ_r is a point of L between z_{r-1} and z_r. If τ_r is the value of the parameter t corresponding to ξ_r, then τ_r lies between t_{r-1} and t_r. Writing $F(t) = \phi(t) + i\psi(t)$, where ϕ and ψ are real, we find that

$$\Sigma = \sum_{r=1}^{n} [\phi(\tau_r) + i\psi(\tau_r)]\, [x_r - x_{r-1}) + i(y_r - y_{r-1})]$$
$$= \sum_{r=1}^{n} \phi(\tau_r)\, (x_r - x_{r-1}) + i \sum_{r=1}^{n} \phi(\tau_r)\, (y_r - y_{r-1})$$

$$+ i \sum_{r=1}^{n} \psi(\tau_r)(x_r - x_{r-1}) - \sum_{r=1}^{n} \psi(\tau_r)(y_r - y_{r-1})$$
$$= \Sigma_1 + i\,\Sigma_2 + i\,\Sigma_3 - \Sigma_4 \text{ (say)}$$
$$= \Sigma_1 - \Sigma_4 + i(\Sigma_2 + \Sigma_3)$$

We consider these four sums separately.

By the mean value theorem of differential calculus, the first sum is

$$\Sigma_1 = \sum_{r=1}^{n} \phi(\tau_r)(x_r - x_{r-1})$$
$$= \sum_{r=1}^{n} \phi(\tau_r)\, \dot{x}(\tau_r')(t_r - t_{r-1})$$

$$(f(a + h) - f(a) = hf'(a + \theta h),\ 0 \le \theta \le 1$$
$$x_r - x_{r-1} = x(t_r) - x(t_{r-1})$$
$$= (t_r - t_{r-1})\ \dot{x}(\tau_r'))$$

where τ_r' his between t_{r-1} and t_r.

We first show that Σ_1 can be made to differ by less than an arbitrary positive number, however small, from the sum

$$\Sigma_1' = \sum_{r=1}^{n} \phi(t_r)\, \dot{x}(t_r)(t_r - t_{r-1})$$

by making the maximum of the differences $t_r - t_{r-1}$sufficiently small.

Now, by hypothesis, the functions $\phi(t)$ and $\dot{x}(t)$ are continuous. As continuous functions are necessarily bounded, there exist a positive number K such that the inequalities

$$|\phi(t)| \le K,\ |\dot{x}(t)| \le K$$

hold for $t_0 \le t \le T$.

Moreover, the functions are also uniformly continuous, we can, therefore, preassign an arbitrary positive number $\in$, as small as we please, and then choose a positive number δ, depending on $\in$, such that

$$|\phi(t) - \phi(t')| < \in,\ |\dot{x}(t) - \dot{x}(t')| < \in,$$

whenever $|t - t'| < \delta$

Hence if the maximum of the differences $t_r - t_{r-1}$ is less than δ, we have

$$|\phi(\tau_r)\, \dot{x}(\tau_r') - \phi(t_r)\, \dot{x}(t_r)|$$
$$= |\ \phi(\tau_r)\ \{\dot{x}(\tau_r') - \dot{x}(t_r)\} + \dot{x}(t_r)\ \{\phi(\tau_r) - \phi(t_r)\}|$$
$$\le |\phi(\tau_r)|.|\ \dot{x}(\tau_r') - \dot{x}(t_r)\ | + |\ \dot{x}(t_r)|.|\phi(\tau_r) - \phi(t_r)|$$
$$< 2K\in$$

and therefore

$$|\Sigma_1 - \Sigma_1'| < 2K\in (T - t_0)$$

By the definition of the integral of a continuous function of a real variable, Σ_1' tends to the limit

$$\int_{t_0}^{T} \phi(t)\ \dot{x}(t)dt \qquad \Big|\ \int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\delta x_i$$

as $n\to\infty$ and the maximum of the differences $t_r - t_{r-1}$ tends to zero. Since $|\Sigma_1 - \Sigma_1'|$ can be made as small as we please by taking δ small enough, Σ_1 must also tend to the same limit.

Similarly the other terms of Σ tend to limits. Combining these results we find that Σ tends to the limit

$$\int_{t_0}^{T} [\phi(t)\ \dot{x}(t) - \psi(t)\ \dot{y}(t)]\, dt$$
$$+ i \int_{t_0}^{T} [\psi(t)\ \dot{x}(t) + \phi(t)\ \dot{y}(t)]\, dt$$
$$= \int_{t_0}^{T} F(t)[\dot{x}(t) + i\ \dot{y}(t)]\, dt$$

and so $f(z)$ is integrable along the regular arc L.

2.9. Remark. The result of the above theorem is not merely of theoretical importance as an existence theorem. It is also of practical use since it reduces the problem of evaluating a complex integral to the integration of two real functions of a real variable.

More generally, it can be shown that if $f(z)$ is continuous on a contour C, it is integrable along C, the value of its integral being the sum of the integrals of $f(z)$ along the regular arcs of which C is composed.

2.10. Theorem. (Absolute value of a complex integral). If $f(z)$ is continuous on a contour C of length l, where it satisfies the inequality

$$|f(z)| \le M, \text{ then } \left|\int_C f(z)dz\right| \le M\, l$$

Proof. Without loss of generality, we assume that C is a regular arc.

Now, if g(t) is any complex continuous function of the real variable t, we have.

$$\left|\sum_{r=1}^{n} g(t_r)(t_r - t_{r-1})\right| \le \sum_{r=1}^{n} |g(t_r)|\,(t_r - t_{r-1})$$

and so, on proceeding to the limit, we get

$$\left|\int_{t_0}^{T} g(t)\,dt\right| \le \int_{t_0}^{T} |g(t)|\,dt$$

Hence, using the result of the previous theorem, we have

$$\left|\int_C f(z)dz\right| = \left|\int_C F(t)[\dot{x}(t) + i\ \dot{y}(t)]dt\right|$$
$$\le \int_{t_0}^{T} |F(t)|\,|\dot{x}(t) + i\ \dot{y}(t)|\,dt$$
$$\le M\int_{t_0}^{T} |\dot{x}(t) + i\ \dot{y}(t)|\,dt$$
$$(f(z) = F(t) \text{ on C } \Rightarrow |F(t)| \le M)$$
$$= M \int_{t_0}^{T} \left|\frac{dz}{dt}\right| dt$$
$$= M\int_{t_0}^{T} |dz| = M\, l.$$

2.11. Remarks. (i) The result of the above theorem (2.10) is also called estimate of the integral.

(ii) So for we had assumed that $f(z)$ is only continuous on the regular arc L along which we take its integral. We now impose the restriction that $f(z)$ is analytic and suppose further that L lies entirely within the simply connected domain D within which $f(z)$ is regular. Then $\int_L f(z)\, dz$ certainly exists, since $f(z)$ is necessarily continuous on L. But we are now in a position to infer much more about this integral i.e. the integral is independent of path of integration. An equivalent form of this result is Cauchy theorem - the keystone in the theory of analytic functions.

2.12. Cauchy Theorem (Elementary Form). First we consider the elementary form of Cauchy theorem which requires the additional assumption that the derivative of *f*(z) is continuous. This form of Cauchy theorem is also known as Cauchy fundamental theorem, which has the following statement.

If *f*(z) is analytic function whose derivative $f'(z)$ exists and is continuous at each point within and on a closed contour C, then

$$\int_C f(z)\, dz = 0$$

Proof. Let D denotes the closed region which consists of all points within and on C. If we write $z = x + iy$, $f(z) = u + iv$, then we have

$$\int_C f(z)dz = \int_C (u + iv)\,(dx + idy)$$

$$= \int_C (u\,dx - vdy) + i \int_C (v\,dx + udy) \qquad (1)$$

Now, we use the Green's theorem for a plane which states that if P(x, y), Q(x, y), $\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ are continuous functions within a domain D and if C is any closed contour in D, then

$$\int_C (P\,dx + Qdy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \qquad (2)$$

By hypothesis $f'(z)$ exists and is continuous in D, so u and v and their partial derivatives u_x, v_x, u_y, v_y are continuous functions of x and y in D. Thus the conditions of Green's theorem are satisfied. Hence applying this theorem in (1), we obtain

$$\int_C f(z)dz = \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$= \iint_D \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dx\,dy$$

(using C−R equations)

$$= 0 + i\,0 = 0$$

Hence the result.

2.13. The General Form of Cauchy's Theorem (Cauchy-Goursat Theorem). An important step was pointed out by Goursat who showed that it is unnecessary to assume the continuity of $f'(z)$, and that Cauchy's theorem is true if it is only assumed that $f'(z)$ exists at each point within and on C. Actually, the continuity of the derivative $f'(z)$ and its differentiability are consequences of Cauchy's theorem. The theorem states as follows:

If a function *f*(z) is analytic and one-valued within and on a simple closed contour C, then

$$\int_C f(z)\, dz = 0$$

Proof. First of all, we observe that the integral certainly exists, since a function which is analytic is continuous and a continuous function is integrable. For the proof of the theorem, we divide up the region inside the closed contour C into a large number of sub-regions by a network of lines parallel to the real and imaginary axes. Suppose that this divides the inside of C into a number of squares $C_1, C_2,\ldots C_M$ say, and a number of irregular regions $D_1, D_2,\ldots, D_N$ say, parts of whose boundaries are parts of C (Fig. 1) .

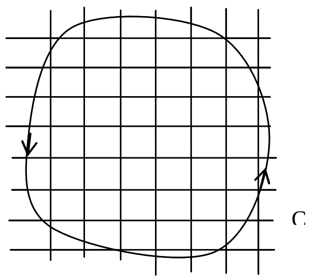


Fig. 1

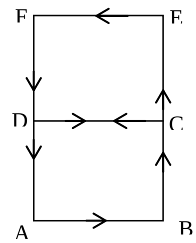


Fig. 2

Then

$$\int_C f(z)\,dz = \sum_{m=1}^{M} \int_{C_m} f(z)\,dz + \sum_{n=1}^{N} \int_{D_n} f(zdz \qquad (1)$$

where each contour is described in positive (anti-clockwise) direction.

Consider, for example, any two adjacent squares ABCD and DCEF with common side CD (Fig.2). The side CD is described from C to D in the first square and from D to C in the second. Hence the two integrals along CD cancel. So all the integrals cancel except those which form part of C itself, since these are described once only. Moreover within the integrals of R.H.S. of (1), there are contained integral along all the parts of the contour C into which C is divided on account of the subdivision. Thus the result (1) is true.

We now use the fact that $f(z)$ is analytic at every point. This means that, if z_0 is any point inside or on C, then

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| < \in$$

provided that $0 < |z - z_0| < \delta = \delta(z_0)$

i.e. if $|z - z_0| < \delta$, then

$$|f(z) - f(z_0) - (z-z_0)\, f'(z_0)| \le \in |z - z_0| \qquad (2)$$

If we consider any particular region C_m or D_n in the above construction, it is evident that we can choose its side so small that (2) is satisfied if z_0 is a given point of the region, and z is any other point. It is not, however, immediately obvious that we can choose the whole network so that the conditions are satisfied in all the partial regions at the same time. We shall prove that this is actually possible. i.e. "having given $\in$, we can choose the network in such a way that, in every C_m or D_n, there is a point z_0 such that (2) holds for every z in this region". This actually, means that the function is uniformly differentiable throughout the interior of C. We prove it by well known process of subdivision.

Suppose that we start with a network of parallel lies at constant distance l between every consecutive pair of lines. Some of the squares formed by these lines may each contain a point z_0 of the desired type. We leave these squares as they are. The rest we subdivide by lines midway between the previous lines. If there still remain any parts which do not have the required property, we subdivide them again in the same way.

Obviously, there are two distinct possibilities. The process may terminate after a finite number of steps and then the result is obtained, or it may go on indefinitely.

In the second case, there is atleast one region which we can subdivide indefinitely without obtaining the required result. We call this region, including its boundaries, R_1. After the first subdivision, we obtain a part R_2 contained in R_1 with the same property. Proceeding in this way, we have an infinity of regions $R_1, R_2, \ldots, R_n$ each contained in the previous one, and in each of which inequality (2) is impossible.

Since $R_1 \supset R_2 \supset R_3, \ldots$, there must be a point z_0 common to all the regions R_n ($n = 1, 2, \ldots$) and since the dimensions of R_n decrease indefinitely, we can have $|z-z_0| < \delta$ for sufficiently large n, say $n > n_0$ and for every z in R_n. But $f(z)$ is analytic at z_0. Hence (2) holds for this z_0 in R_n if $n > n_0$. This contradicts the statement that in no R_n, there exists a point z_0 satisfying inequality (2). Thus the second possibility is ruled out and (2) is satisfied for every point in the region C.

Now, let us consider one of the squares C_m of side l_m. In C_m, by inequality (2), we have

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + \phi(z),$$

where

$$|\phi(z)| \le \in |z - z_0|$$

Hence,

$$\int_{C_m} f(z)\, dz = \int_{C_m} [f(z_0) + (z-z_0) f'(z_0)]\, dz + \int_{C_m} \phi(z)dz \qquad (3)$$

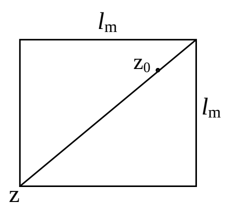
The first integral in (3) simplifies to

$$[f(z_0) - z_0 f'(z_0)] \int_{C_m} dz + f'(z_0) \int_{C_m} zdz$$

and therefore vanishes, since $\int_{C_m} dz = 0$, $\int_{C_m} z\, dz = 0$ (By definition). Also, by virtue of the result regarding absolute value of a complex integral, we obtain.

$$\left|\int_{C_m} \phi(z)dz\right| < \in \int_{C_m} |z-z_0|\, |dz|$$
$$< \in \sqrt{2}\, l_m.\, 4l_m,$$

since $|z-z_0| \le \sqrt{2}\, l_m$ for z_0 inside C_m and z on C_m and the length of C_m is $4l_m$.



In the case of any one of the irregular region D_n, the length of the contour is not greater than $ul_n + \delta_n$, where δ_n is the length of the curved part of the boundary. Hence

$$\left|\int_{D_n} \phi(z)dz\right| < \in \sqrt{2}\, l_n (4l_n + \delta_n).$$

Adding all the parts, we obtain

$$\left|\int_C f(z)dz\right| \le \sum_{m=1}^{M} \left|\int_{C_m} f(z)dz\right| + \sum_{n=1}^{N} \left|\int_{D_n} f(z)dz\right|$$
$$= \sum_{m=1}^{M} \left|\int_{C_m} \phi(z)\, dz\right| + \sum_{n=1}^{N} \left|\int_{D_n} \phi(z)dz\right|$$
$$< \Sigma \in \sqrt{2}\, 4l_m^2 + \Sigma \in \sqrt{2}\, l_n (4l_n + \delta_n)$$
$$< 4\sqrt{2} \in \Sigma(l_m^2 + l_n^2) + \in \sqrt{2}\, l\, \Sigma\, \delta_n \qquad (4)$$

where l denotes some constant greater than every one of the l_n's. Now $\Sigma(l_m^2 + l_n^2)$ is the area of a region which just includes C and is therefore bounded. Also $\Sigma\, \delta_n$ is the length of the contour C. Hence the R.H.S. of (4) is less than a constant multiple of $\in$. But the L.H.S. is independent of $\in$, and $\in$ is arbitrarily small, it follows therefore that

$$\int_C f(z)\, dz = 0$$

which proves the theorem.

2.14. Cor. Suppose $f(z)$ is analytic in a simply connected domain D, then the integral along any rectifiable curve in D joining any two points of D is the same i.e. it does not depend on the curve joining the two points i.e. integral is independent of path.

Proof. Suppose the two points $A(z_1)$ and $B(z_2)$ of the simply connected domain D are joined by the curves C_1 and C_2 as shown in the figure.

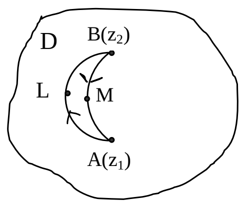
Then, by Cauchy's theorem.

$$\int_{ALBMA} f(z)\,dz = 0$$

i.e. $\int_{ALB} f(z)dz + \int_{BMA} f(z)dz = 0$

i.e. $\int_{ALB} f(z)dz - \int_{AMB} f(z)dz = 0$

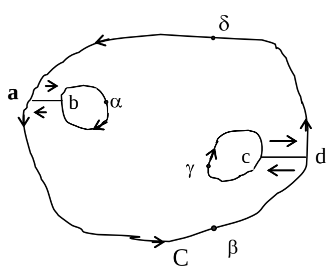
i.e. $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$.



2.15. Extension of Cauchy's Theorem to Contours Defining Multiply Connected Regions. By adopting a suitable convention as to the sense of integration, Cauchy's theorem can be extended to the case of contours which are made up of several distinct closed contours. Consider, for example, a function $f(z)$ which is analytic in the multiply connected region R bounded by the closed contour C and the two interior contours C_1, C_2 as well as on these contours themselves. The complete contour C* which is the boundary of the region R is made up of the three contours C, C_1 and C_2 and we adopt the convention that C* is described in the positive sense if the region R is on the L.H.S. w.r.t. this sense of describing it. Then by Cauchy's theorem

$$\int_{C^*} f(z)\,dz = 0$$

where the integral is taken round the complete contour C* in the positive sense.



Practically, we deal with this case by drawing transversals like ab, cd and by applying Cauchy's theorem for a simple closed contour abαbaβdcγcdδa. It is found convenient in applications to express the same result in the form

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

where all the three integrals are now taken in the same (positive) sense.

An exactly similar result holds in case there are any finite number of closed contours $C_1, C_2, \ldots, C_m$ inside a closed contour C and $f(z)$ is analytic in the multiply connected region bounded by them as well as on them. We then have

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \ldots + \int_{C_m} f(z)dz.$$

where all the contours are described in positive sense.

2.16. Theorem. (Cauchy's Integral Formula). Let $f(z)$ be analytic inside and on a closed contour C and let z_0 be any point inside C. Then

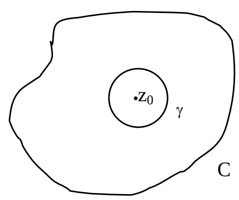
$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz$$

Proof. We consider the function $\frac{f(z)}{z-z_0}$ This function is analytic throughout the region bounded by C except at $z = z_0$.

Then, by 2.15, we have

$$\int_C \frac{f(z)}{z-z_0}\,dz = \int_\gamma \frac{f(z)}{z-z_0}\,dz$$

where γ is any closed contour inside C including the point z_0 as an interior point.



Let us choose γ to be the circle with centre z_0 and radius ρ. Since $f(z)$ is continuous, we can take ρ so small that on γ,

$$| f(z) - f(z_0) | < \in$$

where $\in$ is any preassigned positive number.

Now,

$$\int_\gamma \frac{f(z)}{z-z_0}\,dz = \int_\gamma \frac{[f(z)-f(z_0)]+f(z_0)}{z-z_0}\,dz$$

$$= f(z_0)\int_\gamma \frac{dz}{z-z_0}\,dz + \int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,dz \qquad (1)$$

For any point z on γ,

$$z - z_0 = \rho\, e^{i\theta} \;\Rightarrow dz = \rho\, i\, e^{i\theta}\, d\theta$$

$$\therefore \qquad \int_\gamma \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{\rho e^{i\theta} d\theta}{\rho e^{i\theta}} = \int_0^{2\pi} i d\theta = 2\pi i$$

and

$$\left|\int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| = \left|\int_0^{2\pi} \frac{[f(z)-f(z_0)]}{\rho e^{i\theta}}\rho e^{i\theta} i d\theta\right|$$

$$= \left|\int_0^{2\pi} [f(z)-f(z_0)] i d\theta\right|$$

$$< \in \int_0^{2\pi} d\theta = 2\pi\in$$

Hence from (1), we get

$$\left|\int_C \frac{f(z)}{z-z_0}\,dz - 2\pi i\, f(z_0)\right| < 2\pi\in$$

Since $\in$ is arbitrarily small and L.H.S. is independent of $\in$, it follows that

$$\int_C \frac{f(z)}{z-z_0}\,dz - 2\pi i f(z_0) = 0$$

i.e.
$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz.$$

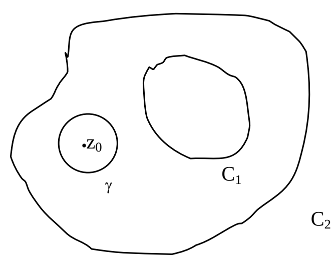
which proves the result.

2.17. Cor. (Extension of Cauchy's Integral Formula to Multiply Connected Regions): If $f(z)$ is analytic in a ring shaped region bounded by two closed contours C_1 and C_2 and z_0 is a point in the region between C_1 and C_2, then

$$f(z_0) = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-z_0}\,dz .$$

where C_2 is the outer contour.

Proof. Describe a circle γ of radius ρ about the point z_0 such that the circle lies in the ring shaped region. The function $\frac{f(z)}{z-z_0}$ is analytic in the region bounded by three close contours C_1, C_2 and γ.



Thus by 2.15, we have.

$$\int_{C_2} \frac{f(z)}{z-z_0}\,dz = \int_{C_1} \frac{f(z)}{z-z_0}\,dz + \int_{\gamma} \frac{f(z)}{z-z_0}\,dz$$

where the integral along each contour is taken in positive sense. Now, using Cauchy's integral formula, we find.

$$\int_{C_2} \frac{f(z)}{z-z_0}\,dz = \int_{C_1} \frac{f(z)}{z-z_0}\,dz + 2\pi i f(z_0)$$

or

$$f(z_0) = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-z_0}\,dz .$$

2.18. Poisson's Integral Formula. Let f (z) be analytic in the region $|z| \le R$, then for $0 < r < R$, we have

$$F(r\,e^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2) f(Re^{i\phi})}{R^2 - 2Rr\cos(\theta-\phi) + r^2}\,d\phi$$

where ϕ is the value of θ on the circle $|z| = R$.

Proof. Let C denote the circle $|z| = R$.
Let $z_0 = r\,e^{i\theta}$, $\theta < r < R$ by any point inside C, then by Cauchy's integral formula,

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}dz \qquad (1)$$

The inverse of z_0 w.r.t. the circle $|z| = R$ is $\frac{R^2}{\bar{z}_0}$ and lies outside the circle, so by Cauchy's theorem, we have

$$0 = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - \frac{R^2}{\bar{z}_0}}dz \qquad (2)$$

Subtracting (2) from (1), we get

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{(z_0 - \frac{R^2}{z_0})f(z)dz}{(z - z_0)(z - \frac{R^2}{\bar{z}_0})}$$

$$= \frac{1}{2\pi i}\int_C \frac{(R^2 - z_0\bar{z}_0)f(z)dz}{(z - z_0)(R^2 - z\bar{z}_0)} \qquad (3)$$

Now, any point on the circle C is expressible as $z = R\,e^{i\phi}$. Also $z_0 = r\,e^{i\theta}$, so $\bar{z}_0 = r\,e^{-i\theta}$
Therefore,

$$R^2 - z_0\bar{z}_0 = R^2 - r^2 \qquad (4)$$

$$(z-z_0)(R^2 - z\bar{z}_0) = z R^2 - z^2\bar{z}_0 - z_0 R^2 + z_0\bar{z}_0 z$$
$$= R^3 e^{i\phi} - R^2 e^{i2\phi} r\,e^{-i\theta} - r\,e^{i\theta}R^2 + r^2 R\,e^{i\phi}$$
$$= R\,e^{i\phi}\,[R^2 - 2rR\cos(\theta-\phi) + r^2] \qquad (5)$$

and

$$dz = R\,i\,e^{i\phi}\,d\phi$$

Thus, (3) becomes

$$f(r\,e^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)f(Re^{i\phi})d\phi}{R^2 - 2Rr\cos(\theta - \phi) + r^2} \qquad (6)$$

which is the required result.
Formula (6) can be separated into real and imaginary parts to get ($f(z) = u + iv$)

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)u(R,\phi)d\phi}{R^2 - 2Rr\cos(\theta - \phi) + r^2}$$

$$v(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)v(R,\phi)d\phi}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\ .$$

2.19. Theorem (The derivative of an analytic function). Let $f(z)$ be analytic within and on a closed contour C and let z_0 be any point inside C, then

$$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^2}dz$$

Proof. Let $z_0 + h$ be a point in the nighbourhood of z_0 and inside C, ($\Delta z = h$). Then Cauchy's Integral formula at these two points, gives

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0} dz.$$

and $$f(z_0 + h) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0 - h} dz$$

Subtracting the first result from second, we get

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)(z - z_0 - h)} dz \qquad (1)$$

We observe in (1) that as $h \to 0$, the required result follows. We have thus only to show that we can proceed to the limit under the integral sign. We consider the difference

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^2} dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)(z - z_0 - h)} - \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^2} dz$$

$$= \frac{h}{2\pi i}\int_C \frac{f(z)dz}{(z - z_0)^2 (z - z_0 - h)} \qquad (2)$$

Since $f(z)$ is analytic on C so $f(z)$ is bounded on C. Thus $|f(z)| \le M$ on C, M being an absolute positive constant. Let us denote the distance of z_0 from the points nearest to it on C by δ and the length of C by l. Then if $|h|<\delta$,

$$\left| h\int_C \frac{f(z)dz}{(z - z_0)^2 (z - z_0 - h)} \right| \le \frac{Ml\,|h|}{\delta^2(\delta - |h|)} \qquad (3)$$

which is bounded and tends to zero as $|h| \to 0$. Thus, taking limit as $|h| \to 0$, it follows from (2) that

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^2} dz$$

Hence $f(z)$ is differentiable at z_0 and

$$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^2} dz$$

which is Cauchy's integral formula for $f'(z)$ at points within C.

2.20. Generalization. This result (2.19) has a very significant consequence in the fact that $f'(z)$ is itself analytic within C. i.e. derivative of an analytic function is also analytic. To prove this, it is enough to show that $f'(z)$ has derivative at any point z_0 inside C.

Using Cauchy's integral formula for $f'(z_0)$ and $f'(z_0+h)$ with the same restriction on h as before, we get

$$f'(z_0+h) - f'(z_0) = \frac{1}{2\pi i}\int_C f(z)\left[\frac{1}{(z - z_0 - h)^2} - \frac{1}{(z - z_0)^2}\right] dz$$

i.e. $$\frac{f'(z_0 + h) - f'(z_0)}{h} = \frac{1}{2\pi i}\int_C \frac{f(z)(2z - 2z_0 - h)dz}{(z - z_0)^2 (z - z_0 - h)^2}$$

by means of arguments parallel to those used in the proof of Cauchy's formula for $f'(z_0)$, we can easily show that as $|h| \to 0$, the integral on R.H.S. tends to the limit

$$\frac{2}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^3}dz$$

Thus $f'(z)$ has a differential co-efficient at z_0, given by the formula

$$f''(z_0) = \frac{\lfloor 2}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^3}dz.$$

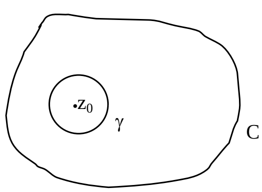
The arguments can obviously be repeated and we get the following result as a generalization. If $f(z)$ is analytic inside and on a closed contour C, it possesses derivatives of all orders which are all analytic inside C. The nth derivative $f^n(z_0)$ at any point z_0 inside C being given by the formula

$$f^n(z_0) = \frac{\lfloor n}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{n+1}}.$$

2.21. Remark. From Cauchy integral formula, we observe a remarkable fact about an analytic function. Its values everywhere inside a closed contour are completely determined by its values on the boundary. In fact the values of each derivative of an analytic function are determined just by the values of the function on the boundary.

4.22. Example. Evaluate $\int_C \frac{dz}{(z-z_0)^m}, m = 1, 2, ...M$ where C is a single closed contour.

Solution. The function $\frac{1}{(z-z_0)^m}$ is analytic except at z = z_0. Hence if C does not enclose z_0, then by Cauchy's theorem, the integral is zero. If C encloses z_0, then we choose a circle γ of small radius ρ with centre z_0.



Thus, we get

$$I = \int_C \frac{dz}{(z-z_0)^m} = \int_\gamma \frac{dz}{(z-z_0)^m}$$

On γ, $z - z_0 = \rho\, e^{i\theta}$, $dz = \rho e^{i\theta}\, i d\theta$

$$\therefore \qquad I = \int_0^{2\pi} \frac{\rho e^{i\theta} i d\theta}{\rho^m e^{im\theta}} = \int_0^{2\pi} i\, \rho^{-m+1}\, e^{-i(m-1)\theta}\, d\theta$$

$$= \begin{cases} 2\pi i, & \text{if } m = 1 \\ 0\ , & \text{if } m \neq 1 \end{cases}$$

Thus $$I = \begin{cases} 0, \text{ if } z = z_0 \text{ is outside C} \\ 0, \text{ if } z = z_0 \text{ is inside C}, m \neq 1 \\ 2\pi i \text{ if } z = z_0 \text{ is inside C}, m = 1 \end{cases}$$

4.23. Example. Evaluate

$$\int_C \frac{e^z}{z(z^2-16)}\,dz$$

where C is a closed contour between the circles of radius 1 and 3, centred at origin.

Solution. The integrand is analytic except at z = 0, z = $\pm$ 4 which are not points of the given region. Therefore, by Cauchy's theorem, the integral vanishes.

2.24. Example. Using Cauchy's Integral Formula show that

$$\int_C \frac{e^{2z}}{(z+1)^4}\,dz = \frac{8\pi e^{-2}}{3}i$$

where C is the circle |z| = 3

Solution. By Cauchy's integral formula for derivatives, we have

$$f^n(z_0) = \frac{\lfloor n}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{n+1}} \qquad (1)$$

where $f(z)$ is analytic inside and on C.

In the present case, C is |z| = 3, $f(z) = e^{2z}$, $z_0 = -1$, n = 3 and $f(z)$ is analytic inside and on the circle |z| = 3.
Also, $f^3(-1) = 8\,e^{-2}$

$\therefore$ (1) becomes

$$8e^{-2} = \frac{\lfloor 3}{2\pi i}\int_C \frac{e^{2z}}{(z+1)^4}\,dz$$

$$\Rightarrow \qquad \int_C \frac{e^{2z}}{(z+1)^4}\,dz = \frac{8\pi e^{-2}}{3}\,i$$

Hence the result.

2.25. Exercise. Using Cauchy's integral formula, prove that

(i) $\displaystyle\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}\,dz = 4\pi i$

where C is the circle |z| = 3

(ii) $\displaystyle\int_C \frac{\sin^6 z}{(z-\pi/6)^3}\,dz = \frac{21}{16}\pi i$

where C is the circle |z| = 1

(iii) $\displaystyle\int_C \frac{\cos z}{z(z-4)}\,dz = \frac{-\pi i}{2}$

where C is the circle |z| = 1

(iv) $\displaystyle\int_C \frac{e^{zt}}{(z^2+1)^2}\,dz = \frac{1}{2}(\sin t - t\cos t)$

where t > 0 and C is the circle |z| = 3.

2.26. A Complex Integral as a function of its upper limit. Let $f(z)$ be analytic in a region D and let

$$F(z) = \int_{z_0}^{z} f(w)\,dw$$

where z_0 is any fixed point in D and the path of integration is any contour from z_0 to z lying entirely in D. It follows from Cor. (2.14) to Cauchy's theorem that the value of F(z) depends on z only and not on the particular path of integration from z_0 to z. F(z) is called the indefinite integral of f(z). We prove below the analogue, in the theory of functions of a complex variable, of the well known 'fundamental theorem of integral calculus.' It asserts that the operations of integration and differentiation are inverse operations.

2.27. Theorem. The function F(z) is analytic in D and its derivative is f(z)

Proof. Since $F(z) = \int_{z_0}^{z} f(w)\,dw$

$\therefore$ $F(z+h) = \int_{z_0}^{z+h} f(w)\,dw$

Thus

$$F(z+h) - F(z) = \int_{z_0}^{z+h} f(w)dw - \int_{z_0}^{z} f(w)dw$$
$$= \int_{z}^{z_0} f(w)dw + \int_{z_0}^{z+h} f(w)dw$$
$$= \int_{z}^{z+h} f(w)dw$$

$$\Rightarrow \quad \frac{F(z+h)-F(z)}{h} = \frac{1}{h}\int_{z}^{z+h} f(w)dw$$

where by Cauchy's theorem, we may suppose that integral is taken along the straight line from z to $z+h$. Thus

$$\frac{F(z+h)-F(z)}{h} - f(z) = \frac{1}{h}\int_{z}^{z+h} f(w)dw - \frac{1}{h}\int_{z}^{z+h} f(z)dw$$
$$= \frac{1}{h}\int_{z}^{z+h} [f(w)-f(z)]dw$$

Since f(z) is analytic so it is continuous, given $\in > 0$, there exists a $\delta > 0$ such that

$$|f(w) - f(z)| < \in \text{ whenever } |w - z| < \delta$$

Therefore, if $0 < |h| < \delta$, we have

$$\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| \le \frac{1}{|h|}\int_{z}^{z+h} |f(w) - f(z)|\,|dw|$$
$$< \frac{1}{|h|}\int_{z}^{z+h} \in |dw| = \frac{1}{|h|} \in |h| = \in$$

Hence

$$\lim_{h\to 0}\frac{F(z+h)-F(z)}{h} - f(z) = 0$$

or $\quad F'(z) = f(z)$

which proves that F(z) is analytic and that its derivative is f(z).

2.28. Morera's Theorem. (Converse of Cauchy's theorem). If f(z) is continuous in a region D and if the integral $\int f(z)dz$ taken round any closed contour in D vanishes, then f(z) is analytic in D.

Proof. When the integral round a closed contour vanishes, then we know that the value of the integral

$$F(z) = \int_{z_0}^{z} f(w)dw$$

is independent of path of integration joining z_0 and z. Also, we have

$$\frac{F(z+h)-F(z)}{h} = \frac{1}{h}\int_z^{z+h} f(w)\,dw$$

and further

$$\frac{F(z+h)-F(z)}{z} - f(z) = \frac{1}{h}\int_z^{z+h} [f(w)-f(z)]\,dw$$

where we are free to assume that the path of integration is the straight line joining the points z and z + h. Since $f(z)$ is continuous in D, we find that (previous theorem 2.27)

$$F'(z) = f(z)$$

i.e. F(z) is analytic with derivative $f(z)$ But we have the result that derivative of an analytic function is analytic. Thus we finally conclude that $F'(z)$ i.e. $f(z)$ is analytic in D.

2.29. Cauchy's Inequality (Cauchy's Estimate). If $f(z)$ is analytic within and on a circle C given by $|z - z_0| = R$ and if $|f(z)| \le M$ for every z on C, then

$$|f^n(z_0)| \le \frac{M\lfloor n}{R^n}$$

Proof. Since $f(z)$ is analytic inside C, we have by Cauchy's integral formula for nth derivative of an analytic function

$$f^n(z_0) = \frac{\lfloor n}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz.$$

Since on the circle $|z - z_0| = R$,

$$z - z_0 = Re^{i\theta},\ dz = Re^{i\theta}\, i d\theta$$

and the length of the circle is $2\pi R$, therefore

$$\begin{aligned}
|f^n(z_0)| &= \frac{\lfloor n}{2\pi}\left|\int_C \frac{f(z)dz}{(z-z_0)^{n+1}}\right| \\
&\le \frac{\lfloor n}{2\pi}\int_C \frac{|f(z)||dz|}{|z-z_0|^{n+1}} \\
&\le \frac{\lfloor n}{2\pi}\int_0^{2\pi} \frac{M|Re^{i\theta}\, id\theta|}{|Re^{i\theta}|^{n+1}} = \frac{\lfloor n}{2\pi}\int_0^{2\pi} \frac{M}{R^n}\,d\theta \\
&= \frac{\lfloor n}{2\pi}\frac{M}{R^n}2\pi = \frac{M\lfloor n}{R^n}
\end{aligned}$$

Hence $$|f^n(z_0)| \le \frac{M\lfloor n}{R^n}$$

2.30. Liouville's Theorem. A function which is analytic in all finite regions of the complex plane, and is bounded, is identically equal to a constant.

or

If an integral function $f(z)$ is bounded for all values of z, then it is constant

or

The only bounded entire functions are the constant functions.

Proof. Let z_1, z_2 be arbitrary distinct points in z-plane and let C be a large circle with centre at origin and radius R such that C encloses z_1 and z_2 i.e. $|z_1| < R$, $|z_2| < R$.

Since $f(z)$ is bounded, there exists a positive number M such that $|f(z)| \le M\ \forall\ z$.
By Cauchy's integral formula,

$$f(z_1) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-z_1}$$

$$f(z_2) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-z_2}$$

$$\therefore \qquad f(z_2) - f(z_1) = \frac{1}{2\pi i}\int_C \frac{f(z)(z_2-z_1)}{(z-z_2)(z-z_1)}dz$$

Thus

$$|f(z_2) - f(z_1)| \le \frac{|z_2-z_1|}{2\pi}\int_C \frac{|f(z)||dz|}{|z-z_1||z-z_2|}$$

$$\le \frac{M|z_2-z_1|}{2\pi}\int_C \frac{|dz|}{|z-z_1||z-z_2|}$$

$$\le \frac{M|z_2-z_1|}{2\pi}\int_C \frac{|dz|}{(|z|-|z_1|)(|z|-|z_2|)} \quad |\because |z-z_1| \ge |z|-|z_1|$$

Now, on the circle C, $z = R\,e^{i\theta}$, $|z| = R$,

$dz = Re^{i\theta}\, id\theta$

Therefore,

$$|f(z_2) - f(z_1)| \le \frac{M|z_2-z_1|}{2\pi}\int_0^{2\pi} \frac{|Re^{i\theta}\, id\theta|}{(R-|z_1|)(R-|z_2|)}$$

$$= \frac{M|z_2-z_1|}{2\pi}\frac{R}{(R-|z_1|(R-|z_2|)}2\pi$$

$$= \frac{M|z_2-z_1|}{\left(1-\frac{|z_1|}{R}\right)\left(1-\frac{|z_2|}{R}\right)}.\frac{1}{R}$$

which tends to zero as $R\to\infty$.

Hence $f(z_2) - f(z_1) = 0$ i.e. $f(z_1) = f(z_2)$

But z_1, z_2 are arbitrary, this holds for all couples of points z_1, z_2 in the z-plane, therefore $f(z)$ = constant.

2.31. The Fundamental Theorem of Algebra. Any polynomial

$P(z) = a_0 + a_1 z + \ldots + z_n z^n$, $a_n \ne 0$, $n \ge 1$ has at least one point $z = z_0$ such that $P(z_0) = 0$ i.e. $P(z)$ has at least one zero.

Proof. We establish the proof by contradiction.

If $P(z)$ does not vanish, then the function $f(z) = \frac{1}{P(z)}$ is analytic in the finite z-plane. Also when $|z|\to\infty$, $P(z)\to\infty$ and hence $f(z)$ is bounded in entire complex plane, including infinity. Liouville's theorem then implies that $f(z)$ and hence $P(z)$ is a constant which violates $n \ge 1$ and thus contradicts the assumption that $P(z)$ does not vanish. Hence it is concluded that $P(z)$ vanishes at some point $z = z_0$

2.32. Remark. The above form of fundamental theorem of algebra does not tell about the number of zeros of $P(z)$. Another form which tells that $P(z)$ has exactly n zeros, will be discussed later on. Of course, here we can prove this result by using the process of algebra as follows :

By the fundamental theorem of algebra, proved above, P(z) has at least one zero say $z = z_0$ such that $P(z_0) = 0$

Then,

$$\begin{aligned} P(z) - P(z_0) &= a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n \\ &\quad - (a_0 + a_1 z_0 + a_2 z_0^2 + \ldots a_n z_0^n) \\ &= a_1 (z - z_0) + a_2(z^2 - z_0^2) + \ldots + a_n (z^n - z_0^n) \\ &= (z - z_0)\, Q(z) \end{aligned}$$

where Q(z) is a polynomial of degree (n–1). Applying the fundamental theorem of algebra again, we note that Q(z) has at least one zero, say z_1 (which may be equal to z_0) and so $P(z) = (z-z_0)(z-z_1) R(z)$, where R(z) is a polynomial of degree (n–2). Continuing in this manner, we see that P(z) has exactly n zeros.

2.33. Taylor's Series. We have observed that a convergent complex power series defines an analytic (holomorphic) function. Here, we discuss its converse i.e. we proceed to prove that if $f(z)$ is an analytic function, regular in a neighbourhood of the point z = a, it can be expanded in a series of powers of (z –a). These two results combine to demonstrate that a function is analytic in a region iff it is locally representable by power series. The following theorem extends Taylor's classical theorem in real analysis to analytic functions of a complex variable.

2.34. Taylor's Theorem. Suppose that $f(z)$ is analytic inside and on a closed contour C and let a be a point inside C. Then

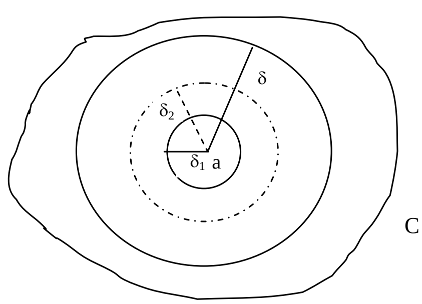
$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{\lfloor 2}(z-a)^2 + \ldots$$

$$\ldots\ldots + \frac{f^n(a)}{\lfloor n}(z-a)^n$$

$$= f(a) + \sum_{n=1}^{\infty} \frac{f^n(a)}{\lfloor n}(z-a)^n$$

The infinite series is convergent if $|z - a| < \delta$ where δ is the distance from a to the nearest point of C. In the region $|z-a| \le \delta_1$ where $\delta_1 < \delta$, the series is uniformly convergent.

Proof. Let $\delta_2 = \dfrac{\delta + \delta_1}{2}$ so that $0 < \delta_1 < \delta_2 < \delta$. Then, by hypothesis, $f(z)$ is analytic within and on the circle γ defined by the equation $|z - a| = \delta_2$. Let a + h be any point of the region defined by $|z - a| \le \delta_1$.



Since a + h lies within the circle γ, using Cauchy's integral formula

$$f(a+h) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a-h}\,dz$$

$$= \frac{1}{2\pi i}\int_\gamma f(z)\frac{1}{(z-a)\left(1-\frac{h}{z-a}\right)}\,dz$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\left[\frac{1}{1-\frac{h}{z-a}}\right]dz$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\left[1+\frac{h}{z-a}+\frac{h^2}{(z-a)^2}+\ldots\frac{h^n}{(z-a)^n}\right.$$

$$\left.+\frac{h^{n+1}}{(z-a)^n(z-a-h)}\right]dz \qquad \left(\because \frac{1}{1-b} = 1+b+b^2+\ldots+b^n+\frac{b^{n+1}}{1-b}\right)$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz + \frac{h}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^2}dz + \frac{h^2}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^3}dz$$

$$+\ldots+\frac{h^n}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{n+1}}dz + \frac{h^{n+1}}{2\pi i}\int_\gamma \frac{f(z)dz}{(z-a)^{n+1}(z-a-h)}$$

Using Cauchy's integral formulae for the derivatives of an analytic function, we get

$$f(a+h) = f(a) + h\,f'(a) + \frac{h^2}{\lfloor 2}f''(a) + \ldots$$

$$+\frac{h^n}{\lfloor n}f^n(a) + \Delta_n$$

where $$\Delta_n = \frac{h^{n+1}}{2\pi i}\int_\gamma \frac{f(z)dz}{(z-a)^{n+1}(z-a-h)}$$

Thus

$$f(a+h) = f(a) + \sum_{r=1}^{n} f^r(a)\frac{h^r}{\lfloor r} + \Delta_n$$

But on account of continuity, $f(z)$ is bounded on the circle γ. Thus there exists a positive constant M such that $|f(z)| \le M$ on γ. Also, when $|z-a| = \delta_2$,

$$|z-a-h| \ge |z-a| - |h| > \delta_2 - \delta_1$$

where a + h lies in the circle and $|z-a| \le \delta_1$ implies $|a+h-a| \le \delta_1$ i.e. $|h| \le \delta_1$.

Now, applying the result regarding the absolute value of a complex integral we have the inequality

$$|\Delta_n| \le \frac{1}{|2\pi i|}\int_\gamma \frac{|f(z)||h|^{n+1}|dz|}{|z-a|^{n+1}|z-a-h|}$$

$$\le \frac{M}{2\pi}\int_\gamma \frac{|h|^{n+1}|dz|}{\delta_2^{\,n+1}(\delta_2-\delta_1)}$$

$$= \frac{M|h|^{n+1}}{2\pi\delta_2^{\,n+1}(\delta_2-\delta_1)}2\pi\,\delta_2$$

$$= \frac{M|h|}{\delta_2-\delta_1}\left(\frac{|h|}{\delta_2}\right)^n$$

Since $|h| \le \delta_1 < \delta_2$, it follows that as $n\to\infty$, $\Delta_n\to 0$
so that we have the identity

$$f(a+h) = f(a) + \sum_{n=1}^{\infty} \frac{f^n(a)}{\lfloor n}h^n$$

Changing over from a + h to z, we thus have the so called Taylor's series (expansion)

$$f(z) = f(a) + \sum_{n=1}^{\infty} \frac{f^n(a)}{\lfloor n}(z-a)^n$$

So far, we have proved only the convergence of this series for all values of z such that $|z - a| \le \delta_1$. It is however possible to prove more i.e. the uniform convergence as follows.
Since $|h| \le \delta_1$, we have.

$$|\Delta_n| \le \frac{M\,\delta_1}{\delta_2-\delta_1}\left(\frac{\delta_1}{\delta_2}\right)^n$$

and we observe that the expression on the right is independent of h. Therefore, given $\in>0$, there exists an integer $N = N(\in)$, independent of h, such that $|\Delta_n| < \in$ for $n \ge N$. This proves the uniform convergence of the Taylor's series of $f(z)$ in the region $|z - z_0| \le \delta_1 < \delta$

2.35. Remarks. (i) The above theorem is sometimes known as the Cauchy-Taylor theorem

(ii) By putting a = 0, Taylor's expansion reduces to

$$f(z) = f(0) + \sum_{n=1}^{\infty} \frac{f^n(0)}{\lfloor n}z^n$$

which is known as Maclaurin's series.

(iii) Taylor's series can be put as

$$f(z) = \sum_{n=1}^{\infty} a_n(z-a)^n$$

where

$$a_n = \frac{f^n(a)}{\lfloor n} = \frac{1}{\lfloor n}.\frac{\lfloor n}{2\pi i}\int\frac{f(z)}{(z-a)^{n+1}}dz$$

$$= \frac{1}{2\pi i}\int\frac{f(z)}{(z-a)^{n+1}}dz\,.$$

(iv) Using $a_n = \dfrac{f^n(a)}{\lfloor n}$, the result of Cauchy's inequality (2.29) can be put as

$$|a_n| = \left|\frac{f^n(a)}{\lfloor n}\right| \le \frac{M\lfloor n}{\lfloor n\, R^n} = \frac{M}{R^n}$$

i.e. $$|a_n| \le \frac{M}{R^n}$$

2. 36. Theorem. On the circumference of the circle of convergence of a power series, there must be at least one singular point of the function represented by the series.

Proof. Suppose that there is no singularity on the circumference $|z - a| = R$ of the radius of convergence of the power series.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

Then, the function $f(z)$ will be regular in a disc $|z-a| < R + \in$, where $\in$ is sufficiently small positive number. But from this it follows that the series $\sum_{n=0}^{\infty} a_n(z-a)^n$ must converge in the disc $|z - a| < R + \in$ and this contradicts the assumption that $|z - a| < R$ is the circle of convergence. Hence there is at least one singular point of the function

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

on the circle of convergence of the power series $\sum_{n=0}^{\infty} a_n (z-a)^n$.

2.37. Example. Expand log (1 + z) in a Taylor's series about the point z = 0 and determine the region of convergence for the resulting series.

Solution. Let $f(z) = \log(1 + z)$

Then $f'(z) = \frac{1}{1+z},\ f''(z) = -\frac{1}{(1+z)^2}$

……………………………

……………………………

$$f^n(z) = \frac{(-1)^{n-1}\lfloor n-1}{(1+z)^n}$$

Hence $f(0) = 0,\ f'(0) = 1,\ f''(0) = -1$

……………………………

……………………………

$f^n(0) = (-1)^{n-1} \lfloor n-1$

∴ By Taylor's theorem,

$$f(z) = \log(1+z) = f(0) + zf'(0) + \frac{z^2}{\lfloor 2} f''(0) + \dots + \frac{z^n}{\lfloor n} f^n(0) + \dots$$

$$= 0 + z + \frac{z^2}{\lfloor 2}(-1) + \dots + \frac{z^n}{\lfloor n}(-1)^{n-1}\lfloor n-1 + \dots\dots$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} \dots\dots + (-1)^{n-1}\frac{z^n}{n} + \dots\dots$$

Now, if u_n denotes the nth term of the series, then

$$u_n = \frac{(-1)^{n-1} z^n}{n},\ u_{n+1} = \frac{(-1)^n z^{n+1}}{n+1}$$

$$\therefore \quad \lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \frac{1}{|z|}$$

Hence by D'Alembert's ratio test, the series converges for $\frac{1}{|z|} > 1$ i.e. $|z| < 1$

2.38. Example. If the function $f(z)$ is analytic when $|z| < R$ and has the Taylor's expansion $\sum_{n=0}^{\infty} a_n z^n$, show that if $r < R$,

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

Hence prove that if $|f(z)| \le M$ when $|z| < R$,

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \le M^2$$

Solution. Since $f(z)$ is analytic for $|z| < R$, so $f(z)$ is analytic within and on a closed contour C defined by $|z| = r$, $r < R$. Thus $f(z)$ can be expanded in a Taylor's series within $|z| = r$ so that

$$\begin{aligned} f(z) &= \sum_{0}^{\infty} a_n z^n \\ &= \sum_{0}^{\infty} a_n r^n e^{in\theta}, \; z = re^{i\theta} \end{aligned}$$

$$\begin{aligned} \therefore \; |f(z)|^2 = f(z)\,\overline{f(z)} &\sum_{n=0}^{\infty} a_n r^n e^{in\theta} \sum_{m=0}^{\infty} \overline{a_m}\, r^m e^{-im\theta} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n \overline{a_m}\, r^{m+n} e^{i(n-m)\theta} \end{aligned}$$

The two series for $f(z)$ and $\overline{f(z)}$ are absolutely convergent and hence their product is uniformly convergent for the range $0 \le 0 \le 2\pi$. Thus, the term by term integration is justified. So, we get

$$\begin{aligned} \int_0^{2\pi} |f(z)|^2 \, d\theta &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n \overline{a_m}\, r^{m+n} \int_0^{2\pi} e^{i(n-m)\theta} \, d\theta \\ &= \sum_{n=0}^{\infty} a_n \overline{a_n}\, r^{n+n}.\, 2\pi, \; \int_0^{\infty} e^{i(n-m)\theta} d\theta = \begin{cases} 0, & n \ne m \\ 2\pi, & n = m \end{cases} \end{aligned}$$

or
$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n} \tag{1}$$

Now, from (1), we get

$$\begin{aligned} \sum_{0}^{\infty} |a_n|^2 r^{2n} &= \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta \\ &\le \frac{1}{2\pi}\int_0^{2\pi} M^2 \, d\theta \\ &= \frac{1}{2\pi} M^2 \, 2\pi = M^2 \end{aligned}$$

which proves the required result.

2.39. Example. If a function $f(z)$ is analytic for all finite values of z and as $|z| \to \infty$, $|f(z)| = A|z|^K$, then $f(z)$ is a polynomial of degree $\le K$.

Solution. Here, $f(z)$ is analytic in the finite part of z-plane. Also, it is given that

$$\lim_{|z|\to\infty} |f(z)| = A|z|^K \qquad (1)$$

We can assume that $f(z)$ is analytic inside a circle C defined by $|z| = R$, where R is large but finite. Hence $f(z)$ can be expanded in a Taylor's series as

$$f(z) = \sum_{0}^{\infty} a_n z^n \qquad (2)$$

where

$$a_n = \frac{f^n(0)}{\lfloor n} = \frac{\lfloor n}{\lfloor n\, 2\pi i} \int_C \frac{f(z)}{(z-0)^{n+1}} dz$$

Thus

$$= \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{n+1}} dz$$

$$|a_n| \le \frac{1}{|2\pi i|} \int_C \frac{|f(z)||dz|}{|z|^{n+1}} = \frac{1}{2\pi R^{n+1}} \int_C |f(z)|\,|dz|$$

$$\le \frac{M}{2\pi R^{n+1}} \int_C |dz|, \text{ M = max. } |f(z)| \text{ on C.}$$

$$= \frac{M}{2\pi R^{n+1}} 2\pi R = \frac{M}{R^n} = \frac{A|z|^K}{R^n} \quad \text{(using (1))}$$

$$= \frac{AR^K}{R^n} = \frac{A}{R^{n-K}}$$

Thus $|a_n| \le A\, R^{K-n} = \dfrac{A}{R^{n-K}}$

which tends to zero as $R\to\infty$, if $n - K > 0$

i.e. $a_n = 0\ \forall$ n such that $n > K$.

Now, from (2), we conclude that $f(z)$ is a polynomial of degree $\le K$. Hence the result.

UNIT-II

1. Zeros of Analytic function

A zero of an analytic function $f(z)$ is the value of z such that $f(z) = 0$.

Suppose $f(z)$ is analytic in a domain D and a is any point in D. Then, by Taylor's theorem,

$f(z)$ can be expanded about z = a in the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad a_n = \frac{f^n(a)}{\lfloor n} \tag{1}$$

Suppose $a_0 = a_1 = a_2 = \ldots\ldots = a_{m-1} = 0,\ a_m \neq 0$ (2)
so that $f(a) = f'(a) = \ldots\ldots = f^{m-1}(a) = 0,\ f^m(a) \neq 0$
In this case, we say that f(z) has a zero of order m at z = a and thus (1) takes the form

$$\begin{aligned} f(z) &= \sum_{n=m}^{\infty} a_n (z-a)^n \\ &= \sum_{n=0}^{\infty} a_{n+m}(z-a)^{n+m} \\ &= (z-a)^m \sum_{n=0}^{\infty} a_{n+m}(z-a)^n \end{aligned}$$

Taking $\sum_{n=0}^{\infty} a_{n+m}(z-a)^n = \phi(z)$ (3)

we get

$$f(z) = (z-a)^m \phi(z) \tag{4}$$

Now
$$\begin{aligned} \phi(a) &= \left[\sum_{n=0}^{\infty} a_{n+m}(z-a)^n\right]_{z=a} \\ &= \left[a_m + \sum_{n=1}^{\infty} a_{n+m}(z-a)^n\right]_{z=a} = a_m \end{aligned}$$

Since $a_m \neq 0$, so $\phi(a) \neq 0$

Thus, an analytic function f(z) is said to have a zero of order m at z = a if f(z) is expressible as

$$f(z) = (z-a)^m \phi(z)$$

where $\phi(z)$ is analytic and $\phi(a) \neq 0$.
Also, $f(z)$ is said to have a simple zero at z = a if z = a is a zero of order one.

1.1. Theorem. Zeros are isolated points.

Proof. Let us take the analytic function $f(z)$ which has a zero of order m at z = a. Then, by definition, $f'(z)$ can be expressed as

$f(z) = (z-a)^m \phi(z)$, where $\phi(z)$ is analytic and $\phi(a) \neq 0$.

Let $\phi(a) = 2K$. Since $\phi(z)$ is analytic in sufficiently small neighbourhood of a, if follows from the continuity of $\phi(z)$ in this neighbourhood that we can choose δ so small that, for $|z-a| < \delta$,

$$|\phi(z) - \phi(a)| < |K|$$

Hence $| \phi(z) | = |\phi(a) + \phi(z) - \phi(a)|$

$\geq | \phi(a)| - |\phi(z) - \phi(a) |$

$> | 2K | - |K|$

$= |K|$, for $|z - a | < \delta$

and thus, since $K \neq 0$, $\phi(z)$ does not vanish in the region $| z - a | < \delta$.
Since $f(z) = (z - a)^m \phi(z)$, it follows at once that $f(z)$ has no zero other than a in the same region. Thus we conclude that there exists a nbd of a in which the only zero of $f(z)$ is the point a itself i.e. a is an isolated zero.

The above theorem can also be stated as "Let $f(z)$ be analytic in a domain D, then unless $f(z)$ is identically zero, there exists a neighbourhood of each point in D throughout which the function has no zero except possibly at the point itself."

From the isolated nature of zeros of an analytic function, we are able to deduce the following remarkable result.

1.2. Theorem. If $f(z)$ is an analytic function, regular in a domain D and if $z_1, z_2, \ldots, z_n, \ldots$ is a sequence of zeros of $f(z)$, having a limiting point in the interior of D, then $f(z)$ vanishes identically in D.

Proof. Let a be the limiting point of the sequence of zeros $z_1, z_2, \ldots, z_n, \ldots$ of f(z). Then virtue of continuity of $f(z)$, $f(a) = 0$. Again, since $f(z)$ is regular in the domain D and a is an interior point of D, we can expand $f(z)$ as a power series in powers of $z - a$ as

$$f(z) = \sum_{n=1}^{\infty} a_n (z-a)^n \qquad (1)$$

which converges in the neighbourhood of a. Now, either $f(z)$ is identically zero in this region on account of the vanishing of all co-efficient a_n, or else there exists a first co-efficient a_m (say) which does not vanish. But if the latter is the case, we have already seen that there is a neighbourhood of which does not contain any zero of $f(z)$ other than a itself. This contradicts the hypothesis that a is the limiting point of the sequence of zeros $z_1, z_2, \ldots, z_n$. We are thus led to the conclusion that $f(z)$ is identically zero in the circle of convergence of the series (1) .

We are now free to repeat the same reasoning, starting with any point inside this circle, as the hypothesis now holds for any such point. In this manner by repeated employment of the same reasoning, it can be shown that $f(z)$ is identically zero throughout the interior of D.

1.3. Remarks. The following two results are direct consequences of the above theorem

(i) If a function is regular in a region and vanishes at all points of a subregion of the given region, or along any arc of a continuous curve in the region, then it must be identically zero throughout the interior of the given region.

(ii) If two functions are regular in a region, and have identical values at an infinite number of points which have a limiting point in the region, they must be equal to each other throughout the interior of the given region.
i.e. If two functions, which are analytic in a domain, coincide in a part of that domain, then they coincide in the whole domain.

For this, we take $f(z) = f_1(z) - f_2(z)$.

2. Laurent's Series. Now, we discuss the functions which are analytic in a punctured disc i.e. an open disc with centre removed. We have seen that a function $f(z)$ which is regular in a of a point $z = a$, can be expanded in a Taylor's series in powers of $(z - a)$ and that this power series is convergent in any circular region with centre a, contained within the given neighbourhood

In case, however, the function is not analytic in the neighbourhood of a point a including it, but analytic only in a ring shaped region (sometimes called annulus) surrounding a, the expansion of f(z) in a Taylor's series in powers of (z −a) ceases to be valid. The question naturally arises as to whether f(z), for values of z in the above said ring shaped region, can be expanded in powers of (z −a) at all. The following theorem answers this question.

2.1. Laurent's Theorem. Let f(z) be analytic in the ring shaped region between two concentric circles C and C′ of radii R and R′ (R′ < R) and centre a, and on the circles themselves, then

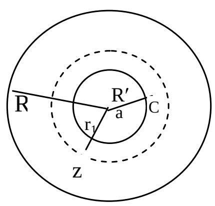
$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

z being any point of the annulus and

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(w)dw}{(w-a)^{n+1}},$$

$$b_n = \frac{1}{2\pi i}\int_{C'} \frac{f(w)dw}{(w-a)^{-n+1}}$$

Proof. Since f(z) is analytic on the circles and within the annulus between the two circles, by Cauchy's integral formula



$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)dw}{w-z} - \frac{1}{2\pi i}\int_{C'} \frac{f(w)dw}{w-z} \qquad (1)$$

consider the identity

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{(w-a)\left[1-\dfrac{z-a}{w-a}\right]} \qquad (2)$$

Applying the result

$$\frac{1}{1-b} = 1 + b + b^2 + \ldots + b^{n-1} + \frac{b^n}{1-b}$$

$$= \sum_{r=0}^{n-1} b^r + \frac{b^n}{1-b}$$

on R.H.S. of (2), we obtain

$$\frac{1}{w-z} = \frac{1}{w-a}\left[\sum_{r=0}^{n-1}\left(\frac{z-a}{w-a}\right)^r + \left(\frac{z-a}{w-a}\right)^n \frac{1}{1-\dfrac{z-a}{w-a}}\right]$$

$$= \sum_{r=0}^{n-1}\frac{(z-a)^r}{(w-a)^{r+1}} + \frac{(z-a)^n}{(w-a)^n}\frac{1}{w-z} \qquad (3)$$

Interchanging z and w, we get

$$\frac{1}{z-w} = \sum_{r=0}^{n-1} \frac{(w-a)^r}{(z-a)^{r+1}} + \frac{(w-a)^n}{(z-a)^n}\frac{1}{z-w} \qquad (4)$$

Equations (3) and (4) can be written as

$$\frac{f(w)}{w-z} = \sum_{r=0}^{n-1} \frac{(z-a)^r f(w)}{(w-a)^{r+1}} + \left(\frac{z-a}{w-a}\right)^n \frac{f(w)}{w-z} \;\forall\; w \text{ on } C \qquad (5)$$

$$\frac{f(w)}{z-w} = \frac{-f(w)}{w-z} = \sum_{r=0}^{n-1} \frac{(w-a)^r f(w)}{(z-a)^{r+1}} + \left(\frac{w-a}{z-a}\right)^n \frac{f(w)}{z-w} \;\forall\; w \text{ on } C' \qquad (6)$$

Let M and M′ be the maximum values of $|f(w)|$ on C and C′ respectively. Also let $|z - a| = r_1$. Equations of circles C and C′ are $|w - a| = R$ and $|w - a| = R'$ respectively.

From the figure, it is clear that

$$\left.\begin{aligned} \left|\frac{w-a}{z-a}\right| &= \frac{R'}{r_1} < 1 \text{ if } w \text{ lies on } C' \\ \left|\frac{z-a}{w-a}\right| &= \frac{r_1}{R} < 1 \text{ if } w \text{ lies on } C \end{aligned}\right\} \qquad (7)$$

The absolute value $|u_n(z)|$ of general term of the series in (5) is

$$|u_n(z)| = \left|\frac{(z-a)^n}{(w-a)^{n+1}} f(w)\right|$$

$$\leq \frac{r_1^n}{R^{n+1}} M = \frac{M}{R}\left(\frac{r_1}{R}\right)^n$$

similarly, the absolute value $|u_n'(z)|$ of general term of the series (6) is

$$|u_n'(z)| \leq \frac{(R')^n}{r_1^{n+1}} M' = \frac{M'}{r_1}\left(\frac{R'}{r_1}\right)^n$$

Hence the series of positive terms

$$\Sigma \frac{M}{R}\left(\frac{r_1}{R}\right)^n \text{ and } \Sigma \frac{M'}{r_1}\left(\frac{R'}{r_1}\right)^n$$

are both convergent as $\frac{r_1}{R} < 1, \frac{R'}{r_1} < 1$.

Consequently by Weierstrass M-test, both the series in (5) and (6) are uniformly convergent. Hence term by term integration is valid. Integrating (5) and (6), we obtain

$$\frac{1}{2\pi i}\int_C \frac{f(w)dw}{w-z} = \sum_{r=0}^{n-1} \frac{(z-a)^r}{2\pi i}\int_C \frac{f(w)dw}{(w-a)^{r+1}}$$

$$+ \frac{(z-a)^n}{2\pi i}\int_C \frac{f(w)dw}{(w-a)^n (w-z)}$$

and

$$-\frac{1}{2\pi i}\int_{C'} \frac{f(w)dw}{w-z} = \sum_{r=0}^{n-1} \frac{(z-a)^{-r-1}}{2\pi i}\int_{C'} f(w)(w-a)^r dw$$

$$+\frac{1}{(z-a)^n 2\pi i}\int_{C'} \frac{(w-a^n) f(w)}{z-w} dw$$

Taking $a_r = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(w)dw}{(w-a)^{r+1}}$,

$$b_{r+1} = \frac{1}{2\pi i}\int_C (w-a)^r f(w)\, dw$$

and

$$U_n = \frac{1}{2\pi i}\int_C \left(\frac{z-a}{w-a}\right)^n \frac{f(w)}{w-z} dw,$$

$$V_n = \frac{1}{2\pi i}\int_{C'} \left(\frac{w-a}{z-a}\right)^n \frac{f(w)}{z-w} dw$$

We get

$$\frac{1}{2\pi i}\int_C \frac{f(w)dw}{w-z} = \sum_{r=0}^{n-1} (z-a)^r a_r + U_n \qquad (8)$$

$$-\frac{1}{2\pi i}\int_{C'} \frac{f(w)dw}{w-z} = \sum_{r=0}^{n-1} \frac{b_{r+1}}{(z-a)^{r+1}} + V_n \qquad (9)$$

Adding (8) and (9) and using (1), we get

$$f(z) = \sum_{r=0}^{n-1} a_r (z-a)^r + \sum_{r=1}^{n} b_r (z-a)^{-r} + U_n + V_n \qquad (10)$$

Now,

$$|U_n| = \left|\frac{1}{2\pi i}\int_C \left(\frac{z-a}{w-a}\right)^n \frac{f(w)}{w-z} dw\right|$$

$$\leq \frac{1}{2\pi}\int_C \left(\frac{r_1}{R}\right)^n \frac{M\,|dw|}{R-r_1} \qquad (\,|w-z| = |(w-a)-(z-a)| \geq |w-a| - |z-a| = R-r_1)$$

$$= \frac{1}{2\pi}\left(\frac{r_1}{R}\right)^n \frac{M}{R-r_1} 2\pi R$$

$$= \frac{M}{1-\dfrac{r_1}{R}}\left(\frac{r_1}{R}\right)^n$$

which tends to zero as $n\to\infty$, since $\dfrac{r_1}{R} < 1$.

Thus $\lim\limits_{n\to\infty} U_n = 0$

Similarly, we can get $\lim\limits_{n\to\infty} V_n = 0$

Making $n\to\infty$ in (10), we obtain

$$f(z) = \sum_{r=0}^{\infty} a_r (z-a)^r + \sum_{r=1}^{\infty} b_r (z-a)^{-r}$$

or

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n} \qquad (11)$$

where

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(w)dw}{(w-a)^{n+1}},$$

$$b_n = \frac{1}{2\pi i}\int_{C'} \frac{f(w)dw}{(w-a)^{-n+1}}$$

which proves the theorem.

2.2. Remarks. (i) The result (ii) can be put in a more compact form as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

where the co-efficients are given by the single formula

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^{n+1}}dw$$

where γ denotes C when $n \ge 0$ and C′ when $n < 0$ since however the integrand is analytic in the annulus $R' < |z-a| < R$, we may take γ to be any closed contour which passes round the ring.

(ii) The function $f(z)$ which is expanded in Laurent's series is one-valued. Laurent's theorem will not provide an expansion for multi-valued function.

(iii) In the particular case when $f(z)$ is analytic inside C′, all the coefficients b_n are zero, by Cauchy's theorem, and the series reduces to Taylor's series.

(iv) The series of positive powers of $z - a$ converges, not merely in the ring, but everywhere inside the circle C. Similarly the series of negative powers of $z - a$ converges everywhere outside C′.

(v) The series of negative powers of $z - a$ i.e., $\sum_{n=1}^{\infty} b_n (z-a)^{-n}$ is called the principal part of Laurent's expansion, which the series of positive powers i.e. $\sum_{n=0}^{\infty} a_n (z-a)^n$ is called the regular part.

(vi) There is no handy method, like that for Taylor's series, for finding the Laurent co-efficients. But if we can find them by any method (generally by direct expansion), their validity is justified due to the fact that Laurent's co-efficients are unique.

2.3. Example. Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in a Laurent's series valid for the regions.

(i) $|z| < 1$, (ii) $1 < |z| < 3$, (iii) $|z| > 3$, (iv) $0 < |z+1| < 2$

Solution. $f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$

(i) for $|z| < |$, we have

$$f(z) = \frac{1}{2}(z+1)^{-1} - \frac{1}{2}(z+3)^{-1}$$

$$= \frac{1}{2}(z+1)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2}\left[1 - z + z^2 - z^3 + \ldots\right] - \frac{1}{6}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \ldots\right]$$

$$= \frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2$$

(ii) for |z| > 1, we have

$$\frac{1}{2(z+1)} = \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} = \frac{1}{2z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\ldots\right]$$

$$= \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \ldots$$

and for |z| < 3, we have.

$$\frac{1}{2(z+3)} = \frac{1}{6}\frac{1}{\left(1+\frac{z}{3}\right)} = \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + \ldots$$

Hence the Laurent's series for $f(z)$, valid for the annulus 1 < |z| < 3, is

$$f(z) = \ldots\ldots - \frac{1}{2z^4} + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{3}{18} - \frac{z^2}{54} + \frac{z^3}{162}$$

(iii) for |z| > 3

$$f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= \frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4}\ldots..$$

(iv) We put z + 1 = u, then 0 < | u | < 2 and we have

$$f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{u(u+2)}$$

$$= \frac{1}{2u\left(1+\frac{u}{2}\right)} = \frac{1}{2u}\left(1+\frac{u}{2}\right)^{-1}$$

$$= \frac{1}{2u} - \frac{1}{4} + \frac{u}{8} - \frac{u^2}{16} + \ldots\ldots\ldots$$

$$= \frac{1}{2(z+1)} - \frac{1}{4} + \frac{z+1}{8} - \frac{(z+1)^2}{16} + \ldots..$$

2.4. Example. Show that $e^{\frac{c}{2}\left(z-\frac{1}{z}\right)} = \sum_{-\infty}^{\infty} a_n z^n$

where $a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - c\sin\theta)\, d\theta$

Solution. The function $f(z) = e^{\frac{c}{2}\left(z-\frac{1}{z}\right)}$ is analytic except at z = 0 and z = ∞. Hence $f(z)$ is analytic in the annulus $r_1 \le |z| \le r_2$, where r_1 is small and r_2 is large. Therefore, $f(z)$ can be expanded in the Laurent's series in the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n} \qquad (1)$$

where

$$a_n = \frac{1}{2\pi i}\int_{C'} \frac{f(z)}{z^{n+1}}\,dz,$$

$$b_n = \frac{1}{2\pi i}\int_{C'} \frac{f(z)}{z^{-n+1}}\,dz,$$

C′ being any circle with centre at the origin for the shake of convenience, let us take C′ to be the unit circle |z| = 1 which gives $z = e^{i\theta}$

Now,

$$a_n = \frac{1}{2\pi i}\int_{C'} \frac{e^{\frac{C}{2}(z-z^{-1})}}{z^{n+1}}\,dz,$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{Ci\sin\theta}ie^{i\theta}d\theta}{e^{i(n+1)\theta}}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{i(C\sin\theta - n\theta)}d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos(c\sin\theta - n\theta)d\theta + \frac{i}{2\pi}\int_0^{2\pi} F(\theta)d\theta \qquad (2)$$

where F(θ) = sin (C sin θ – nθ).

Since F(2π –θ) = sin [C sin (2π –θ) –n (2π – θ)]

= – sin (C sin θ – nθ + 2nπ)

= – sin (C sinθ – nθ) = –F(θ)

$$\therefore \qquad \int_0^{2\pi} F(\theta)\,d\theta = 0$$

Thus, from (2), we have

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(C\sin\theta - n\theta)\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - C\sin\theta)\,d\theta$$

We note that if z is replaced by z^{-1}, the function $f(z)$ remains unaltered so that

$$b_n = (-1)^n a_n$$

Hence from (1), we get

$$f(z) = \sum_0^{\infty} a_n z^n + \sum_1^{\infty} (-1)^n a_n z^{-n}$$

$$= \sum_{-\infty}^{\infty} a_n z^n$$

where $a_n = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \cos(n\theta - C\sin\theta)\,d\theta$

2.5. Example. Prove that the function $f(z) = \cosh(z + z^{-1})$ can be expanded in a series of the type

$$\sum_{0}^{\infty} a_n z^n + \sum_{1}^{\infty} b_n z^{-n}$$

in which the co-efficients of z^n and z^{-n}, both are given by $\frac{1}{2\pi}\int_0^{2\pi}$ cos nθ cosh (2 cosθ) dθ .

Solution. The function $f(z) = \cosh(z + z^{-1})$ is analytic except at z = 0 and z = ∞. Hence $f(z)$ is analytic in the annulus $r_1 \le |z| \le r_2$, where r_1 is small and r_2 is large. Therefore, $f(z)$ can be expanded in the Laurent's series as

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n} \qquad (1)$$

where

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z^{n+1}},$$

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^{-n+1}}dz$$

C being any circle with centre at the origin. We take C to be the unit circle $|z| = 1$ which gives $z = e^{i\theta}$

Now,

$$a_n = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\cosh(z+\bar{z})}{z^{n+1}}dz$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\cosh(2\cos\theta)e^{i\theta}id\theta}{e^{i(n+1)\theta}}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\, e^{-in\theta}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos n\theta\, d\theta - \frac{i}{2\pi}\int_0^{2\pi} F(\theta)d\theta \qquad (2)$$

where $F(\theta) = \cosh(2\cos\theta)\sin n\theta$

We note that $F(2\pi - \theta) = -F(\theta)$

$$\Rightarrow \quad \int_0^{2\pi} F(\theta)\, d\theta = 0$$

Thus (2) becomes

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos n\theta\, d\theta \qquad (3)$$

It is clear that

$$b_n = a_{-n} = \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos(-n\theta)\, d\theta = a_n$$

Thus, from (1), we find

$$\cosh(z + z^{-1}) = \sum_{0}^{\infty} a_n z^n + \sum_{1}^{\infty} a_n z^{-n} = a_0 + \sum_{1}^{\infty} a_n (z^n + z^{-n})$$

where a_n is give by (3)

3. Isolated Singularities

The point where the function ceases to be analytic is called the singularity of the function. Suppose that a function $f(z)$ is analytic throughout the neighbourhood of a point z = a, say for

$|z - a| < \delta$, except at the point a itself. Then the point a is called an isolated singularity of the function $f(z)$. In other words, the point $z = a$ is said to be isolated singularity of $f(z)$ if

(i) $f(z)$ is not analytic at $z = a$

(ii) there exists a deleted neighbourhood of $z = a$ containing no other singularity.

For example, the function $f(z) = \dfrac{z+1}{(z-1)(z+2)(z+3)}$ has three isolated singularities at $z = 1, -2, 3$ respectively.

It is not difficult to construct examples of singularities which are not isolated. For instance, the function

$$f(z) = \frac{1}{\sin\frac{1}{z}} \text{ has such a singularity at } z = 0$$

In fact, $\sin\dfrac{1}{z} = 0$ when $\dfrac{1}{z} = n\pi$

or $\quad z = \dfrac{1}{n\pi}$. Thus, $f(z)$ ceases to be analytic when $z = \dfrac{1}{n\pi}$. When $n\to\infty$, $z\to 0$. Therefore in every neighbourhood of origin there lies an infinite number of points of the form $\dfrac{1}{n\pi}$ and hence $z = 0$ is not an isolated singularity.

In case of isolated singularity at $z = a$ $f(z)$ can be expanded in a Laurent's series in positive and negative powers of $z - a$ in the region defined by $r < |z - a| < R$ and r may be taken as small as we please. Thus, with suitable definitions of a_n and b_n in this region, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n},\ 0 < |z-a| < R$$

where $\sum_{n=1}^{\infty} b_n (z-a)^{-n}$ is the principal part of the expansion of $f(z)$ at the singular point $z = a$. There are now three possible cases, discussed as follows.

3.1. Removable Singularity. If the principal part of $f(z)$ at $z = a$ contains nθ term i.e. $b_n = 0\ \forall\ n$, then the singularity $z = a$ is called a removable singularity of $f(z)$. In such a case we can make $f(z)$ regular when $|z-a| < R$ by suitably defining its value at $z = a$. For example, the function $f(z) = \dfrac{\sin z}{z}$ is undefined at $z = 0$. Also we have

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{\lfloor 3} + \frac{z^5}{\lfloor 5} \ldots\ldots\right)$$

$$= 1 - \frac{z^2}{\lfloor 3} + \frac{z^4}{\lfloor 5} \ldots$$

Thus $\dfrac{\sin z}{z}$ contains no negative powers of z.

If it were the case $f(0) \neq 1$, then $z = 0$ is a removable singularity which can be removed by simply redefining $f(0) = 1$. Singularities of this type are of little importance.

3.2. Pole. If the principal part of $f(z)$ at $z = a$ contains a finite number of terms, say m, i.e. $b_n = 0$ $\forall$ n such that $n > m$, then the singularity is called a pole of order m. Poles of order 1,2,3… are called simple, double, triple poles. The coefficient b_1 is called the residue of $f(z)$ at the pole a. Thus, if z = a is a pole of order m of the function $f(z)$, then $f(z)$ has the expansion of the form

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{m} b_n (z-a)^{-n} \\ &= \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \ldots + \frac{b_m}{(z-a)^m} \\ &= \frac{1}{(z-a)^m}\left[b_m + b_{m-1}(z-a) + \ldots + b_1 (z-a)^{m-1} + \sum_{n=0}^{\infty} a_n (z-a)^{m+n}\right] \\ &= \frac{\phi(z)}{(z-a)^m} \end{aligned}$$

where $\phi(z)$ is analytic for $|z-a| < R$ and $\phi(a) = b_m \neq 0$

We can therefore find a neighbourhood $|z - a| < \delta$ of the pole in which $|f(z)| \geq \frac{1}{2}|b_m|\ |z-a|^{-m}$.

Hence if $f(z)$ has a pole of order m at z = a, then $|f(z)| \to \infty$ as $z \to a$ in any manner, i.e., an analytic function cannot be bounded in the neighbourhood of a pole.

For example, the function

$$f(z) = \frac{z-2}{(z-5)^2(z+4)^3}$$

has z = 5, z = – 4 as poles of order two and three respectively. Moreover, if $f(z)$ has a pole of order m at a, then $\frac{1}{f(z)}$ is regular and has a zero of order m there, since

$$\frac{1}{f(z)} = \frac{(z-a)^m}{\phi(z)},$$ where $\phi(z)$ is regular and does not vanish when $|z-a| < \delta$.

Similarly, we note that the converse is also true, i.e., if $f(z)$ has a zero of order m at z = a, $\frac{1}{f(z)}$ has a pole of order m there.

Further, note that poles are isolated, since zeros are isolated.

3.3. Isolated Essential Singularity. If the principal part of $f(z)$ at z = a has an infinite number of terms, i.e., $b_n \neq 0$ for infinitely many values of n, then the singularity a is called isolated essential singularity or essential singularity. In this case, a is evidently also a singularity of $\frac{1}{f(z)}$

For example, $e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{\lfloor 2 z^2} + \frac{1}{\lfloor 3 z^3} + \ldots$

has z = 0 as an isolated essential singularity.

3.4. Example. Find the singularities of the function

$$f(z) = \frac{e^{\frac{c}{z-a}}}{e^{\frac{z}{a}} - 1}$$

indicating the character of each singularity.

Solution. $f(z) = \dfrac{e^{\frac{c}{z-a}}}{e^{\frac{z}{a}} - 1} = \dfrac{\exp(c/z) - a}{\exp\left(1 + \dfrac{z-a}{a}\right) - 1}$

$$= \frac{e^{c/z-a}}{e.e^{z-a/a} - 1} = -e^{c/z-a}\left[1 - e.e.^{\frac{z-a}{a}}\right]^{-1}$$

$$= -\,e^{c/z-a}\left[1 - e\left\{1 + \frac{z-a}{a} + \frac{(z-a)^2}{\lfloor 2 a^2} + ...\right\}\right]^{-1}$$

$$= -\left[1 + \frac{c}{z-a} + \left(\frac{c}{z-a}\right)^2 \frac{1}{\lfloor 2} + ...\right]$$

$$\times\left[1 + e\left\{1 + \frac{z-a}{a} + ...\right\} + e^2\left\{1 + \frac{z-a}{a} + ...\right\}^2 + ...\right]$$

Clearly, this expansion contains positive and negative powers of (z − a). Moreover, terms containing negative powers of (z −a) are infinite in number.

Hence by definition, z = a is an isolated essential singularity.

Again, $f(z) = \dfrac{e^{c/z-a}}{e^{z/a} - 1}$

Evidently, denominator has zero of order one at

$$e^{z/a} = 1 = e^{2n\pi i}$$

i.e., $z = 2n\pi ia$

Thus, $f(z)$ has a pole of order one at each point $z = 2n\pi ia$, where $n = 0, \pm 1, \pm 2 \ldots.$

3.5 Behaviour of an Analytic Function near an isolated Essential Singularity. As we know that if z = a is a pole of an analytic function $f(z)$, then $|f(z)| \to \infty$ as $z \to a$ in any manner. The behaviour of an analytic function near an isolated essential singularity is of a much complicated character. The following theorem is a precise statement of this complicated nature of $f(z)$ near an isolated essential singularity and this theorem is called Weierstress theorem.

3.6. Theorem. If a is an isolated essential singularity of $f(z)$, then given positive numbers l, $\in$, however small, and any number K, however large, there exists a point z in the circle $|z - a| < l$ at which $| f(z) - K < \in$.

or

In any neighbourhood of an isolated essential singularity, an analytic function approaches any given value arbitrarily closely.

Proof. We first observe that if l and M are any positive numbers, then there are values of z in the circle $|z - a| < l$ at which $|f(z)| > M$ (1)

For, if this were not true, then we would have $|f(z)| \le M$ for $|z - a| < l$. If the principal part in the Laurent expansion of $f(z)$ about a is

$$\sum_{n=1}^{\infty} b_n (z-a)^{-n},$$

where $$b_n = \frac{1}{2\pi i}\int_\gamma \frac{f(w)dw}{(w-a)^{-n+1}}$$

and γ is the circle $| w - a | = r$, r being sufficiently small, then

$$| b_n | = \left| \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{-n+1}} dw \right|$$
$$= \left| \frac{1}{2\pi i} \int_\gamma (w-a)^{n-1} f(w)dw \right|$$
$$\leq \frac{M}{2\pi} r^{n-1} \int_\gamma | dw |$$
$$= \frac{M}{2\pi} r^{n-1} 2\pi r = Mr^n$$

By the result of the absolute value of a complex integral, this holds for all $n \geq 1$ and r, so that, making $r \to 0$, we find that $b_n = 0$ for $n \geq 1$.

This implies that there is no isolated essential singularity at z = a. But this contradicts the hypothesis that a is an isolated essential singularity of $f(z)$. Thus, the observed result (1) is true, i.e., "in the neighbourhood of an isolated essential singularity, $f(z)$ cannot be bounded."

Now, let us take any finite, but arbitrary positive number K. There are now two distinct possibilities, either $f(z) - K$ has zeros inside every circle $|z - a| = l$ or else we can find a sufficiently small ρ such that $f(z) - K$ has no zero for $| z - a | < l$. In the first case, the result follows immediately. In the second case, choosing a sufficiently small l, we have $| f(z) - K | \neq 0$ in $| z - a | < l$, so that

$$\phi(z) = \frac{1}{f(z) - K}$$ is regular for $|z - a| < l$, except at a, where as we shall just see, $\phi(z)$ has an essential singularity. We have.

$$f(z) = \frac{1}{\phi(z)} + K$$

If $\phi(z)$ were analytic at a, $f(z)$ would either be analytic or have a pole at a. On the other hand if $\phi(z)$ has a pole at a, $f(z)$ would be obviously analytic there. Thus we reach at the contradiction and therefore, $\phi(z)$ has an essential singularity at a. So, due to (1), given $\in > 0$, there exists a point z in the circle $|z - a| < l$ such that

$$|\phi(z) > \frac{1}{\in}, \text{ i.e. } | f(z) - K| < \in$$

and hence the theorem is proved.

3.7. Remark. The above theorem helps us to understand clearly the distinction between poles and isolated essential singularities. While $| f(z)| \to \infty$, as z tends to a pole in any manner, at an isolated essential singularity $f(z)$ has no unique limiting value, and it comes arbitrarily close to any arbitrarily pre assigned value at infinity of points in every neighbourhood of the isolated essential singularity.

4. Maximum Modulus Principle. Here, we continue the study of properties of analytic functions. Contrary to the case of real functions, we cannot speak of maxima and minima of a complex function $f(z)$, since $\forall$ is not an ordered field. However, it is meaningful to consider maximum and minimum values of the modulus $| f(z)|$ of the complex function $f(z)$, real part of

f(z) and imaginary part of f(z). The following theorem known as maximum modulus principle, is also true if f(z) is not one-valued, provided $|f(z)|$ is one-valued.

4.1. Theorem. Let f(z) be analytic within and on a simple closed contour C. If $|f(z)| \le M$ on C, then the inequality $|f(z)| < M$ holds every where within C. Moreover $|f(z)| = M$ at a point within C if and only if f(z) is constant.

In other words, $|f(z)|$ attains the max. value on the boundary C and not at any interior point of the region D bounded by C.

Proof. We prove the theorem by contradiction. If possible, let $|f(z)|$ attains the maximum value at an interior point $z = z_0$ of the region D enclosed by C. Since f(z) is analytic inside C, we can expand f(z) by Taylor's theorem in the nbd. of z_o as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where $$a_n = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{(z-z_0)^n} dz$$

and γ is the circle $|z - z_0| = r$, r being small.

We have $z - z_0 = r e^{i\theta}$ i.e. $z = z_0 + r e^{i\theta}$, $0 \le \theta \le 2\pi$.

Also $$|f(z)|^2 = f(z)\,\overline{f(z)}$$

$$= \sum_{n=0}^{\infty} a_n r^n e^{in\theta} \sum_{m=0}^{\infty} \overline{a_m}\, r^m e^{-im\theta}$$

$$= \sum_{0}^{\infty} \sum_{0}^{\infty} a_n \overline{a_m}\, r^{n+m} e^{i(n-m)\theta}$$

Integrating both sides from 0 to 2π, we get

$$\int_0^{2\pi} |f(z)|^2\, d\theta = \sum_{0}^{\infty}\sum_{0}^{\infty} a_n \overline{a_m} r^{m+n} \int_0^{2\pi} e^{i(n-m)\theta} d\theta$$

$$= \sum_{n=0}^{\infty} a_n\, \overline{a}_n\, r^{2n}\, 2\pi, \; n = m$$

$$= \sum_{0}^{\infty} |a_n|^2 r^{2n}\, 2\pi \qquad (1)$$

where $$\int_0^{2\pi} e^{i(n-m)\theta} d\theta = \begin{cases} 0 \text{ if } n \ne m \\ 2\pi \text{ if } n = m \end{cases}$$

From (1), we have for n = 0,

$$\int_0^{2\pi} |f(z)|^2\, d\theta = |a_0|^2\, 2\pi$$

and putting $z = z_0$ in this, we find

$$\int_0^{2\pi} |f(z_0)|^2\, d\theta = |a_0|^2\, 2\pi$$

or

$$|f(z_0)|^2 \int_0^{2\pi} d\theta = |a_0|^2\, 2\pi$$

or

$$|f(z_0)|^2\, 2\pi = |a_0|^2\, 2\pi \;\Rightarrow\; |f(z_0)|^2 = |a_0|^2 \qquad (2)$$

Also, since f(z) has max. value at $z = z_0$, so

$$|f(z)|^2 \le |f(z_0)|^2 = |a_0|^2$$

Hence from (1), we get

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi}\int_0^{2\pi} |f(z)|^2 \, d\theta$$
$$\leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|^2 \, d\theta$$
$$= \frac{1}{2\pi} |a_0|^2 \, 2\pi = |a_0|^2$$

Thus,

$$|a_0|^2 + |a_1|^2 r^2 + |a_2|^2 r^4 + \ldots \leq |a_0|^2 \text{ for positive values of r.}$$

Hence $|a_1| = |a_2| = |a_3| = \ldots = 0$

i.e. $a_1 = a_2 = a_3 = 0$

which implies

$$f(z) = a_0 = \text{constant.}$$

Hence $|f(z)|$ cannot attain a max. value at an interior point of D which is a contradiction to our supposition.

Also $|f(z)|$ attains a max. value at an interior point of D if it is constant and in that case. $|f(z)| = M$ throughout D.

4.2. Theorem. Let $f(z)$ be analytic within and on a simple closed contour C and let $f(z) \neq 0$ inside C. Further suppose that $f(z)$ is not constant, then $|f(z)|$ cannot attain a minimum value inside C.

Proof. Since $f(z)$ is analytic within and on C and also $f(z) \neq 0$ inside C, so $\frac{1}{f(z)}$ is also analytic within and on C.

Therefore, by maximum modulus principle, $\left|\frac{1}{f(z)}\right|$ cannot attain a maximum value inside C which implies that $|f(z)|$ cannot have a minimum value inside C.

4.3. Theorem. Let $f(z)$ be an analytic function, regular for $|z| < R$ and let $M(r)$ denote the maximum of $|f(z)|$ on $|z| = r$, then $M(r)$ is a steadily increasing function of r for $r < R$.

Proof. By maximum modulus principle, for two circles

$|z| = r_1$ and $|z| = r_2$, we have

$|f(z)| \leq M(r)$, where $r_1 < r_2$

which implies $M(r_1) \leq M(r_2)$, $r_1 < r_2$

and $M(r_1) = M(r_2)$ if $f(z)$ is constant.

Also $M(r)$ cannot be bounded because if it were so, then $f(z)$ is a constant (by Lioureille's theorem). Hence $M(r)$ is a steadily increasing function of r.

4.4. Schwarz's Lemma. Let $f(z)$ be analytic in a domain D defined by $|z| < R$ and let $|f(z)| \leq M$ for all z in D and $f(0) = 0$, then $|f(z)| \leq \frac{M}{R}|z|$.

Also, if the equality holds for any one z, then $f(z) = \frac{M}{R} z e^{i\alpha}$, where α is real constant.

Proof. Let C be the circle $|z| = r < R$.

Since $f(z)$ is analytic within and on C, therefore by Taylor's theorem

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ at any point z within C.}$$

i.e. $f(z) = a_0 + a_1 z + a_2 z^2 +$

Since $f(0) = 0$, we get $a_0 = 0$

$\therefore \quad f(z) = a_1 z + a_2 z^2 + a_3 z^3 \qquad (1)$

Let $\quad g(z) = \dfrac{f(z)}{z} \qquad (2)$

then we have

$g(z) = a_1 + a_2 z \ \ a_3 z^2 \qquad (3)$

The function g(z) in (2) has a singularity at $z = 0$ which can be removed by defining $g(0) = a_1$.

Now, g(z) is analytic within and on C and so by maximum modulus principle, |g(z)| attains maximum value on C, say at $z = z_0$ and not within C.

Thus $|z_0| = r < R$ and

$$|g(z_0)| = \left|\frac{f(z_0)}{z_0}\right| = \text{max. value of } |g(z)| = \left|\frac{f(z)}{z}\right| \le \frac{M}{r} \qquad (4)$$

and thus for any z inside C, we have.

$$|g(z)| < \frac{M}{r}$$

i.e. $$\left|\frac{f(z)}{z}\right| < \frac{M}{r} \Rightarrow |f(z)| < \frac{M}{r}\ |z| \qquad (5)$$

This inequality holds for all r. s. t. $r < R$.

Now, L. H. S. is free from r, making $r \to R$ in (5), we find

$$|f(z)| < \frac{M}{R}\ |z| \ \forall \ z \text{ s. t } |z| < R.$$

Also, from (4), we note that for the point z_0 on C,

$$|f(z_0)| = \frac{M}{r}\ |z_0|$$

Making $r \to R$, we get

$$|f(z_0)| = \frac{M}{R}\ |z_0|$$

i.e., $$f(z) = \frac{M}{R}\ ze^{i\alpha} \text{ for z lying on } |z| = R.$$

which proves the result.

4.5. Remarks. (i) If we take $M = 1$, $R = 1$, then Schwarz's lemma takes the form as follows.

"If $f(z)$ is analytic in a domain D defined by $|z| < 1$ and $|f(z)| \le 1$ for all z in D and $f(0) = 0$, then $|f(z)| \le |z|$. Also if the equality holds for any one z, then $f(z) = z\, e^{i\alpha}$, where α is a real constant."

(ii) In view of the power series expansion,

$$f(z) = f(0) + z f'(0) + \frac{z^2}{\lfloor 2}\ f''(0) + \ldots$$

we get

$$\frac{f(z)}{z} = f'(0) + \frac{z^2}{\lfloor 2}\ f''(0) + \ldots$$

where we have assumed that $f(z)$ satisfies the conditions of Schwarz's lemma so that

$$\left|\frac{f(z)}{z}\right| \le \frac{M}{R}$$

This implies that

$$|f'(0) + \frac{z^2}{\lfloor 2} f''(0) + \ldots\ldots| \le \frac{M}{R}$$

By setting z = 0, we obtain

$$f'(0) \le \frac{M}{R}$$

(iii) Let $f(z)$ be analytic inside and on the unit circle, $|f(z)| \le M$ on the circle and $f(0) = a$ where $0 < a < m$. Then

$$|f(z)| \le M\frac{M|z|+a}{a|z|+M}$$

inside the circle.

For its proof, we consider

$$\phi(z) = M\frac{f(z)-a}{a\,f(z)-M^2}$$

Then $$\phi(0) = M\frac{f(0)-a}{a\,f(0)-M^2} = M\frac{a-a}{a^2-M^2} = 0$$

Also, $\phi(z)$ is regular at every point on the unit circle.

Also, $$|\phi(z)| = \left|M\frac{f(z)-a}{a\,f(z)-M^2}\right| \le M\left|\frac{M-a}{aM-M^2}\right| = 1$$

Thus, $\phi(z)$ satisfies all the conditions of Schwarz's lemma.

Therefore,

$$|\phi(z)| = \left|M\frac{f(z)-a}{af(z)-M^2}\right| \le |z|$$

which gives $$|f(z)| \le M\frac{M|z|+a}{a|z|+M}.$$

5. Meromorphic Function. A function $f(z)$ is said to be meromorphic in a region D if it is analytic in D except at a finite number of poles. In other words, a function $f(z)$ whose only singularities in the entire complex plane are poles, is called a meromorphic function. The word meromorphic is used for the phrase "analytic except for poles". The concept of meromorphic is used in contrast to holomorphic. A meromorphic function is a ratio of entire functions.

Rational functions are meromorphic functions.

e.g. $$f(z) = \frac{z^2-1}{z^5+2z^3+z}$$

$$= \frac{(z+1)(z-1)}{z(z^4+2z^2+1)} = \frac{(z+1)(z-1)}{z(z^2+1)^2}$$

$$= \frac{(z+1)(z-1)}{z(z+i)^2(z-i)^2}$$

has poles at z = 0 (simple), at z = $\pm$ i (both double) and zeros at z = $\pm$ 1 (both simple)

Since only singularities of $f(z)$ are poles, therefore $f(z)$ is a meromorphic function.

Similarly, tan z, cot z, sec z are all meromorphic functions.

A meromorphic function does not have essential singularity. The following theorem tells about the number of zeros and poles of a meromorphic function.

5.1. Theorem. Let $f(z)$ be analytic inside and on a simple closed contour C except for a finite number of poles inside C and let $f(z) \neq 0$ on C, then $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N - P$

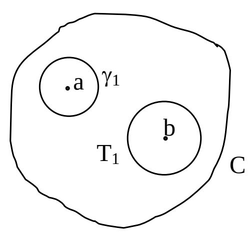
where N and P are respectively the total number of zeros and poles of $f(z)$ inside C, a zero (pole) of order m being counted m times.

Proof. Suppose that $f(z)$ is analytic within and on a simple closed contour C except at a pole z = a of order p inside C and also suppose that $f(z)$ has a zero of order n at z = b inside C.

Then, we have to prove that

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = n-p$$

Let γ_1 and T_1 be the circles inside C with centre at z = a and z = b respectively.



Then, by cor. to Cauchy's theorem, we have

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f'(z)}{f(z)}\,dz + \frac{1}{2\pi i}\int_{T_1} \frac{f'(z)}{f(z)}\,dz \qquad (1)$$

Now, $f(z)$ has pole of order p at z = a, so

$$f(z) = \frac{g(z)}{(z-a)^p} \qquad (2)$$

where g(z) is analytic and non-zero within and on γ_1. Taking logarithm of (2) and differentiating, we get

$$\log f(z) = \log g(z) - p \log (z-a)$$

i.e.,
$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} - \frac{p}{z-a}$$

Therefore,

$$\int_{\gamma_1} \frac{f'(z)}{f(z)}\,dz = \int_{\gamma_1} \frac{g'(z)}{g(z)}\,dz - p\int_{\gamma_1} \frac{dz}{z-a} \qquad (3)$$

Since $\frac{g'(z)}{g(z)}$ is analytic within and on γ_1, by Cauchy theorem,

$$\int_{\gamma_1} \frac{g'(z)}{g(z)}\,dz = 0$$

Thus (3) gives $\int_{\gamma_1} \frac{f'(z)}{f(z)}\,dz = -\,2\pi ip$ (4)

Again, $f(z)$ has a zero of order n at z = b, so we can write

$$f(z) = (z - b)^n\, \phi(z) \qquad (5)$$

where $\phi(z)$ is analytic and non-zero within and on T_1

Taking logarithm, then differentiating, we get

$$\frac{f'(z)}{f(z)} = \frac{n}{z-b} + \frac{\phi'(z)}{\phi(z)}$$

or

$$\int_{T_1} \frac{f'(z)}{f(z)}\,dz = n\int_{T_1} \frac{dz}{z-b} + \int_{T_1} \frac{\phi'(z)}{\phi(z)}\,dz \qquad (6)$$

Since $\frac{\phi'(z)}{\phi(z)}$ is analytic within and on T_1, therefore

$\int_{T_1} \frac{\phi'(z)}{\phi(z)}\,dz = 0$ and thus (6) becomes

$$\int_{T_1} \frac{f'(z)}{f(z)}\,dz = 2\pi in \qquad (7)$$

Writing (1) with the help of (4) and (7), we get

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = -p + n = n-p \qquad (8)$$

Now, suppose that $f(z)$ has poles of order p_m at $z = a_m$ for m = 1, 2,…,r and zeros of order n_m at $z = b_m$ for m = 1, 2,…,s within C. We enclose each pole and zero by circles $\gamma_1, \gamma_2, \ldots, \gamma_r$ and $T_1, T_2, \ldots, T_s$. Thus (8) becomes

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \sum_{m=1}^{s} n_m - \sum_{m=1}^{r} p_m$$

Taking $\sum_{m=1}^{s} n_m = N$, $\sum_{m=1}^{r} p_m = P$, we obtain

$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N - P$ which proves the theorem. This theorem is also known as the argument principle which can be put in a more explicit manner as follows :

5.2. Theorem (The Argument Principle). Let $f(z)$ be meromorphic inside a closed contour C and analytic on C where $f(z) \neq 0$. When $f(z)$ describes C, the argument of $f(z)$ increases by a multiple of 2π, namely

$$\Delta_C \arg f(z) = 2\pi\,(N-P)$$

where N and P are respectively the total number of zeros and poles of $f(z)$ inside C, a zero (pole) of order m being counted m times.

Proof. Let $\arg f(z) = \phi$

So, we can write

$$f(z) = |f(z)|\, e^{i\phi}$$

i.e. $\log f(z) = \log |f(z)| + i\phi$ (1)

Then as proved in the above theorem 4.1,

$$\begin{aligned} N - P &= \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz \\ &= \frac{1}{2\pi i}\int_C d\,(\log f(z)) \\ &= \frac{1}{2\pi i}\int_C d\,(\log |f(z)| + i\,\phi) \\ &= \frac{1}{2\pi i}\int_C d\,(\log |f(z)| + \frac{1}{2\pi}\int_C d\phi \qquad (2) \end{aligned}$$

The first integral in (2) vanishes, since $\log |f(z)|$ is single valued, i.e., it returns to its original value at z goes round C. Now, $\int_C d\phi$ is the variation in the argument of $f(z)$ in describing the contour C,

Therefore $\int d\phi = \Delta_C \arg f(z)$

Thus, (2) becomes

$$\Delta_C \arg f(z) = 2\pi\,(N - P)$$

This formula makes it possible to compute the number N−P from the variation of the argument of $f(z)$ along the boundary of the closed contour C and is known as argument principle.

In particular, if $f(z)$ is analytic inside and on C, then P = 0

and $$N = \frac{1}{2\pi}\Delta_C \arg f(z).$$

5.3. Rouche's Theorem. If $f(z)$ and $g(z)$ are analytic inside and on a closed contour C and $|g(z)| < |f(z)|$ on C, then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C.

Proof. First we prove that neither $f(z)$ nor $f(z) + g(z)$ has a zero on C.

If $f(z)$ has a zero at z = a on C, then $f(a) = 0$

Thus $|g(z) < |f(z)|$ $\Rightarrow |g(a)| < f(a) = 0$

$\Rightarrow$ $g(a) = 0$ $\Rightarrow |f(a)| = |g(a)|$

i.e. $|f(z)| = |g(z)|$ at z = a

which is contrary to the assumption that

$$|g(z)| < |f(z)| \text{ on C.}$$

Again, if $f(z) + g(z)$ has a zero at z = b on C,

then $f(b) + g(b) = 0$ $\Rightarrow f(b) = -g(b)$

i.e. $|f(b)| = |g(b)|$

again a contradiction.

Thus, neither $f(z)$ nor $f(z) + g(z)$ has a zero on C.

Now, let N and N′ be the number of zeros of $f(z)$ and $f(z) + g(z)$ respectively inside C. We are to prove that N = N′.

Since $f(z)$ and $f(z) + g(z)$ both are analytic within and on C and have no pole inside C, therefore the argument principle

$$\frac{1}{2\pi i}\int_C \frac{f'}{f}dz = N - P, \text{ with } P = 0, \text{ gives}$$

$$\frac{1}{2\pi i}\int_C \frac{f'}{f}dz = N, \quad \frac{1}{2\pi i}\int_C \frac{f'+g'}{f+g}dz = N'$$

Subtracting these two results, we get

$$\frac{1}{2\pi i}\int_C \left[\frac{f'+g'}{f+g} - \frac{f'}{f}\right]dz = N' - N$$

Let us take $\phi(z) = \dfrac{g(z)}{f(z)}$ so that $g = f\phi$

Now, $|g| < |f| \Rightarrow |g/f| < 1$ i.e. $|\phi| < |$

Therefore,

$$\frac{f'+g'}{f+g} = \frac{f'+f'\phi+f\phi'}{f+f\phi} = \frac{f'(1+\phi)+f\phi'}{f(1+\phi)}$$

$$= \frac{f'}{f} + \frac{\phi'}{1+\phi}$$

i.e. $$\frac{f'+g'}{f+g} - \frac{f'}{f} + \frac{\phi'}{1+\phi} \qquad (2)$$

Using (2) in (1), we get

$$N' - N = \frac{1}{2\pi i}\int_C \frac{\phi'}{1+\phi}dz = \frac{1}{2\pi i}\int_C \phi'(1+\phi)^{-1}dz \qquad (3)$$

Since we have observed that $|\phi| < 1$, so binomial expansion of $(1 + \phi)^{-1}$ is possible and this expansion in powers of ϕ is uniformly convergent and hence term by term integration is possible.

Thus, $\int_C \phi'(1+\phi)^{-1}dz = \int_C \phi'(1-\phi+\phi^2-\phi^3+\ldots)\,dz$

$$= \int_C \phi'\,dz - \int_C \phi\phi'\,dz + \int_C \phi^2\phi'\,dz \qquad (4)$$

Now, the functions f and g both are analytic within and on C and $f \neq 0$ $g \neq 0$ for any point on C, therefore $\phi = g/f$ is analytic and non-zero for any point on C. Thus ϕ and its all derivatives are analytic and so by Cauchy's theorem, each integral on R.H.S. of (4) vanishes. Thus

$$\int_C \phi'(1+\phi)^{-1}dz = 0$$

and therefore from (3), we conclude $N' - N = 0$

i.e. $N = N'$

5.4. Theorem (Fundamental Theorem of Algebra). Every polynomial of degree n has exactly n zeros.

Proof. Let us consider the polynomial

$$a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n,\ a_n \neq 0$$

We take $f(z) = a_n z^n$, $g(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_{n-1} z^{n-1}$

Let C be a circle $|z| = r$, where $r > 1$.

Now, $|f(z)| = |a_n z^n| = |a_n| r^n$

$|g(z)| \le |a_0| + |a_1| r + |a_2| r^2 + \ldots + |a_{n-1}| r^{n-1}$

$\le (|a_0| + |a_1| + \ldots + |a_{n-1}| r^{n-1}$

Therefore,

$$\left|\frac{g(z)}{f(z)}\right| \le \frac{(|a_0|+|a_1|+...+|a_{n-1}|)r^{n-1}}{|a_n| r^n}$$

$$= \frac{|a_0|+|a_1|+...+|a_{n-1}|}{|a_n| r}$$

Hence $|g(z) < |f(z)|$, provided that

$$\frac{|a_0|+|a_1|+...+|a_{n-1}|}{|a_n| r} < 1$$

i.e. $$r > \frac{|a_0|+|a_1|+...+|a_{n-1}|}{|a_n|} \tag{1}$$

Since r is arbitrary, therefore we can choose r large enough so that (1) is satisfied. Now, applying Rouche's theorem, we find that the given polynomial $f(z + g(z)$ has the same number of zeros as $f(z)$. But $f(z)$ has exactly n zeros all located at z = 0. Hence the given polynomial has exactly n zeros.

5.5. Example. Determine the number of roots of the equation

$$z^8 - 4z^5 + z^2 - 1 = 0$$

that lie inside the circle $|z| = 1$

Solution. Let C be the circle defined by $|z| = 1$

Let us take $f(z) = z^8 - 4z^5$, $g(z) = z^2 - 1$.
On the circle C,

$$\left|\frac{g(z)}{f(z)}\right| = \left|\frac{z^2-1}{z^8-4z^5}\right| \le \frac{|z|^2+1}{|z|^5|4-z^3|}$$

$$\le \frac{1+1}{4-|z|^3} = \frac{2}{4-1} = \frac{2}{3} < 1$$

Thus $|g(z)| < |f(z)|$ and both $f(z)$ and $g(z)$ are analytic within and on C, Rouche's theorem implies that the required number of roots is the same as the number of roots of the equation $z^8 - 4z^5 = 0$ in the region $|z| < 1$. Since $z^3 - 4 \ne 0$ for $|z| < 1$, therefore the required number of roots is found to be 5.

5.6. Inverse Function. If $f(z) = w$ has a solution $z = F(w)$, then we may write

$f\{F(w)\} = w$, $F\{f(z)\} = z$. The function F defined in this way, is called inverse function of *f*.

5.7. Theorem. (Inverse Function Theorem). Let a function $w = f(z)$ be analytic at a point $z = z_0$ where $f'(z_0) \ne 0$ and $w_0 = f(z_0)$.

Then there exists a neighbourhood of w_0 in the w-plane in which the function $w = f(z)$ has a unique inverse $z = F(w)$ in the sense that the function F is single-valued and analytic in that neighbourhood such that $F(w_0) = z_0$ and

$$F'(w) = \frac{1}{f'(z)}.$$

Proof. Consider the function $f(z) - w_0$. By hypothesis, $f(z_0) - w_0 = 0$. Since $f'(z_0) \neq 0$, f is not a constant function and therefore, neither $f(z) - w_0$ not $f'(z)$ is identically zero. Also $f(z) - w_0$ is analytic at $z = z_0$ and so it is analytic in some neighbourhood of z_0. Again, since zeros are isolated, neither $f(z) - w_0$ nor $f'(z)$ has any zero in some deleted neighbourhood of z_0. Hence there exists $\in > 0$ such that $f(z) - w_0$ is analytic for $|z - z_0| \leq \in$ and $f(z) - w_0 \neq 0$, $f'(z) \neq 0$ for $0 < |z-z_0| \leq \in$. Let D denote the open disc

$$\{z : |z - z_0| < \in\}$$

and C denotes its boundary

$$\{z : |z-z_0| = \in\}.$$

Since $f(z) - w_0$ for $|z - z_0| \leq \in$, we conclude that $|f(z) - w_0|$ has a positive minimum on the circle C. Let

$$\min_{z\in C} |f(z) - w_0| = m$$

and choose δ such that $0 < \delta < m$.

We now show that the function $f(z)$ assumes exactly once in D every value w_1 in the open disc $T = \{ w : |w - w_0| < \delta\}$. We apply Rouche's theorem to the functions $w_0 - w_1$ and $f(z)-w_0$. The condition of the theorem are satisfied, since

$$|w_0-w_1| < \delta < m = \min_{z\in C} |f(z) - w_0| \leq |f(z) - w_0| \text{ on C.}$$

Thus we conclude that the functions.

$$f(z)-w_0 \text{ and } (f(z) - w_0) + (w_0 - w_1) = f(z) - w_1$$

have the same number of zeros in D. But the function $f(z) - w_0$ has only one zero in D i.e. a simple zeros at z_0, since $(f(z) - w_0)' = f'(z) \neq 0$ at z_0.

Hence $f(z)-w_1$ must also have only one zero, say z_1 in D. This means that the function $f(z)$ assumes the value w, exactly once in D. It follows that the function $w = f(z)$ has a unique inverse, say $z = F(w)$ in D such that F is single-valued and $w = f\{F(w)\}$. We now show that the function F is analytic in D. For fix w_1 in D, we have $f(z) = w_1$ for a unique z_1 in D. If w is in T and $F(w) = z$, then

$$\frac{F(w) - F(w_1)}{w - w_1} = \frac{z - z_1}{f(z) - f(z_1)} \qquad (1)$$

It is noted that T is continuous. Hence $z \to z_1$ whenever $w \to w_1$. Since $z_1 \in D$, as shown above $f'(z_1)$ exists and is zero. If we let $w \to w$, then (1) shows that

$$F'(w_1) = \frac{1}{f'(z_1)}.$$

Thus $F'(w)$ exists in the neighbourhood T of w_0 so that the function F is analytic there.

6. Calculus of Residues

The main result to be discussed here is Cauchy's residue theorem which does for meromorphic functions what Cauchy's theorem does for holomorphic functions. This theorem is extremely important theoretically and for practical applications.

6.1. The Residue at a Singularity. We know that in the neighbourhood of an isolated singularity $z = a$, a one valued analytic function $f(z)$ may be expanded in a Laurent's series as

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

The co-efficient b_1 is called the residue of $f(z)$ at $z = a$ and is given by the formula

$$\text{Res } (z = a) = b_1 = \frac{1}{2\pi i}\int_{\gamma} f(z)\,dz \qquad \Big|\because b_n \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)dz}{(z-a)^{-n+1}}$$

Where γ is any circle with centre z = a, which excludes all other singularities of $f(z)$. In case, z = a is a simple pole, then we have

$$\text{Res } (z = a) = b_1 = \lim_{z\to a}(z-a)\, f(z) \qquad \Big|\because f(z) = \sum_{0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a}$$

A more general definition of the residue of a function $f(z)$ at a point z = a is as follows.

If the point z = a is the only singularity of an analytic function $f(z)$ inside a closed contour C, then the value $\frac{1}{2\pi i}\int_{C} f(z)\,dz$ is called the residue of $f(z)$ at a.

6.2. Residue at Infinity. If $f(z)$ is analytic or has an isolated singularity at infinity and if C is a circle enclosing all its singularities in the finite parts of the z-plane, the residue of $f(z)$ at infinity is defined by

$$\text{Res } (z = \infty) = \frac{1}{2\pi i}\int_{C} f(z)\,dz, \qquad \Big| \text{ or Res } (z = \infty) - \frac{1}{2\pi i}\int_{C} f(z)\,dz$$

Integration taken in positive sense

the integration being taken round C in the negative sense w.r.t. the origin, provided that this integral has a definite value. By means of the substitution $z = w^{-1}$, the integral defining the residue at infinity takes the form

$$\frac{1}{2\pi i}\int [-f(w^{-1})]\, \frac{dw}{w^2},$$

taken in positive sense round a sufficiently small circle with centre at the origin.

Thus, we also say if

$$\lim_{w\to 0}\,[-f(w^{-1})\, w^{-1}] \qquad \text{or} \qquad \lim_{z\to\infty}\,[-z\, f(z)]$$

has a definite value, that value is the residue of $f(z)$ at infinity.

6.3. Remarks. (i) The function may be regular at infinity, yet has a residue there.

For example, consider the function $f(z) = \frac{b}{z-a}$ for this function

$$\begin{aligned}\text{Res } (z = \infty) &= -\frac{1}{2\pi i}\int_{C} f(z)\,dz \\ &= -\frac{1}{2\pi i}\int_{C} \frac{b}{z-a}\,dz \\ &= -\frac{b}{2\pi i}\int_0^{2\pi} \frac{re^{i\theta} i d\theta}{re^{i\theta}}, \text{ C being the circle } |z-a| = r \\ &= -\frac{b}{2\pi}\int_0^{2\pi} d\theta = -b\end{aligned}$$

$\therefore \qquad \text{Res } (z = \infty) = -b$

Also, z = a is a simple pole of $f(z)$ and its residue there is $\frac{1}{2\pi i}\int_{C} f(z)\,dz = b$

$$\Big| \text{ or } \lim_{z\to a} (z-a)\, f(z) = b$$

Thus Res (z = a) = b – Res (z = ∞)

(ii) If the function is analytic at a point z = a, then its residue at z = a is zero but not so at infinity.

(iii) In the definition of residue at infinity, C may be any closed contour enclosing all the singularities in the finite parts of the z-plane.

6.4. Calculation of Residues. Now, we discuss the method of calculation of residue in some special cases.

(i) If the function $f(z)$ has a simple pole at z = a, then, Res (z = a) = $\lim_{z\to a} (z-a) f(z)$.

(ii) If $f(z)$ has a simple pole at z = a and $f(z)$ is of the form $f(z) = \dfrac{\phi(z)}{\psi(z)}$ i.e. a rational function, then

$$\text{Res}\,(z = a) = \lim_{z\to a} (z-a) f(z) = \lim_{z\to a} (z-a) \frac{\phi(z)}{\psi(z)}$$

$$= \lim_{z\to a} \frac{\phi(z)}{\dfrac{\psi(z)-\psi(a)}{z-a}}$$

$$= \frac{\phi(a)}{\psi'(a)},$$

where $\psi(a) = 0$, $\psi'(a) \neq 0$, since $\psi(z)$ has a simple zero at z = a

(iii) If $f(z)$ has a pole of order m at z = a then we can write

$$f(z) = \frac{\phi(z)}{(z-a)^m} \qquad (1)$$

where $\phi(z)$ is analytic and $\phi(a) \neq 0$.

Now, $\text{Res}\,(z = a) = b_1 = \dfrac{1}{2\pi i}\displaystyle\int_C f(z)\,dz = \dfrac{1}{2\pi i}\int_C \dfrac{\phi(z)}{(z-a)^m}\,dz$

$$= \frac{1}{\lfloor m-1}\, \frac{\lfloor m-1}{2\pi i}\int_C \frac{\phi(z)}{(z-a)^{m-1+1}}\,dz$$

$$= \frac{1}{\lfloor m-1}\, \phi^{m-1}(a) \qquad \text{[By Cauchy's integral formula for derivatives]} \qquad (2)$$

Using (1), formula (2) take the form

$$\text{Res}\,(z = a) = \frac{1}{\lfloor m-1}\, \frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)] \text{ as } z\to a$$

i.e.
$$\text{Res}\,(z = a) = \lim_{z\to a} \frac{1}{\lfloor m-1}\, \frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)] \qquad (3)$$

Thus, for a pole of order m, we can use either formula (2) or (3).

(iv) If z = a is a pole of any order for $f(z)$, then the residue of $f(z)$ at z = a is the co-efficient of $\dfrac{1}{z-a}$ in Laurent's expansion of $f(z)$

(v) Res (z = ∞) = Negative of the co-efficient of $\dfrac{1}{z}$ in the expansion of $f(z)$ in the neighbourhood of z = ∞.

6.5. Examples. (a) Find the residue of $\frac{z^4}{z^2+a^2}$ at z = –ia

Solution. Let *f*(z) = $\frac{z^4}{z^2+a^2}$.

Poles of *f*(z) are z = $\pm$ ia

Thus z = –ia is a simple pole, so

$$\text{Res }(z=-ia) = \lim_{z\to -ia} (z+ia) f(z)$$

$$= \lim_{z\to -ia} (z+ia)\frac{z^4}{(z+ia)(z-ia)}$$

$$= \lim_{z\to -ia} \frac{z^4}{z-ia} = \frac{a^4}{-2ia} \qquad \Bigg| \begin{array}{l} \text{or } \phi(z) = \frac{z^4}{z-ia} \\ \text{as } f(z) = \frac{z^4/(z-ia)}{(z+ia)} \end{array}$$

$$= \frac{a^4 i}{2a} = \frac{ia^3}{2}$$

(b) Find the residues of $e^{iz} z^{-4}$ at its poles.

Solution. Let *f*(z) = $\frac{e^{iz}}{z^4}$

f(z) has pole of order 4 at z = 0, so

$$\text{Res }(z=0) = \frac{1}{\lfloor 3}\left[\frac{d^3}{dz^3}(e^{iz})\right]_{z=0} = -\frac{i}{6} \qquad | \ \phi(z) = e^{iz}$$

Alternatively, by the Laurent's expansion

$$\frac{e^{iz}}{z^4} = \frac{1}{z^4} + \frac{i}{z^3} - \frac{1}{\lfloor 2 z^2} - \frac{i}{\lfloor 3 z} + \ldots$$

we find that

$$\text{Res }(z=0) = \text{co-efficient of } \frac{1}{z}$$

$$= -\frac{i}{6}\ .$$

(c) Find the residue of $\frac{z^3}{z^2-1}$ at z = ∞.

Solution. Let *f*(z) = $\frac{z^3}{z^2-1} = \frac{z^3}{z^2\left(1-\frac{1}{z^2}\right)} = z\left(1-\frac{1}{z^2}\right)^{-1}$

$$= z\left(1+\frac{1}{z^2}+\frac{1}{z^4}+\ldots\right)$$

$$= z + \frac{1}{z} + \frac{1}{z^3} + \ldots$$

Therefore,

$$\text{Res}\ (z = \infty) = -(\text{co-efficient of } \frac{1}{z}) = -1$$

(d) Find the residues of $\dfrac{z^3}{(z-1)^4(z-2)(z-3)}$ at its poles.

Solution. Let $f(z) = \dfrac{z^3}{(z-1)^4(z-2)(z-3)}$

Poles of $f(z)$ are z = 1 (order four) and z = 2, 3 (simple)
Therefore,

$$\text{Res}\ (z = 2) = \lim_{z\to 2}(z-2)\, f(z) = \lim_{z\to 2} \frac{z^3}{(z-1)^4(z-3)} = -8$$

$$\text{Res}\ (z = 3) = \lim_{z\to 3}(z-3)\, f(z) = \frac{27}{16}$$

For z = 1, we take $\phi(z) = \dfrac{z^3}{(z-2)(z-3)}$,

where $f(z) = \dfrac{\phi(z)}{(z-1)^4}$ and thus $\text{Res}\ (z = 1) = \dfrac{\phi^3(1)}{\lfloor 3}$

$$\text{Now, } \phi(z) = z + 5 - \frac{8}{z-2} + \frac{27}{z-3}$$

$$\phi^3(z) = \frac{48}{(z-2)^4} - \frac{162}{(z-3)^4}$$

$$\phi^3(1) = \frac{303}{8}$$

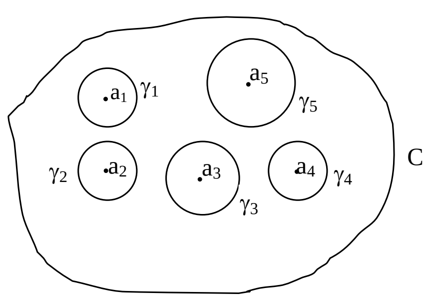
Thus,

$$\text{Res}\ (z = 1) = \frac{303}{8\lfloor 3} = \frac{101}{16}$$

6.6. Theorem. (Cauchy Residue Theorem). Let $f(z)$ be one-valued and analytic inside and on a simple closed contour C, except for a finite number of poles within C. Then

$$\int_C f(z)\, dz = 2\pi i\ [\text{Sum of residues of } f(z) \text{ at its poles within C}]$$

Proof. Let $a_1, a_2, \ldots, a_n$ be the poles of $f(z)$ inside C. Draw a set of circles γ_r of radii $\in$ and centre a_r (r = 1, 2,…, n) which do not overlap and all lie within C. Then $f(z)$ is regular in the domain bounded externally by C and internally by the circles γ_r.



Then by cor. to Cauchy's Theorem, we have

$$\int_C f(z)\,dz = \sum_{r=1}^{n} \int_{\gamma_r} f(z)\,dz \qquad (1)$$

Now, if a_r is a pole of order m, then by Laurent's theorem, $f(z)$ can be expressed as

$$f(z) = \phi(z) + \sum_{s=1}^{m} \frac{b_s}{(z-a_r)^s}$$

where $\phi(z)$ is regular within and on γ_r.

Then

$$\int_{\gamma_r} f(z)\,dz = \sum_{s=1}^{m} \int_{\gamma_r} \frac{b_s}{(z-a_r)^s}\,dz \qquad (2)$$

where $\int_{\gamma_r} f(z)\,dz = 0$, by Cauchy's theorem

Now, on γ_r $|z-a_r| = \in$ i.e. $z = a_r + \in e^{i\theta}$

$\Rightarrow \quad dz = \in ie^{i\theta}\,d\theta$

where θ varies from 0 to 2π as the point z moves once round γ_r.

Thus, $\displaystyle\int_{\gamma_r} f(z)\,dz = \sum_{s=1}^{m} b_s \in^{1-s} \int_0^{2\pi} e^{(1-s)i\theta} i\,d\theta$

$= 2\pi i\, b_1$

$= 2\pi i$ [Residue of $f(z)$ at a_r]

where $\displaystyle\int_0^{2\pi} e^{(1-s)i\theta}d\theta = \begin{cases} 0, \text{ if } s \neq 1 \\ 2\pi \text{ if } s = 1 \end{cases}$

Hence, from (1), we find

$$\int_C f(z)dz = \sum_{r=1}^{n} 2\pi i \text{ [Residue of } f(z) \text{ at } a_r]$$

$$= 2\pi i \left[\sum_{r=1}^{n} \text{Residue of } f(z) \text{ at } a_r\right]$$

$= 2\pi i$ [sum of Residues of $f(z)$ at its poles inside C.]

which proves the theorem.

6.7. Remark. If $f(z)$ can be expressed in the form $f(z) = \dfrac{\phi(z)}{(z-a)^m}$ where $\phi(z)$ is analytic and $\phi(a) \neq 0$, then the pole z = a is a pole of type I or overt.

If $f(z)$ is of the form $f(z) = \dfrac{\phi(z)}{\psi(z)}$, where $\phi(z)$ and $\psi(z)$ are analytic and $\phi(a) \neq 0$ and $\psi(z)$ has a zero of order m at z = a, then z = a is a pole of type II or covert. Actually, whether a pole of $f(z)$ is overt or covert, is a matter of how $f(z)$ is written

7. Evaluation of Integrals

Cauchy's residue theorem provides the natural way to deal with a contour integral $\int_C f(z)\,dz$ where $f(z)$ has poles inside C. For an integral round just one type I pole Cauchy's integral formula or Cauchy's formula for derivatives should be used. Of course, when $f(z)$ has no poles inside or on C, Cauchy's theorem applies. In Cauchy's residue theorem's applications to evaluation of definite integrals, we frequently need to find the limiting value of an integral along a path as that path shrinks or expands indefinitely.

For evaluation of integrals of the type $\int_0^{2\pi} f(\cos\theta, \sin\theta)$ where $f(\cos\theta, \sin\theta)$ is a rational function of cos θ and sin θ and bounded for the entire range of integration, we use the transformation $z = e^{i\theta}$ i.e. $\cos\theta = \frac{z+z^{-1}}{2}, \sin\theta = \frac{z-z^{-1}}{2i}$ This will reduce the integral to the form $\int_C g(z)\,dz$ where g(z) is a rational function of z, with no singularity on the unit circle |z| = 1, denoted by C.

Another type of integrals, we shall consider, are $\int_{-\infty}^{\infty} f(x)\,dx$, which are obtained by using complex function $f(z)$.

7.1. Example. Evaluate $\int_0^{\pi} \frac{a\,d\theta}{a^2+\sin^2\theta}$, where a > 0

Solution. Let $I = \int_0^{\pi} \frac{a\,d\theta}{a^2+\sin^2\theta}$

$$= \int_0^{\pi} \frac{2a\,d\theta}{2a^2+2\sin^2\theta} = \int_0^{\pi} \frac{2a\,d\theta}{2a^2+1-\cos 2\theta}$$

$$= \int_0^{2\pi} \frac{a\,dt}{2a^2+1-\cos t}, \; 2\theta = t$$

$$= \int_0^{2\pi} \frac{a\,dt}{2a^2+1-\left(\frac{e^{it}+e^{-it}}{2}\right)}$$

Putting $z = e^{it}$ so that $dz = e^{it}\,i dt$, we get

$$I = \int_C \frac{2a}{2(2a^2+1)+(z+z^{-1})}\frac{dz}{iz}$$

$$= \frac{2a}{i}\int_C \frac{dz}{-z^2+2(2a^2+1)z-1}$$

$$= 2ai\int_C \frac{dz}{z^2-2(2a^2+1)z+1}$$

or

$$I = 2ai \int_C f(z)\,dz \qquad (1)$$

where $f(z) = \frac{1}{z^2-2(2a^2+1)z+1}$ and C is the unit circle |z| = 1

Now, poles of $f(z)$ are given by

$$z^2 - 2(2a^2+1)z + 1 = 0$$

i.e. $z = (2a^2+1) \pm 2a\sqrt{a^2+1}$

we take $\alpha = (2a^2+1) + 2a\sqrt{a^2+1}$

$\beta = (2a^2+1) - 2a\sqrt{a^2+1}$

Thus, poles of $f(z)$ are $z = \alpha_1\beta$

Clearly, $|\alpha| > 1$ and since $|\alpha\beta| = 1 \Rightarrow |\beta| < 1$

Thus, $f(z)$ has only one simple pole $z = \beta$ lying within C.

$$\text{Res}\,(z = \beta) = \lim_{z\to\beta} (z-\beta)\, f(z)$$

$$= \frac{1}{\beta-\alpha} = \frac{-1}{4a\sqrt{a^2+1}}$$

Hence by Cauchy's Residue theorem

$$\int_C f(z)\, dz = 2\pi i \text{ (sum of residues of } f(z) \text{ at its poles within C)}$$

$$= 2\pi i \left(\frac{-1}{4a\sqrt{a^2+1}}\right)$$

Thus, from (1), we get

$$I = 2ai \left(\frac{-2\pi i}{4a\sqrt{a^2+1}}\right) = \frac{\pi}{\sqrt{a^2+1}}$$

Similarly, we have $\int_0^{\pi} \dfrac{a d\theta}{a^2+\cos^2\theta} = \dfrac{\pi}{\sqrt{a^2+1}}$

7.2. Example. Prove that $\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta - n\theta) d\theta = \dfrac{2\pi}{\lfloor n}$, n is a +ve integer.

Solution. Let $I = \int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta - n\theta)\, d\theta$

$$= \text{R.P. } \int_0^{2\pi} e^{\cos\theta} e^{(\sin\theta - n\theta)i}\, d\theta$$

$$= \text{R.P. } \int_0^{2\pi} e^{\cos\theta + i\sin\theta - in\theta}\, d\theta$$

$$= \text{R.P. } \int_0^{2\pi} e^{e^{i\theta} - in\theta} d\theta$$

$$= \text{R.P. } \int_0^{2\pi} e^{e^{i\theta}} (e^{-i\theta})^n\, d\theta$$

putting $z = e^{i\theta}$, we get

$$I = \text{R.P. } \int_C e^z z^{-n} \frac{dz}{iz} = \text{R.P.} \frac{1}{i} \int_C \frac{e^z}{z^{n+1}} dz$$

$$= \text{R.P. } \frac{1}{i} \int_C f(z) dz \qquad (1)$$

where $f(z) = \dfrac{e^z}{z^{n+1}}$, C is the unit circle $|z| = 1$

Evidently, $z = 0$ is a pole of order $n + 1$ for $f(z)$, lying within C.

$$\therefore \quad \text{Res}\,(z = 0) = \frac{1}{\lfloor n}\left[\frac{d^n}{dz^n} e^z\right]_{z=0} = \frac{1}{\lfloor n}$$

Thus, using Cauchy's residue theorem, we get

$$I = \text{R.P. } \frac{1}{i} 2\pi i \frac{1}{\lfloor n} = \frac{2\pi}{\lfloor n}$$

Note that $\int_0^{2\pi} e^{\cos\theta} \sin(\sin\theta - n\theta)\, d\theta = 0$

Similarly, we have

$$\int_0^{2\pi} e^{-\cos\theta}\cos(\sin\theta + n\theta)d\theta = \frac{2\pi(-1)^n}{\lfloor n}.$$

7.3. Theorem. Let $f(z)$ be a function of the complex variable z satisfying the conditions

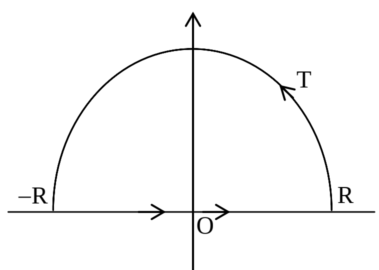
(i) $f(z)$ is meromorphic in the upper half of the complex plane i.e. $I_m. z \geq 0$.
(ii) $f(z)$ has no pole on the real axis
(iii) $z f(z) \to 0$ uniformly as $|z| \to \infty$ for $0 \leq \arg z \leq \pi$.
(iv) $\int_0^\infty f(x)\,dx$ and $\int_{-\infty}^{\infty} f(x)\,dx$ both converge.

Then $\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\, \Sigma$ Res;

where Σ Res. denotes the sum of residues of $f(z)$ at its poles in the upper half of the z-plane.

Proof. Let us consider the integral $\int_C f(z)dz$, where C is the contour consisting of the segment of the real axis from $-R$ to R and the semi-circle in the upper half plane on it as diameter.

Let the semi-circular part of the contour C be denoted by T and let R be chosen so large that C includes all the poles of $f(z)$.



Then by Cauchy's residue theorem,

$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_T f(z)\,dz = 2\pi i\, \Sigma \text{ Res.} \quad (1)$$

By hypothesis (iii), $|z f(z)| < \in$ for all points z on T. If R is chosen sufficiently large, however small be the +ve number $\in$, then for such R,

$$\left|\int_T f(z)\,dz\right| = \left|\int_0^\pi f(Re^{i\theta})\, Re^{i\theta}\, i d\theta\right|$$

$$= \left|\int_0^\pi z f(z)\,d\theta\right|$$

$$< \in \int_0^\pi d\theta = \in\pi$$

It follows that as $|z| = R \to \infty$, $\int_T f(z)\,dz \to 0$

Now, since hypothesis (iv) holds,

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$$

Taking limit as $R \to \infty$ in (1), we get

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\, \Sigma\text{Res}.$$

Hence the result

7.4. Example. By method of contour integration, prove that

$$\int_0^\infty \frac{dx}{(1+x^2)^2} = \pi/4$$

Solution. Consider the integral

$$\int_C f(z)\,dz, \text{ where } f(z) = \frac{1}{(1+z^2)^2}$$

C being the closed contour consisting of T, the upper half of the large circle |z| = R and the real axis from –R to R.

Poles of $f(z)$ are z = $\pm$ i(each of order two) $f(z)$ has only one pole of order two at z = i within C.

We can write

$$f(z) = \frac{\phi(z)}{(z-i)^2}, \text{ where } \phi(z) = \frac{1}{(z+i)^2}$$

$$\therefore \qquad \text{Res}\,(z = i) = \frac{1}{\lfloor 1}\phi'(i) = \frac{1}{4i}$$

Hence, by Cauchy's residue theorem

$$\int_C f(z)\,dz = 2\pi i\,\frac{1}{4i} = \frac{\pi}{2}$$

or $$\int_T f(z)\,dz + \int_{-R}^{R} f(x)\,dx = \frac{\pi}{2}$$

i.e. $$\int_T \frac{dz}{(z^2+1)^2} + \int_{-R}^{R} \frac{dx}{(1+x^2)^2} = \frac{\pi}{2} \qquad (1)$$

Now, using the inequality,

$$|z_1 + z_2| \geq |z_1| - |z_2|,\ \frac{1}{|z_1+z_2|} \leq \frac{1}{|z_1|-|z_2|}, \text{ we get}$$

$$\left|\int_T \frac{dz}{(1+z^2)^2}\right| \leq \int_T \frac{|dz|}{(z^2-1)^2} = \int_T \frac{|dz|}{(|z|^2-1)^2}$$

$$= \frac{1}{(R^2-1)^2}\int_T |dz| = \frac{\pi R}{(R^2-1)^2} \to 0 \text{ as } |z| = R \to \infty$$

so $$\lim_{R\to\infty}\int_T \frac{dz}{(1+z^2)^2} = 0$$

Making R→∞ in (1), we obtain

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \pi/2$$

or $$\int_0^{\infty} \frac{dx}{(1+x^2)^2} = \pi/4.$$

7.5. Example. Prove that $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)^2} = \frac{\pi(a+2b)}{2ab^3(a+b)^2}$ a > 0, b > 0

Solution. Consider the integral $\int_C f(z)dz$,

$$\text{where } f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)^2}$$

and c is the closed contour consisting of T, the upper half of the large circle |z| = R and the real axis from −R to R.
Poles of $f(z)$ are $z = \pm ia$ (simple) and $z = \pm ib$ (double)
Only poles of $f(z)$ lying within C are z = ia (simple) and z = ib (double)

$$f(z) = \frac{\phi(z)}{(z-ib)^2}, \qquad \phi(z) = \frac{1}{(z^2+a^2)(z+ib)^2}$$

$$\text{Res } (z = ia) = \frac{1}{2ia(a^2-b^2)^2}$$

$$\text{Res } (z = ib) = \frac{(3b^2-a^2)i}{4b^3(a^2-b^2)^2}$$

Thus, sum of residues $= \dfrac{i}{4(a^2-b^2)^2}\left[-\dfrac{2}{a}+\dfrac{3b^2-a^2}{b^3}\right]$

$$= \frac{i\left[-2b^3+a(3b^2-a^2)\right]}{4ab^3(a^2-b^2)^2}$$

$$= \frac{i[(b^3-a^3)-3b^2(b-a)]}{4ab^3(a^2-b^2)^2}$$

$$= \frac{i(b-a)[b^2+a^2+ab-3b^2]}{4ab^3(a^2-b^2)^2}$$

$$= \frac{i(b-a)(a-b)(a+2b)}{4ab^3(a^2-b^2)^2} = \frac{-i(a+2b)}{4ab^3(a+b)^2}$$

So, by Cauchy residue theorem,

$$\int_C f(z)\, dz = \int_T f(z) + \int_{-R}^{R} f(x)\, dx$$

$$= 2\pi i\left[\frac{-i(a+2b)}{4ab^3(a+b)^2}\right] = \frac{\pi(a+2b)}{2ab^3(a+b)^2} \qquad (1)$$

Now, $\left|\int_T f(z)dz\right| = \left|\int_T \dfrac{dz}{(z^2+a^2)(z^2+b^2)^2}\right|$

$$= \left|\int_T \frac{|dz|}{(|z|^2-a^2)(|z|^2-b^2)^2}\right|$$

$$= \frac{\pi R}{(R^2-a^2)(R^2-b^2)} \to 0 \text{ as } |z| = R \to \infty.$$

Making $R\to\infty$ in (1), we obtain

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+z^2)(x^2+b^2)^2} = \frac{\pi(a+2b)}{2ab^3(a+b)^2}.$$

Hence the result.

Deductions. (i) putting a = 1, b = 2, we get

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)^2} = \frac{5\pi}{144} \qquad \text{or} \qquad \int_0^{\infty} \frac{dx}{(x^2+1)(x^2+4)^2} = \frac{5\pi}{288}$$

(ii) Putting a = 2, b = 1, we get

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+2)(x^2+1)^2} = \frac{\pi}{9} \qquad \text{or} \quad \int_{0}^{\infty} \frac{dx}{(x^2+2)(x^2+1)^2} = \frac{\pi}{18}.$$

(iii) Putting a = 3, b = 2, we get

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+9)(x^2+4)^2} = \frac{7\pi}{1200} \qquad \text{or} \quad \int_{0}^{\infty} \frac{dx}{(x^2+9)(x^2+4)^2} = \frac{7\pi}{2400}$$

7.6 Jordan's Inequality. If $0 \le \theta \le \pi/2$, the $\dfrac{2\theta}{\pi} \le \sin\theta \le \theta$

This inequality is called Jordan inequality. We know that as θ increases from 0 to π/2, cosθ decreases steadily and consequently, the mean ordinate of the graph of y = cos x over the range $0 \le x \le \theta$ also decreases steadily. But this mean ordinate is given by

$$\frac{1}{\theta}\int_0^{\theta} \cos x \, dx = \frac{\sin\theta}{\theta}$$

It follows that when $0 \le \theta \le \pi/2$,

$$\frac{2}{\pi} \le \frac{\sin\theta}{\theta} \le 1$$

7.7. Jordan's Lemma. If *f*(z) is analytic except at a finite number of singularities and if $f(z)\to 0$ uniformly as $z\to\infty$, then

$$\lim_{R\to\infty} \int_T e^{imz} f(z)\, dz = 0 \text{ , } m > 0$$

where T denotes the semi-circle $|z| = R$, $I_m. z \ge 0$, R being taken so large that all the singularities of *f*(z) lie within T.

Proof. Since $f(z)\to 0$ uniformly as $|z|\to\infty$, there exists $\in > 0$ such that $|f(z)| < \in \ \forall$ z on T.

Also $|z| = R \quad \Rightarrow z = Re^{i\theta} \quad \Rightarrow dz = Re^{i\theta}\, id\theta \quad \Rightarrow |dz| = Rd\theta$

$$|e^{imz}| = |e^{imRe^{i\theta}}| = |e^{imR\cos\theta}\, e^{-mR\sin\theta}|$$
$$= e^{-mR\sin\theta}$$

Hence, using Jordan inequality,

$$\left|\int_T e^{imz} f(z)\, dz\right| \le \int_T |e^{imz} f(z)|\, |dz|$$
$$< \int_0^{\pi} e^{-mR\sin a} \in R\, d\theta$$
$$= 2\in R\int_0^{\pi/2} e^{-mR\sin\theta}\, d\theta$$
$$= 2\in R\int_0^{\pi/2} e^{-2mR\theta/\pi} d\theta \qquad \left|\ \because \frac{2a}{\pi} \le \sin\theta \quad \text{i.e.} -\sin\theta \le \frac{-2\theta}{\pi}\right.$$
$$= 2\in R\ \frac{(1-e^{-mR})}{2mR/\pi}$$
$$= \frac{\in\pi}{m}(1-e^{-mR}) < \frac{\in\pi}{m}$$

Hence $\lim\limits_{R\to\infty} \int_T e^{imz} f(z)\, dz = 0$

7.8. Example. By method of contour integration prove that

$$\int_0^\infty \frac{\cos mx}{x^2+a^2}dx = \frac{\pi}{2a}e^{-ma}, \text{ where } m \geq 0,\ a > 0$$

Solution. We consider the integral

$$\int_C f(z)dz, \text{ where } f(z) = \frac{e^{imz}}{z^2+a^2}$$

and C is the closed contour consisting of T, the upper half of the large circle $|z| = R$ and real axis from $-R$ to R.

Now, $\frac{1}{z^2+a^2} \to 0$ as $|z| = R\to\infty$

Hence by Jordan lemma,

$$\lim_{R\to\infty}\int_T \frac{e^{im}z}{z^2+a^2}dz = 0$$

i.e. $$\lim_{R\to\infty}\int_T f(z)dz = 0 \quad (1)$$

Now, poles of $f(z)$ are given by $z = \pm ia$ (simple), out of which $z = ia$ lies within C.

$\therefore$ Res $(z = ia) = \frac{e^{-ma}}{2ia}$

Hence by Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i\ \frac{e^{-ma}}{2ia} = \frac{\pi}{a}e^{-ma}$$

or

$$\int_T f(z)dz + \int_{-R}^{R} f(x)dx = \frac{\pi}{a}e^{-ma}$$

Making $R\to\infty$ and using (1), we get

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{x^2+a^2}dx = \frac{\pi}{a}e^{-ma}$$

Equating real parts, we get

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+a^2}dx = \frac{\pi}{a}e^{-ma}$$

or $$\int_0^{\infty} \frac{\cos mx}{x^2+a^2}dx = \frac{\pi}{2a}e^{-ma}$$

Hence the result.

Deduction. (i) Replacing m by a and a by 1 in the above example, we get

$$\int_0^{\infty} \frac{\cos ax}{x^2+1}dx = \frac{\pi}{2}e^{-a}$$

Putting a = 1, we get

$$\int_0^{\infty} \frac{\cos x}{x^2+1} = \frac{\pi}{2}e^{-1} = \frac{\pi}{2e}$$

(ii) Taking m = 1, a = 2, we get

$$\int_0^{\infty} \frac{\cos x}{x^2+4}dx = \frac{\pi}{4e^2}.$$

7.9. Example. Prove that $\int_{-\infty}^{\infty} \dfrac{x^3 \sin mx}{x^4 + a^4} = \dfrac{\pi}{2} e^{-ma/\sqrt{2}} \cos\left(\dfrac{ma}{\sqrt{2}}\right)$ m > 0, a > 0

Solution. Consider the integral $\int_C f(z)dz$, where

$$f(z) = \frac{z^3 e^{imz}}{z^4 + a^4}$$

and C is the closed contour….

Since $\dfrac{z^3}{z^4 + a^4} \to 0$ as $|z| = R \to \infty$, so by Jordan lemma,

$$\lim_{R\to\infty} \int_T \frac{z^3 e^{imz}}{z^4 + a^4} dz = 0 \qquad (1)$$

Poles of $f(z)$ are given by

$$z^4 + a^4 = 0$$

or $\qquad z^4 = -a^4 = e^{2n\pi i} e^{\pi i} a^4$

or $\qquad z = a\, e^{(2n+1)\pi i/4}$, n = 0, 1, 2, 3.

Out of these four simple poles, only

$z = ae^{i\pi/4}$, $a\, e^{i3\pi/4}$ lie within C.

If $f(z) = \dfrac{\varphi(z)}{\psi(z)}$, then Res (z = α) = $\lim_{z\to\alpha} \dfrac{\varphi(z)}{\psi'(z)}$, α being simple pole.

∴ For the present case,

$$\text{Res } (z = \alpha) = \lim_{z\to\alpha} \frac{z^3 e^{imz}}{4z^3} = \lim_{z\to\alpha} \frac{e^{imz}}{4}.$$

Thus, $\qquad$ Res $(z = ae^{i\pi/4})$ + Res $(z = a\, e^{iz\pi/4})$

$$= \frac{1}{4} [\exp (ima\, e^{i\pi/4}) + \exp (ima\, e^{iz\pi/4})]$$

$$= \frac{1}{4}\left[\exp\left\{ima\left(\frac{i+1}{\sqrt{2}}\right)\right\} + \exp\left\{ima\left(\frac{-1+i}{\sqrt{2}}\right)\right\}\right]$$

$$= \frac{1}{4} \exp\left(\frac{-ma}{\sqrt{2}}\right)\left[\exp\left(\frac{ima}{\sqrt{2}}\right) + \exp\left(\frac{-ima}{\sqrt{2}}\right)\right]$$

$$= \frac{1}{2}\exp\left(\frac{-ma}{\sqrt{2}}\right)\cos\left(\frac{ma}{\sqrt{2}}\right)$$

Hence by Cauchy's residue theorem,

$$\int_C f(z)\, dz = \int_T f(z)\, dz + \int_{-R}^{R} f(x)\, dx = \pi i \exp\left(\frac{-ma}{\sqrt{2}}\right)\cos\left(\frac{ma}{\sqrt{2}}\right)$$

Taking limit as R→∞ and using (1), we get

$$\int_{-\infty}^{\infty} \frac{x^3 e^{imx}}{x^4 + a^4} dx = \pi i \exp\left(\frac{-ma}{\sqrt{2}}\right)\cos\left(\frac{ma}{\sqrt{2}}\right)$$

Equating imaginary parts, we obtain

$$\int_{-\infty}^{\infty} \frac{x^3 \sin mx}{x^4 + a^4} dx = \pi \exp\left(\frac{-ma}{\sqrt{2}}\right)\cos\left(\frac{ma}{\sqrt{2}}\right)$$

or $$\int_0^\infty \frac{x^3 \sin mx}{x^4 + a^4}\,dx = \frac{\pi}{2} \exp\left(\frac{-ma}{\sqrt{2}}\right)\cos\left(\frac{ma}{\sqrt{2}}\right)$$

8. Multivalued Function and its Branches

The familiar fact that sin θ and cos θ are periodic functions with period 2π, is responsible for the non-uniqueness of θ in the representation $z = |z|\, e^{i\theta}$ i.e. $z = re^{i\theta}$. Here, we shall discuss non-uniqueness problems with reference to the function arg z, log z and z^a. We know that a function w = *f*(z) is multivalued when for given z, we may find more than one value of w. Thus, a function *f*(z) is said to be single-valued if it satisfies

$$f(z) = f(z(r, \theta)) = f(z(r, \theta + 2\pi))$$

otherwise it is classified as multivalued function.

For analytic properties of a multivalued function, we consider domains in which these functions are single valued. This leads to the concept of branches of such functions. Before discussing branches of a many valued function, we give a brief account of the three functions arg z, log z and z^a.

8.1. Argument Function. For each $z \in \mathbb{C}$, $z \neq 0$, we define the argument of z to be

$$\arg z = [\arg z] = \{\theta \in R : z = |z|e^{i\theta}\}$$

the square bracket notation emphasizes that arg z is a set of numbers and not a single number. i.e. [arg z] is multivalued. Infact, it is an infinite set of the form $\{\theta + 2n\pi : n \in I\}$, where θ is any fixed number such that $e^{i\theta} = \dfrac{z}{|z|}$.

For example, $\arg i = \{(4n+1)\,\pi/2 : n \in I\}$

Also, $\arg\left(\dfrac{1}{z}\right) = \{-\theta : \theta \in \arg z\}$

Thus, for $z_1, z_2 \neq 0$, we have

$$\arg (z_1 z_2) = \{\theta_1 + \theta_2 : \theta_1 \in \arg z_1, \theta_2 \in \arg z_2\}$$
$$= \arg z_1 + \arg z_2$$

and $$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

For principal value determination, we can use Arg z = θ, where $z = |z|\, e^{i\theta}$, $-\pi < \theta \leq \pi$ (or $0 \leq \theta < 2\pi$). When z performs a complete anticlockwise circuit round the unit circle, θ increases by 2π and a jump discontinuity in Arg z is inevitable. Thus, we cannot impose a restriction which determines θ uniquely and therefore for general purpose, we use more complicated notation arg z or [arg z] which allows z to move freely about the origin with θ varying continuously. We observe that

$$\arg z = [\arg z] = \text{Arg } z + 2n\pi,\ n \in I.$$

8.2. Logarithmic Function. We observe that the exponential function e^z is a periodic function with a purely imaginary period of 2πi, since

$$e^{z+2\pi i} = e^z.\, e^{2\pi i} = e^z,\ e^{2\pi i} = 1.$$

i.e. exp (z + 2πi) = exp z for all z.

If w is any given non-zero point in the w-plane then there is an infinite number of points in the z-plane such that the equation

$$w = e^z \qquad (1)$$

is satisfied. For this, we note that when z and w are written as z = x + iy and $w = \rho\, e^{i\phi}$ $(-\pi < \phi \leq \pi)$, equation (1) can be put as

$$e^z = e^{x+iy} = e^x e^{iy} = \rho\, e^{i\phi} \qquad (2)$$

From here, $e^x = \rho$ and $y = \phi + 2n\pi$, $n \in I$.

Since the equation $e^x = \rho$ is the same as $x = \log_e \rho = \log \rho$ (base e understood), it follows that when $w = \rho\, e^{i\phi}(-\pi < \phi \le \pi)$, equation (1) is satisfied if and only if z has one of the values

$$z = \log \rho + i\,(\phi + 2n\pi),\ n \in I \qquad (3)$$

Thus, if we write

$$\log w = \log \rho + i\,(\phi + 2n\pi),\ n\in I \qquad (4)$$

we see that exp (log w) = w, this motivates the following definition of the (multivalued) logarithmic function of a complex variable.

The logarithmic function is defined at non-zero points $z = re^{i\theta}$ $(-\pi < \theta \le \pi)$ in the z-plane as

$$\log z = \log r + i\,(\theta + 2n\pi),\ n \in I \qquad (5)$$

The principal value of log z is the value obtained from (5) when n = 0 and is denoted by Log z. Thus

$$\text{Log } z = \log r + i\theta \text{ i.e. Log } z = \log |z| + i \text{ Arg } z \qquad (6)$$

Also, from (5) & (6), we note that

$$\log z = \text{Log } z + 2n\pi i,\ n \in I \qquad (7)$$

The function Log z is evidently well defined and single-valued when $z \ne 0$.
Equation (5) can also be put as

$$\log z = \{\log |z| + i\theta : \theta\in \arg z\}$$

or $$[\log z] = \{\log |z| + i\theta : \theta\in [\arg z]\} \qquad (8)$$

or $$\log z = \log |z| + i\theta = \log |z| + i \arg z \qquad (9)$$

where $\theta = \theta + 2n\pi$, $\theta = \text{Arg } z$.

From (8), we find that

$$\log 1 = \{2n\pi i,\ n \in I\},\ \log (-1) = \{(2n+1)\ \pi i,\ n\in I\}$$

In particular, Log 1 = 0, Log (–1) = πi . Similarly πlog, log i = {(un+1) πi/2, n∈I}, log (–i) = {un–1) πi/2, n∈I} In particular, Log i = πi/2, Log (–i) = –πi/2.

Thus, we conclude that complex logarithm is not a bona fide function, but a multifunction. We have assigned to each $z \ne 0$ infinitely many values of the logarithm.

8.3. Complex Exponents. When $z \ne 0$ and the exponent a is any complex number, the function z^a is defined by the equation.

$$w = z^a = e^{\log z^a} = e^{a \log z} = \exp (\,a \log z) \qquad (1)$$

where log z denotes the multivalued logarithmic function. Equation (1) can also be expressed as

$$w = z^a = \{e^{a(\log |z| + i\theta)} : \theta\in \arg z\}$$

or $$[z^a] = \{e^{a(\log |z| + i\theta)} : \theta\in [\arg z]\}$$

Thus the manivalued nature of the function log z will generally result in the many-valuedness of z^a. Only when a is an integer, z^a does not produce multiple values. In this case z^a contains a single point z^n. When $a = \dfrac{1}{n}$ (n = 2, 3,…), then

$$w = z^{1/n} = (r\, e^{i\theta})^{1/n} = r^{1/n}\, e^{i(\theta + 2m\pi)/n},\ m\in I$$

We note that in particular, the complex nth roots of $\pm$ 1 are obtained as

$$w^n = 1 \quad \Rightarrow w = e^{2m\pi i/n},\ w^n = -1 \quad \Rightarrow w = e^{(2m+1)\pi i/n},\ m = 0,1,\ldots, n-1.$$

For example, $i^{-2i} = \exp(-2\, i \log i) = \exp[-2i\,(4n+1)\ \pi i/2]$

$= \exp [(4n+1) \pi], n \in I$

In should be observed that the formula

$x^a x^b = x^{a+b}, x, a, b, \in R$

can be shown to have a complex analogue (in which values of the multi-functions involved have to be appropriately selected) but the formula

$x_1^2 x_2^a = (x_1 x_2)^a, x_1, x_2, a \in R$

has no universally complex generalization.

8.4. Branches, Branch Points and Branch Cuts. We recall that a multifunction w defined on a set $S \subseteq \forall$ is an assignment to each $z \in S$ of a set [w(z)] of complex numbers. Our main aim is that given a multifunction w defined on S, can we select, for each $z \in S$, a point $f(z)$ in [w(z)] so that $f(z)$ is analytic in an open subset G of S, where G is to be chosen as large as possible? If we are to do this, then $f(z)$ must vary continuously with z in G, since an analytic function is necessarily continuous.

Suppose w is defined in some punctured disc D having centre a and radius R i.e. $0 < |z - a| < R$ and that $f(z) \in [w(z)]$ is chosen so that $f(z)$ is at least continuous on the circle γ with centre a and radius r $(0 < r < R)$. As z traces out the circle γ starting from, say z_0, $f(z)$ varies continuously, but must be restored to its original value $f(z_0)$ when z completes its circuit, since $f(z)$ is, by hypothesis, single valued. Notice also that if $z - a = r\, e^{i\theta(z)}$, where $\theta(z)$ is chosen to vary continuously with z, then $\theta(z)$ increases by 2π as z performs its circuit, so that $\theta(z)$ is not restored to its original value. The same phenomenon does not occur if z moves round a circle in the punctured disc D not containing a, in this case $\theta(z)$ does return to its original value. More generally, our discussion suggests that if we are to extract an analytic function from a multi-function w, we shall meet to restrict to a set in which it is impossible to encircle, one at a time, points a such that the definition of [w(z)] involves the argument of $(z-a)$. In some cases, encircling several of these 'bad' points simultaneously may be allowable.

A branch of a multiple-valued function $f(z)$ defined on $S \subseteq \not\subset$ is any single-valued function F(z) which is analytic in some domain $D \subset S$ at each point of which the value F(z) is one of the values of $f(z)$. The requirement of analyticity, of course, prevents F(z) from taking on a random selection of the values of $f(z)$.

A branch cut is a portion of a line or curve that is introduced in order to define a branch F(z) of a multiple-valued function $f(z)$.

A multivalued function $f(z)$ defined on $S \subseteq \forall$ is said to have a branch point at z_0 when z describes an arbitrary small circle about z_0, then for every branch F(z) of $f(z)$, F(z) does not return to its original value. Points on the branch cut for F(z) are singular points of F(z) and any point that is common to all branch cuts of $f(z)$ is called a branch point. For example, let us consider the logarithmic function

$$\log z = \log r + i\theta = \log |z| + i \arg z \qquad (1)$$

If we let α denote any real number and restrict the values of θ in (1) to the interval $\alpha < \theta < \alpha + 2\pi$, then the function

$$\log z = \log r + i\theta \ (r > 0, \alpha < \theta < \alpha + 2\pi) \qquad (2)$$

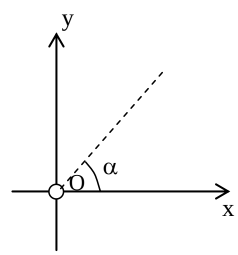
with component functions

$$u(r, \theta) = \log r \text{ and } v(r, \theta) = \theta \qquad (3)$$

is single-valued, continuous and analytic function. Thus for each fixed α, the function (2) is a branch of the function (1). We note that if the function (2) were to be defined on the ray $\theta = \alpha$, it would not be continuous there. For, if z is any point on that ray, there are points arbitrarily close to z at which the values of v are near to α and also points such that the values of v are near to $\alpha + 2\pi$. The origin and the ray $\theta = \alpha$ make up the branch cut for the branch (2) of the logarithmic function. The function

$$\text{Log } z = \log r + i\theta \ (r > 0, -\pi < \theta < \pi) \tag{4}$$

is called the principal branch of the logarithmic function in which the branch cut consists of the origin and the ray $\theta = \pi$. The origin is evidently a branch point of the logarithmic function.



For analyticity of (2), we observe that the first order partial derivatives of u and v are continuous and satisfy the polar form

$$u_r = \frac{1}{r} v_\theta, \qquad v_r = -\frac{1}{r} u_\theta$$

of the C−R equations. Further

$$\frac{d}{dz}(\log z) = e^{-i\theta}(u_r + iv_r)$$

$$= e^{-i\theta}\left(\frac{1}{r} + i\,0\right) = \frac{1}{re^{i\theta}}$$

Thus $\frac{d}{dz}(\log z) = \frac{1}{z}$ ($|z| = r > 0$, $\alpha < \arg z < \alpha + 2\pi$)

In particular

$$\frac{d}{dz}(\log z) = \frac{1}{z} \ (|z| > 0, -\pi < \text{Arg } z < \pi).$$

Further, since $\log \frac{1}{z} = -\log z$, ∞ is also a branch point of log z. Thus a cut along any half-line from 0 to ∞ will serve as a branch cut.

Now, let us consider the function $w = z^a$ in which a is an arbitrary complex number. We can write

$$w = z^a = e^{\log z^a} = e^{a \log z} \quad (5)$$

where many-valued nature of log z results is many-valuedness of z^a. If Log z denotes a definite branch, say the principal value of log z, then the various values of z^a will be of the form

$$z^a = e^{a(\text{Log } z + 2n\pi i)} = e^{a \text{ Log } z}\, e^{2\pi i a n} \quad (6)$$

where $\log z = \text{Log } z + 2n\pi i$, $n \in I$.

The function in (6) has infinitely many different values. But the number of different values of z^a will be finite in the cases in which only a finite number of the values $e^{2\pi\, ian}$, $n \in I$, are different from one another. In such a case, there must exist two integers m and m′ (m′ = m) such that $e^{2\pi\, iam} = e^{2\pi\, iam'}$

or $e^{2\pi\, ia(m - m')} = 1$. Since $e^z = 1$ only if $z = 2\pi in$, thus we get $a(m - m') = n$ and therefore it follows that a is a rational number. Thus z^a has a finite set of values iff a is a rational number. If a is not rational, z^a has infinity of values.

We have observed that if $z = re^{i\theta}$ and α is any real number, then the branch

$$\log z = \log r + i\theta \; (r > 0, \alpha < \theta < \alpha + 2\pi) \quad (7)$$

of the logarithmic function is single-valued and analytic in the indicated domain. When this branch is used, it follows that the function (5) is single valued and analytic in the said domain. The derivative of such a branch is obtained as

$$\frac{d}{dz}(z^a) = \frac{d}{dz}[\exp(a \log z)] = \exp(a \log z)\, \frac{a}{z}$$

$$= a\, \frac{\exp(a \log z)}{\exp(\log z)} = a \exp[(a - 1) \log z]$$

$$= az^{a-1}.$$

As a particular case, we consider the multivalued function $f(z) = z^{1/2}$ and we define

$$z^{1/2} = \sqrt{r}\, e^{i\theta/2},\; r > 0,\; \alpha < \theta < \alpha + 2\pi \quad (8)$$

where the component functions

$$u(r, \theta) = \sqrt{r} \cos \theta/2,\; v(r, \theta) = \sqrt{r} \sin \theta/2 \quad (9)$$

are single valued and continuous in the indicated domain. The function is not continuous on the line $\theta = \alpha$ as there are points arbitrarily close to z at which the values of $v(r, \theta)$ are nearer to $\sqrt{r}\, \sin \alpha/2$ and also points such that the values of $v(r, \theta)$ are nearer to $-\sqrt{r}\, \sin \alpha/2$. The function (8) is differentiable as C-R equations in polar form are satisfied by the functions in (9) and

$$\frac{d}{dz}(z^{1/2}) = e^{-i\theta}(u_r + iv_r) = e^{i\theta}\left(\frac{1}{2\sqrt{r}}\cos\theta/2 + i\frac{1}{2\sqrt{r}}\sin\theta/2\right)$$

$$= \frac{1}{2\sqrt{r}} e^{-i\theta/2} = \frac{1}{2z^{1/2}}$$

Thus (8) is a branch of the function $f(z) = z^{1/2}$ where the origin and the line $\theta = \alpha$ form branch cut. When moving from any point $z = re^{i\theta}$ about the origin, one complete circuit to reach again, at z, we have changed arg z by 2π. For original position $z = re^{i\theta}$, we have $w = \sqrt{r}\, e^{i\theta/2}$, and after one complete circuit, $w = \sqrt{r}\; e^{i(\theta+2\pi)/2} = -\sqrt{r}\, e^{i\theta/2}$. Thus w has not returned to its original value and hence change in branch has occurred. Since a complete circuit about z = 0 changed the branch of the function, z = 0 is a branch point for the function $z^{1/2}$.

UNIT – III

1. Transformations

Here, we shall study how various curves and regions are mapped by elementary analytic function. We shall work in $\mathbb{C}_\infty$ i.e. the extended complex plane. We start with the linear function.

$$w = Az \qquad \ldots(1)$$

where A is non zero complex constant and $z \neq 0$. We write A and z in exponential form as

$$A = ae^{i\alpha}, \qquad z = re^{i\theta}$$

Then
$$w = (ar)\, e^{i(\alpha + \theta)} \qquad \ldots(2)$$

Thus we observe from (2) that transformation (1) expands (or contracts) the radius vector representing z by the factor $a = |A|$ and rotates it through an angle $\alpha = \arg A$ about the origin. The image of a given region is, therefore, geometrically similar to that region. The general linear transformation

$$w = Az + B \qquad \ldots(3)$$

is evidently an expansion or contraction and a rotation, followed by a translation. The image region mapped by (3) is geometrically congruent to the original one.

Now we consider the function

$$w = \frac{1}{z} \qquad \ldots(4)$$

which establishes a one to one correspondence between the non zero points of the z-plane and the w-plane. Since $z\bar{z} = |z|^2$, the mapping can be described by means of the successive transformations

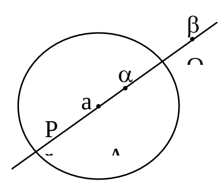
$$Z = \frac{1}{|z|^2} z, \qquad w = \bar{Z} \qquad \ldots(5)$$

Geometrically, we know that if P and Q are inverse points w.r.t. a circle of radius r with centre A, then

$$(AP)\,(AQ) = r^2$$

Thus α and β are inverse points w.r.t. the circle $|z - a| = r$ if

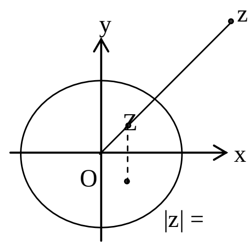
$$(\alpha - a)\,\overline{(\beta - a)} = r^2$$



where the pair $\alpha = a$, $\beta = \infty$ is also included. We note that α, β, a are collinear. Also points α and β are inverse w.r.t a straight line *l* if β is the reflection of α in *l* and conversely. Thus the first of the transformation in (5) is an inversion w.r.t the unit circle $|z| = 1$ i.e. the image of a non zero point z is the point Z with the properties

$$z\bar{Z} = 1, \qquad |Z| = \frac{1}{|z|} \quad \text{and } \arg Z = \arg z$$

Thus the point exterior to the circle $| z | = 1$ are mapped onto the non zero points interior to it and conversely. Any point on the circle is mapped onto itself. The second of the transformation in (5) is simply a reflection in the real axis.



Since $\lim_{z \to 0} \frac{1}{z} = \infty$ and $\lim_{z \to \infty} \frac{1}{z} = 0$,

it is natural to define a one-one transformation $w = T(z)$ from the extended z plane onto the extended w plane by writing

$$T(0) = \infty, \quad T(\infty) = 0$$

and $T(z) = \frac{1}{z}$

for the remaining values of z. It is observed that T is continuous throughout the extended z plane.

When a point $w = u + iv$ is the image of a non zero point $z = x + iy$ under the transformation $w = \frac{1}{z}$, writing $w = \frac{\bar{z}}{| z |^2}$ results in

$$u = \frac{x}{x^2 + y^2} \quad , \quad v = \frac{-y}{x^2 + y^2} \qquad \ldots(6)$$

Also, since $z = \frac{1}{w} = \frac{\bar{w}}{| w |^2}$, we get

$$x = \frac{u}{u^2 + v^2}, \quad y = \frac{-v}{u^2 + v^2} \qquad \ldots(7)$$

The following argument, based on these relations (6) and (7) between co-ordinates shows the important result that the mapping $w = \frac{1}{z}$ transforms circles and lines into circles and lines.

When a, b, c, d are real numbers satisfying the condition $b^2 + c^2 > 4ad$, then the equation

$$a(x^2 + y^2) + bx + cy + d = 0 \qquad \ldots(8)$$

represents an arbitrary circle or line, where $a \neq 0$ for a circle and $a = 0$ for a line.

If x and y satisfy equation (8), we can use relations (7) to substitute for these variables. Thus using (7) in (8), we obtain

$$d(u^2 + v^2) + bu - cv + a = 0 \qquad \ldots(9)$$

which also represents a circle or a line. Conversely, if u and v satisfy (9), it follows from (6) that x and y satisfy (8). From (8) and (9), it is clear that

(i) a circle ($a \neq 0$) not passing through the origin ($d \neq 0$) in the z plane is transformed into a circle not passing through the origin in the w plane.

(ii) a circle ($a \neq 0$) through the origin ($d = 0$) in the z plane is transformed into a line which does not pass through the origin in the w plane.
(iii) a line ($a = 0$) not passing through the origin ($d \neq 0$) in the z plane is transformed into a circle through the origin in the w plane.
(iv) a line ($a = 0$) through the origin ($d = 0$) in the z plane is transformed into a line through the origin in the w plane.

Hence we conclude that $w = \frac{1}{z}$ transforms circles and lines into circles and lines.

Remark : In the extended complex plane, a line may be treated as a circle with infinite radius.

1.1. Bilinear Transformation. The transformation

$$w = \frac{az+b}{cz+d},\ ad - bc \neq 0 \qquad \ldots(1)$$

where a, b, c, d are complex constants, is called bilinear transformation or a linear fractional transformation or Möbius transformation. We observe that the condition $ad - bc \neq 0$ is necessary for (1) to be a bilinear transformation, since if

$$ad - bc = 0, \text{ then } \frac{b}{a} = \frac{d}{c} \text{ and we get}$$

$$w = \frac{a(z+b/a)}{c(z+d/c)} = \frac{a}{c}$$ i.e. we get a constant function which is not linear.

Equation (1) can be written in the form

$$cwz + dw - az - b = 0 \qquad \ldots(2)$$

Since (2) is linear in z and linear in w or bilinear in z and w, therefore (1) is termed as bilinear transformation.

When $c = 0$, the condition $ad - bc \neq 0$ becomes $ad \neq 0$ and we see that the transformation reduces to general linear transformation. When $c \neq 0$, equation (1) can be written as

$$w = \frac{a}{c}\frac{(z+b/a)}{(z+d/c)} = \frac{a}{c}\left[1+\frac{b/a-d/c}{z+d/c}\right]$$

$$= \frac{a}{c} + \frac{bc-ad}{c^2} = \frac{1}{z+d/c} \qquad \ldots(3)$$

We note that (3) is a composition of the mappings

$$z_1 = z + \frac{d}{c}, \qquad z_2 = \frac{1}{z_1}, \qquad z_3 = \frac{bc-ad}{c^2} z_2$$

and thus we get $\qquad w = \frac{a}{c} + z_3.$

The above three auxiliary transformations are of the form

$$w = z + \alpha, \qquad w = \frac{1}{z}, \qquad w = \beta z \qquad \ldots(4)$$

Hence every bilinear transformation is the resultant of the transformations in (4).

But we have already discussed these transformations and thus we conclude that a bilinear transformation always transforms circles and lines into circles and lines because the transformations in (4) do so.

From (1), we observe that if c = 0, a, d $\neq$ 0, each point in the w plane is the image of one and only one point in the z-plane. The same is true if c $\neq$ 0, except when $z = -\frac{d}{c}$ which makes the denominator zero. Since we work in extended complex plane, so in case $z = -\frac{d}{c}$, $w = \infty$ and thus we may regard the point at infinity in the w-plane as corresponding to the point $z = -\frac{d}{c}$ in the z-plane.

Thus if we write

$$T(z) = w = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0 \qquad \ldots(5)$$

Then $\quad T(\infty) = \infty, \quad \text{if } c = 0$

and $\quad T(\infty) = \frac{a}{c}, \quad T\left(-\frac{d}{c}\right) = \infty, \quad \text{if } c \neq 0$

Thus T is continuos on the extended z-plane. When the domain of definition is enlarged in this way, the bilinear transformation (5) is one-one mapping of the extended z-plane onto the extended w-plane.

Hence, associated with the transformation T, there is an inverse transformation T^{-1} which is defined on the extended w-plane as

$$T^{-1}(w) = z \text{ if and only if } T(z) = w.$$

Thus, when we solve equation (1) for z, then

$$z = \frac{-dw+b}{cw-a}, \; ad - bc \neq 0 \qquad \ldots(6)$$

and thus

$$T^{-1}(w) = z = \frac{-dw+b}{cw-a}, \; ad - bc \neq 0$$

Evidently T^{-1} is itself a bilinear transformation, where

$$T^{-1}(\infty) = \infty \quad \text{if } c = 0$$

and $\quad T^{-1}\left(\frac{a}{c}\right) = \infty, \quad T^{-1}(\infty) = -\frac{d}{c}, \text{ if } c \neq 0$

From the above discussion, we conclude that inverse of a bilinear transformation is bilinear. The points $z = -\frac{d}{c}$ $(w = \infty)$ and $z = \infty$$(w = \frac{a}{c})$ are called critical points.

1.2. Theorem. Composition (or resultant or product) of two bilinear transformations is a bilinear transformation.

Proof. We consider the bilinear transformations

$$w = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0 \qquad \ldots(1)$$

and $$w_1 = \frac{a_1w+b_1}{c_1w+d_1}, \quad a_1d_1 - b_1c_1 \neq 0 \qquad \ldots(2)$$

Putting the value of w from (1) in (2), we get

$$w_1 = \frac{a_1\left(\frac{az+b}{cz+d}\right)+b_1}{c_1\left(\frac{az+b}{cz+d}\right)+d_1} = \frac{(a_1a+b_1c)z+(b_1d+a_1b)}{(c_1a+d_1c)z+(d_1d+c_1b)}$$

Taking $A = a_1a + b_1c, \quad B = b_1d + a_1b,$

$C = c_1a + d_1c, \quad D = d_1d + c_1b,$ we get

$$w_1 = \frac{Az+B}{Cz+D}$$

Also

$$\begin{aligned} AD - BC &= (a_1a + b_1c)(d_1d + c_1b) - (b_1d + a_1b)(c_1a + d_1c) \\ &= (a_1ad_1d + a_1ac_1b + b_1cd_1d + b_1cc_1b) \\ &\qquad - (b_1dc_1a + b_1d\,d_1c + a_1bc_1a + a_1bd_1c) \\ &= a_1ad_1d + b_1bc_1c - b_1dc_1a - a_1bd_1c \\ &= ad(a_1d_1 - b_1c_1) - bc(a_1d_1 - b_1c_1) \\ &= (ad - bc)(a_1d_1 - b_1c_1) \neq 0 \end{aligned}$$

Thus $w_1 = \dfrac{Az+B}{Cz+D}, \quad AD - BC \neq 0$

Is a bilinear transformation.

This bilinear transformation is called the resultant (or product or composition) of the bilinear transformations (1) and (2).

The above property is also expressed by saying that bilinear transformations form a group.

1.3. Definitions. (i) The points which coincide with their transforms under bilinear transformation are called its fixed points. For the bilinear transformation $w = \dfrac{az+b}{cz+d}$, fixed points are given by $w = z$ i.e. $z = \dfrac{az+b}{cz+d}$ …(1)

Since (1) is a quadratic in z and has in general two different roots, therefore there are generally two invariant points for a bilinear transformation.

(ii) If z_1, z_2, z_3, z_4 are any distinct points in the z-plane, then the ratio

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$

is called cross ratio of the four points z_1, z_2, z_3, z_4. This ratio is invariant under a bilinear transformation i.e.

$$(w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$$

1.4. Transformation of a Circle. First we show that if p and q are two given points and K is a constant, then the equation

$$\left|\frac{z-p}{z-q}\right| = K, \tag{1}$$

represents a circle. For this, we have

$$|z-p|^2 = K^2|z-q|^2$$

$\Rightarrow \quad (z-p)\,\overline{(z-p)} = K^2(z-q)\,\overline{(z-q)}$

$\Rightarrow \quad (z-p)(\bar{z}-\bar{p}) = K^2(z-q)(\bar{z}-\bar{q})$

$\Rightarrow \quad z\bar{z} - \bar{p}z - p\bar{z} + p\bar{p} = K^2(z\bar{z} - \bar{q}z - q\bar{z} + q\bar{q})$

$\Rightarrow \quad (1 - K^2) z\bar{z} - (p - qK^2)\bar{z} - (\bar{p} - \bar{q}K^2)z = K^2 q\bar{q} - p\bar{p}$

$$\Rightarrow \quad z\bar{z} - \left(\frac{p - qK^2}{1-K^2}\right)\bar{z} - \left(\frac{\bar{p} - \bar{q}K^2}{1-K^2}\right)z + \frac{|p|^2 - K^2|q|^2}{1-K^2} = 0 \qquad \ldots(2)$$

Equation (2) is of the form

$$z\bar{z} + b\bar{z} + \bar{b}z + c = 0 \qquad \text{(c is being a real constant)}$$

which always represents a circle.
Thus equation (2) represents a circle if $K \neq 1$.
If $K = 1$, then it represents a straight line

$$|z - p| = |z - q|$$

Further, we observe that in the form (1), p and q are inverse points w.r.t. the circle. For this, if the circle is $|z - z_0| = \rho$ and p and q are inverse points w.r.t. it, then

$$z - z_0 = \rho e^{i\theta}, \; p - z_0 = qe^{i\lambda},$$

$$q - z_0 = \frac{\rho^2}{a} e^{i\lambda}$$

Therefore,

$$\left|\frac{z-p}{z-q}\right| = \left|\frac{\rho e^{i\theta} - ae^{i\lambda}}{\rho e^{i\theta} - \dfrac{\rho^2}{a} e^{i\lambda}}\right| = \frac{a}{\rho}\left|\frac{\rho e^{i\theta} - ae^{i\lambda}}{ae^{i\theta} - \rho e^{i\lambda}}\right|$$

$$= K \left|\frac{\rho(\cos\theta + i\sin\theta) - a(\cos\lambda + i\sin\lambda)}{a(\cos\theta + i\sin\theta) - \rho(\cos\lambda + i\sin\lambda)}\right|, \; K = \frac{a}{\rho}$$

$$= K\left|\frac{(\rho\cos\theta - a\cos\lambda) + i(\rho\sin\theta - a\sin\lambda)}{(a\cos\theta - \rho\cos\lambda) + i(a\sin\theta - \rho\sin\lambda)}\right|$$

$$= K\left[\frac{(\rho\cos\theta - a\cos\lambda)^2 + (\rho\sin\theta - a\sin\lambda)^2}{(a\cos\theta - \rho\cos\lambda)^2 + (a\sin\theta - \rho\sin\lambda)^2}\right]^{1/2}$$

$$= K, \text{ where } K \neq 1, \text{ since } a \neq \rho$$

Thus, if p and q are inverse points w.r.t. a circle, then its equation can be written as

$$\left|\frac{z-p}{z-q}\right| = K, \quad K \neq 1, \text{ K being a real constant.}$$

1.5 Theorem. In a bilinear transformation, a circle transforms into a circle and inverse points transform into inverse points. In the particular case in which the circle becomes a straight line, inverse points become points symmetric about the line.

Proof : We know that $\left|\frac{z-p}{z-q}\right| = K$ represents a circle in the z-plane with p and q as inverse points, where $K \neq 1$. Let the bilinear transformation be

$$w = \frac{az+b}{cz+d} \qquad \text{so that} \qquad z = \frac{dw-b}{-cw+a}$$

Then under this bilinear transformation, the circle transforms into

$$\left|\frac{\frac{dw-b}{-cw+a}-p}{\frac{dw-b}{-cw+a}-q}\right| = K \quad \Rightarrow \quad \left|\frac{dw-b-p(q-cw)}{dw-b-q(a-cw)}\right| = K$$

$$\Rightarrow \quad \left|\frac{w(d+cp)-(ap+b)}{w(d+cq)-(aq+b)}\right| = K \quad \Rightarrow \quad \left|\frac{w-\frac{ap+b}{cp+d}}{w-\frac{aq+b}{cq+d}}\right| = K\left|\frac{cq+d}{cp+d}\right| \qquad \ldots(1)$$

The form of equation (1) shows that it represents a circle in the w-plane whose inverse points are $\frac{ap+b}{cp+d}$ and $\frac{aq+b}{cq+d}$. Thus, a circle in the z-plane transforms into a circle in the w-plane and the inverse points transform into the inverse points.

Also if $K\left|\frac{cq+d}{cp+d}\right| = 1$, then equation (1) represents a straight line bisecting at right angle the join of the points $\frac{ap+b}{cp+d}$ and $\frac{aq+b}{cq+d}$ so that these points are symmetric about this line. Thus in a particular case, a circle in the z-plane transforms into a straight line in the w-plane and the inverse points transform into points symmetrical about the line.

1.6. Example. Find all bilinear transformations of the half plane Im $z \geq 0$ into the unit circle $|w| \leq 1$.

Solution. We know that two points z, $\bar{z}$, symmetrical about the real z-axis(Im z = 0) correspond to points w, $\frac{1}{\bar{w}}$, inverse w.r.t. the unit w-circle. ($|w|\ \frac{1}{|\bar{w}|} = 1$). In particular, the origin and the point at infinity in the w-plane correspond to conjugate values of z.

Comment [a1]:

Let

$$w = \frac{az+b}{cz+d} = \frac{a}{c}\frac{(z+b/a)}{(z+d/c)} \qquad \ldots(1)$$

be the required transformation.

Clearly $c \neq 0$, otherwise points at ∞ in the two planes would correspond.

Also, $w = 0$ and $w = \infty$ are the inverse points w.r.t. $|w| = 1$. Since in (1), $w = 0$, $w = \infty$ correspond respectively to $z = -\frac{b}{a}$, $z = -\frac{d}{c}$, therefore these two values of z-plane must be conjugate to each other. Hence we may write

$$-\frac{b}{a} = \alpha,\ -\frac{d}{c} = \bar{\alpha} \text{ so that}$$

$$w = \frac{a}{c}\frac{z-\alpha}{z-\bar{\alpha}} \qquad (2)$$

The point z = 0 on the boundary of the half plane Im $z \geq 0$ must correspond to a point on the boundary of the circle $|w| = 1$, so that

$$1 = |w| = \left|\frac{a}{c}\right|\left|\frac{0-\alpha}{0-\bar{\alpha}}\right| = \left|\frac{a}{c}\right|$$

$\Rightarrow \qquad \dfrac{a}{c} = e^{i\lambda} \quad \Rightarrow \quad a = ce^{i\lambda}$, where λ is real.

Thus, we get

$$w = e^{i\lambda}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right) \qquad \ldots(3)$$

Since $z = \alpha$ gives $w = 0$, α must be a point of the upper half plane i.e. $\text{Im}\,\alpha > 0$. With this condition, (3) gives the required transformation. In (3), if z is real, obviously $|w| = 1$ and if $\text{Im}\,z > 0$, then z is nearer to α than to $\bar{\alpha}$ and so $|w| < 1$. Hence the general linear transformation of the half plane $\text{Im}\,z \geq 0$ on the circle $|w| \leq 1$ is

$$w = e^{i\lambda}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right),\ \text{Im}\,\alpha > 0.$$

1.7. Example. Find all bilinear transformations of the unit $|z| \leq 1$ into the unit circle $|w| \leq 1$.

OR

Find the general homographic transformations which leaves the unit circle invariant.

Solution. Let the required transformation be

$$w = \frac{az+b}{cz+d} = \frac{a}{c}\frac{(z+b/a)}{(z+d/c)} \qquad \ldots(1)$$

Here, $w = 0$ and $w = \infty$, correspond to inverse points

$$z = -\frac{b}{a},\ z = -\frac{d}{c}, \qquad \text{so we may write}$$

$$-\frac{b}{a} = \alpha, \qquad -\frac{d}{c} = \frac{1}{\bar{\alpha}} \qquad \text{such that } |\alpha| < 1.$$

So,
$$w = \frac{a}{c}\left(\frac{z-\alpha}{z-1/\bar{\alpha}}\right) = \frac{a\bar{\alpha}}{c}\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right) \qquad \ldots(2)$$

The point $z = 1$ on the boundary of the unit circle in z-plane must correspond to a point on the boundary of the unit circle in w-plane so that

$$1 = |w| = \left|\frac{a\bar{\alpha}}{c}\frac{1-\alpha}{\bar{\alpha}-1}\right| = \left|\frac{a\bar{\alpha}}{c}\right|$$

or $a\,\bar{\alpha} = c\,e^{i\lambda}$, where λ is real.

Hence (2) becomes,

$$w = e^{i\lambda}\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right), \qquad |\alpha| < 1 \qquad \ldots(3)$$

This is the required transformation, for if $z = e^{i\theta}$, $\alpha = be^{i\beta}$, then

$$|w| = \left|\frac{e^{i\theta} - be^{i\beta}}{be^{i(\theta-\beta)} - 1}\right| = 1.$$

If $z = re^{i\theta}$, where $r < 1$, then

$$|z-\alpha|^2 - |\bar{\alpha}z - 1|^2$$
$$= r^2 - 2rb\cos(\theta-\beta) + b^2 - \{b^2r^2 - 2br\cos(\theta-\beta) + 1\}$$
$$= (r^2-1)(1-b^2) < 0$$

and so

$$|z-\alpha|^2 < |\bar{\alpha}z - 1|^2 \quad \Rightarrow \quad \frac{|z-\alpha|^2}{|\bar{\alpha}z-1|^2} < 1$$

i.e. $\qquad |w| < 1$

Hence the result.

1.8. Example. Show that the general transformation of the circle $|z| \le \rho$ into the circle $|w| \le \rho'$ is

$$w = \rho\rho' e^{i\lambda}\left(\frac{z-\alpha}{\bar{\alpha}z-\rho^2}\right), \qquad |\alpha| < \rho.$$

Solution. Let the transformation be

$$w = \frac{az+b}{cz+d} = \frac{a}{c}\left(\frac{z+b/a}{z+d/c}\right) \qquad \ldots(1)$$

The points $w = 0$ and $w = \infty$, inverse points of $|w| = \rho'$ correspond to inverse point $z = -b/a$, $z = -d/c$ respectively of $|z| = \rho$, so we may write

$$-\frac{b}{a} = \alpha, \quad -\frac{d}{c} = \frac{\rho^2}{\bar{\alpha}}, \qquad |\alpha| < \rho$$

Thus, from (1), we get

$$w = \frac{a}{c}\left(\frac{z-\alpha}{z-\dfrac{\rho^2}{\bar{\alpha}}}\right) = \frac{a\bar{\alpha}}{c}\left(\frac{z-\alpha}{\bar{\alpha}z-\rho^2}\right) \qquad \ldots(2)$$

Equation (2) satisfied the condition $|z| \le \rho$ and $|w| \le \rho'$. Hence for $|z| = \rho$, we must have $|w| = \rho'$ so that (2) becomes

$$\rho' = |w| = \left|\frac{a\bar{\alpha}}{c}\right|\left|\frac{z-\alpha}{\bar{\alpha}z-z\bar{z}}\right|, \qquad z\bar{z} = \rho^2$$

$$= \left|\frac{a\bar{\alpha}}{c}\right|\left|\frac{1}{z}\right|\left|\frac{z-\alpha}{\bar{z}-\bar{\alpha}}\right| = \left|\frac{a\bar{\alpha}}{c}\right|\left|\frac{1}{z}\right|\left|\frac{z-\alpha}{\overline{z-\alpha}}\right|$$

$$= \left|\frac{a\bar{\alpha}}{c}\right|\frac{1}{\rho}, \qquad |z-\alpha| = |\overline{z-\alpha}|$$

$$\Rightarrow \qquad \rho\rho' = \left|\frac{a\bar{\alpha}}{c}\right| \qquad \Rightarrow \qquad \frac{a\bar{\alpha}}{c} = \rho\rho' e^{i\lambda}, \ \lambda \text{ being real.}$$

Thus, the required transformation becomes

$$w = \rho\rho' e^{i\lambda}\left(\frac{z-\alpha}{\bar{\alpha}z-\rho^2}\right), \quad |\alpha| < \rho.$$

1.9. Example. Find the bilinear transformation which maps the point 2, i, –2 onto the points 1, i, –1.

Solution. Under the concept of cross-ratio, the required transformation is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

Using the values of z_i and w_i , we get

$$\frac{(w-1)(i+1)}{(1-i)(-1-w)} = \frac{(z-2)(i+2)}{(2-i)(-2-z)}$$

or

$$\frac{w-1}{w+1} = \left(\frac{z-2}{z+2}\right)\left(\frac{2+i}{2-i}\right)\left(\frac{1-i}{1+i}\right)$$

or

$$\frac{w-1}{w+1} = \frac{4-3i}{5}\,\frac{z-2}{z+2}$$

or $$\frac{w-1+w+1}{w-1-(w+1)} = \frac{(4-3i)(z-2)+5(z+2)}{(4-3i)(z-2)-5(z+2)}$$

or $$-w = \frac{3z(3-i)+2i(3-i)}{-iz(z-i)-6(3-i)} = \frac{3z+2i}{-(iz+6)}$$

or $$w = \frac{3z+2i}{iz+6}$$

which is the required transformation.

1.10. Exercise

Find the bilinear transformation which maps

(i) 1, –i, 2 onto 0, 2, –i respectively.
(ii) 1, i, 0 onto 1, i, −1 respectively.
(iii) 0, 1, ∞ onto ∞, –i, 1 respectively.
(iv) –1, ∞, i into 0, ∞, 1 respectively.
(v) ∞, i, 0 onto 0, i, ∞ respectively.
(vi) 1, 0, −1 onto i, ∞, 1 respectively.
(vii) 1, i, –1 onto i, 0, –i respectively.

Hint : For the terms like $\frac{i-\infty}{\infty-w}$ etc, we use

$$\frac{i-\infty}{\infty-w} = \lim_{n\to\infty}\frac{i-n}{n-w} \qquad \Big| \frac{\infty}{\infty} \text{ form}$$
$$= \lim_{n\to\infty}\frac{0-1}{1-0}$$
$$= -1$$

Answers :

(i) $w = \frac{2(z-1)}{(1+i)z-2}$ (ii) $w = \frac{(i-2)z+1}{iz-1}$ (iii) $w = \frac{z-(1+i)}{z}$

(iv) $w = \frac{z+1}{1+i}$ (v) $w = -\frac{1}{z}$ (vi) $w = \frac{(1+i)z+(i-1)}{2z}$

(vii) $w = \frac{z+i}{z-i}$

2. Conformal Mappings

Let S be a domain in a plane in which x and y are taken as rectangular Cartesian co-ordinates. Let us suppose that the functions u(x, y) and v(x, y) are continuous and possess continuous partial derivatives of the first order at each point of the domain S. The equations

$$u = u(x, y), \quad v = v(x, y)$$

set up a correspondence between the points of S and the points of a set T in the (u, v) plane. The set T is evidently a domain and is called a map of S. Moreover, since the first order partial derivatives of u and v are continuous, a curve in S which has a continuously turning tangent is mapped on a curve with the same property in T. The correspondence between the two domains is not, however, necessarily a one-one correspondence.

For example, if we take $u = x^2$, $v = y^2$, then the domain $x^2 + y^2 < 1$ is mapped on the triangle bounded by u = 0, v = 0, u + v = 1, but there are four points of the circle corresponding to each point of the triangle.

2.1 Definition : A mapping from S to T is said to be isogonal if it has a one-one transformation which maps any two intersecting curves of S into two curves of T which cut at the same angle. Thus in an isogonal mapping, only the magnitude of angle is preserved.

An isogonal transformation which also conserves the sense of rotation is called conformal mapping. Thus in a conformal transformation, the sense of rotation as well as the magnitude of the angle is preserved.

The following theorem provides the necessary condition of conformality which briefly states that if f(z) is analytic, mapping is conformal.

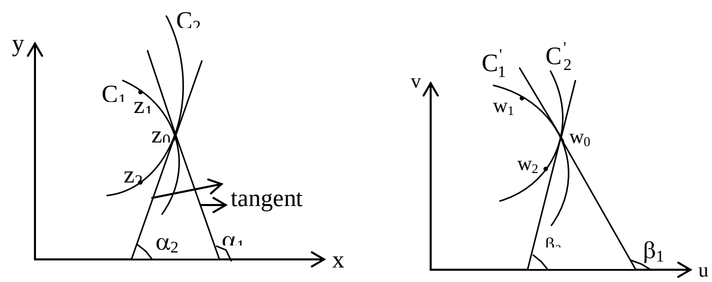
2.2. Theorem : Prove that at each point z of a domain D where f(z) is analytic and $f'(z) \neq 0$, the mapping $w = f(z)$ is conformal.

Proof. Let $w = f(z)$ be an analytic function of z, regular and one valued in a region D of the z-plane. Let z_0 be an interior point of D and let C_1 and C_2 be two continuous curves passing through z_0 and having definite tangents at this point, making angles α_1, α_2, say, with the real axis.

We have to discover what is the representation of this figure in the w-plane. Let z_1 and z_2 be points on the curves C_1 and C_2 near to z_0. We shall suppose that they are at the same distance r from z_0, so we can write

$$z_1 - z_0 = re^{i\theta 1},\ z_2 - z_0 = re^{i\theta 2}.$$

Then as $r \to 0$, $\theta_1 \to \alpha_1$, $\theta_2 \to \alpha_2$. The point z_0 corresponds to a point w_0 in the w-plane and z_1 and z_2 correspond to point w_1 and w_2 which describe curves C'_1 and C_2', making angles β_1 and β_2 with the real axis.



Let $w_1 - w_0 = \rho_1\ e^{i\phi_1}$, $w_2 - w_0 = \rho_2\ e^{i\phi_2}$,
where $\rho_1, \rho_2 \to 0 \Rightarrow \phi_1, \phi_2 \to \beta_1, \beta_2$
respectively.
Now, by the definition of an analytic function,

$$\lim_{z \to z_0} \frac{w_1 - w_0}{z_1 - z_0} = f'(z_0)$$

Since $f'(z_0) \neq 0$, we may write it in the form $Re^{i\lambda}$ and thus

$$\lim \frac{\rho_1 e^{i\phi_1}}{re^{i\theta_1}} = Re^{i\lambda} \text{ i.e. } \lim \frac{\rho_1}{r} e^{i(\phi_1 - \theta_1)} = Re^{i\lambda}$$

$$\Rightarrow \qquad \lim \frac{\rho_1}{r} = R = |f'(z_0)|$$

and $\lim (\phi_1 - \theta_1) = \lambda$
i.e. $\lim \phi_1 - \lim \theta_1 = \lambda$
i.e. $\beta_1 - \alpha_1 = \lambda \quad \Rightarrow \quad \beta_1 = \alpha_1 + \lambda$
Similarly, $\beta_2 = \alpha_2 + \lambda$.

Hence the curves C'_1 and C'_2 have definite tangents at w_0 making angles $\alpha_1 + \lambda$ and $\alpha_2 + \lambda$ respectively with the real axis. The angle between C'_1 and C'_2 is

$$\beta_1 - \beta_2 = (\alpha_1 + \lambda) - (\alpha_2 - \lambda) = \alpha_1 - \alpha_2$$

which is the same as the angle between C_1 and C_2. Hence the curve C_1' and C_2' intersect at the same angle as the curves C_1 and C_2. Also the angle between the curves has the same sense in the two figures. So the mapping is conformal.

Special Case : When $f'(z_0) = 0$, we suppose that $f'(z)$ has a zero of order n at the point z_0. Then in the neighbourhood of this point (by Taylor's theorem)

$$f(z) = f(z_0) + a(z - z_0)^{n+1} + \ldots, \text{ where } a \neq 0$$

Hence $w_1 - w_0 = a(z - z_0)^{n+1} + \ldots.$

i.e. $\rho_1 \, e^{i\phi_1} = | a | r^{n+1} e^{i[\delta + (n+1)\theta_1]} + \ldots$

where $\delta = \arg a$

Hence $\lim \phi_1 = \lim [\delta + (n + 1) \theta_1] = \delta + (n + 1) \alpha_1 \qquad | \delta$ is constant

Similarly $\lim \phi_2 = \delta + (n + 1) \alpha_2$

Thus the curves C'_1 and C'_2 still have definite tangent at w_0, but the angel between the tangents is

$$\lim(\phi_2 - \phi_1) = (n + 1) (\alpha_2 - \alpha_1)$$

Thus, the angle is magnified by $(n + 1)$.

Also the linear magnification, $R = \lim \dfrac{\rho_1}{r} = 0 \qquad |\because \lim \dfrac{\rho_1}{r} = R = |f'(z_0)| = 0$

Therefore, the conformal property does not hold at such points where $f'(z) = 0$

A point z_0 at which $f'(z_0) = 0$ is called a critical point of the mapping. The following theorem is the converse of the above theorem and is sufficient condition for the mapping to be conformal.

2.3. Theorem : If the mapping $w = f(z)$ is conformal then show that $f(z)$ is an analytic function of z.

Proof. Let $w = f(z) = u(x, y) + iv(x, y)$

Here, $u = u(x, y)$ and $v = v(x, y)$ are continuously differentiable equations defining conformal transformation from z-plane to w-plane. Let ds and $d\sigma$ be the length elements in z-plane and w-plane respectively so that

$$ds^2 = dx^2 + dy^2, \quad d\sigma^2 = du^2 + dv^2 \qquad \ldots(1)$$

Since u, v are functions of x and y, therefore

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy, \qquad dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$\therefore \qquad du^2 + dv^2 = \left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy\right)^2 + \left(\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy\right)^2$$

i.e. $$d\sigma^2 = \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] dx^2 + \left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] dy^2$$

$$+\,2\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right)dxdy \qquad \ldots(2)$$

Since the mapping is given to be conformal, therefore the ratio $d\sigma^2 : ds^2$ is independent of direction, so that from (1) and (2), comparing the coefficients, we get

$$\frac{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2}{1}=\frac{\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2}{1}=\frac{\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}}{0}$$

$$\Rightarrow \qquad \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2=\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2 \qquad \ldots(3)$$

and $$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}=0 \qquad \ldots(4)$$

Equations (3) and (4) are satisfied if

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y} \qquad \ldots(5)$$

or $$\frac{\partial u}{\partial x}=-\frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x}=\frac{\partial u}{\partial y} \qquad \ldots(6)$$

Equation (6) reduces to (5) if we replace v by $-v$ i.e. by taking as image figure obtained by the reflection in the real axis of the w-plane.

Thus the four partial derivatives u_x, u_y, v_x, v_y exist, are continuous and they satisfy C-R equations (5). Hence f(z) is analytic.

2.4. Remarks

(i) The mapping $w = f(z)$ is conformal in a domain D if it is conformal at each point of the domain.

(ii) The conformal mappings play an important role in the study of various physical phenomena defined on domains and curves of arbitrary shapes. Smaller portions of these domains and curves are conformally mapped by analytic function to well-known domains and curves.

2.5. Example : Discuss the mapping $w = \bar{z}$.

Solution. We observe that the given mapping replaces every point by its reflection in the real axis. Hence angles are conserved but their signs are changed and thus the mapping is isogonal but not conformal. If the mapping $w = \bar{z}$ is followed by a conformal transformation, then resulting transformation of the form $w = f(\bar{z})$ is also isogonal but not conformal, where f(z) is analytic function of z.

2.6. Example : Discuss the nature of the mapping $w = z^2$ at the point $z = 1 + i$ and examine its effect on the lines Im z = Re z and Re z = 1 passing through that point.

Solution. We note that the argument of the derivative of $f(z) = z^2$ at $z = 1 + i$ is

$$[\arg 2z]_{z=1+i} = \arg(2 + 2i) = \pi/4$$

Hence the tangent to each curve through $z = 1 + i$ will be turned by the angle $\pi/4$. The co-efficient of linear magnification is $|f'(z)|$ at $z = 1 + i$, i..e $|2 + 2i| = 2\sqrt{2}$. The mapping is

$$w = z^2 = x^2 - y^2 + 2ixy = u(x, y) + iv(x, y)$$

We observe that mapping is conformal at the point $z = 1 + i$, where the half lines $y = x (y \geq 0)$ and $x = 1 (y \geq 0)$ intersect. We denote these half lines by C_1 and C_2, with positive sense upwards and observe that the angle from C_1 to C_2 is $\pi/4$ at their point of intersection. We have

$$u = x^2 - y^2, \qquad v = 2xy$$

The half line C_1 is transformed into the curve C'_1 given by

$$u = 0, \qquad v = 2y^2 (y \geq 0)$$

Thus C'_1 is the upper half $v \geq 0$ of the v-axis.

The half line C_2 is transformed into the curve C'_2 represented by

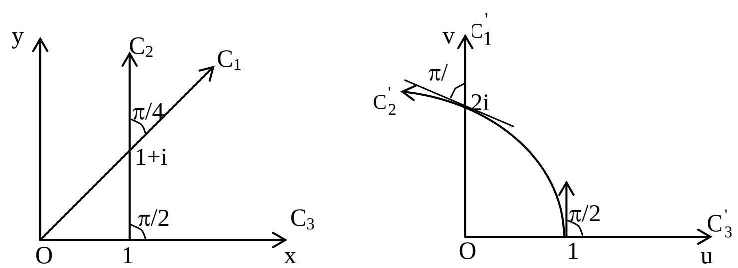
$$u = 1 - y^2, \qquad v = 2y \ (y \geq 0)$$

Hence C'_2 is the upper half of the parabola $v^2 = -4(u - 1)$. We note that, in each case, the positive sense of the image curve is upward.

For the image curve C'_2,

$$\frac{dv}{du} = \frac{dv/dy}{du/dy} = \frac{2}{-2y} = -\frac{2}{v}$$

In particular $\frac{dv}{du} = -1$ when $v = 2$. Consequently, the angle from the image curve C'_1 to the image curve C'_2 at the point $w = f(1 + i) = 2i$ is $\frac{\pi}{4}$, as required by the conformality of the mapping there.



Note. The angle of rotation and the scalar factor (linear magnification) can change from point to point. We note that they are 0 and 2 respectively, at the point $z = 1$, since $f'(1) = 2$, where the curves C_2 and C'_2 are the same as above and the non-negative x-axis (C_3) is transformed into the non-negative u-axis (C'_3).

2.7. Example. Discuss the mapping $w = z^a$, where a is a positive real number.

Solution. Denoting z and w in polar as

$$z = re^{i\theta}, \ w = \rho e^{i\phi}, \text{ the mapping gives } \rho = r^a, \ \phi = a\theta.$$

Thus the radii vectors are raised to the power a and the angles with vertices at the origin are multiplied by the factor a. If $a > 1$, distinct lines through the origin in the z-plane are not mapped onto distinct lines through the origin in the w-plane, since, e.g. the straight line through the origin at an angle $\frac{2\pi}{a}$ to the real axis of the z-plane is mapped onto a line through the origin in the w-plane at an angle 2π to the real axis i.e. the positive real axis itself. Further $\frac{dw}{dz} = az^{a-1}$, which vanishes at the origin if $a > 1$ and has a singularity at the origin if $a < 1$. Hence the mapping is conformal and the angles are therefore preserved, excepting at the origin. Similarly the mapping $w = e^z$ is conformal.

2.8. Example. Prove that the quadrant $|z| < 1$, $0 < \arg z < \frac{\pi}{2}$ is mapped conformally onto a domain in the w-plane by the transformation $w = \frac{4}{(z+1)^2}$.

Solution. If $w = f(z) = \frac{4}{(z+1)^2}$, then $f'(z)$ is finite and does not vanish in the given quadrant. Hence the mapping $w = f(z)$ is conformal and the quadrant is mapped onto a domain in the w-plane provided w does not assume any value twice i.e. distinct points of the quadrant are mapped to distinct points of the w-plane. We show that this indeed is true. If possible, let $\frac{4}{(z_1+1)^2} = \frac{4}{(z_2+1)^2}$, where $z_1 \neq z_2$ and both z_1 and z_2 belong to the quadrant in the z-plane. Then, since $z_1 \neq z_2$, we have $(z_1 - z_2)(z_1 + z_2 + 2) = 0$
$\Rightarrow z_1 + z_2 + 2 = 0$ i.e. $z_1 = -z_2 - 2$. But since z_2 belongs to the quadrant, $-z_2 - 2$ does not, which contradicts the assumption that z_1 belongs to the quadrant. Hence w does not assume any value twice.

3. Space of Analytic Functions

We start with the following definition

3.1. Definition. **A metric space is a pair (X, d) where X is a set and d is a function from X × X into R, called the distance function or metric, which satisfy the following conditions for x, y, z ∈ X**

(i) $d(x, y) \geq 0$
(ii) $d(x, y) = 0$ if $x = y$
(iii) $d(x, y) = d(y, x)$
(iv) $d(x, z) \leq d(x, y) + d(y, z)$

Conditions (iii) and (iv) are called 'symmetry' and 'triangle inequality' respectively. A metric space (X, d) is said to be bounded if there exists a positive number K such that

$d(x, y) \leq K$ for all $x, y \in X$.

The metric space (X, d), in short, is also denoted by X, the metric being understood. If x and r > 0 are fixed then let us define

$B(x; r) = \{x \in X : d(x, y) < r\}$
$\bar{B}(x; r) = \{y \in X : d(x, y) \leq r\}$

B(x; r) and $\bar{B}$(x; r) are called open and closed balls (spheres) respectively, with centre x and radius r. B(x; ∈) is also referred to as the ∈-neighbourhood of x.

Let X = R or ∀ and define $d(z, w) = |z-w|$. This makes both (R,d) and (∀, d) metric spaces. (∀, d) is the case of principal interest for us. In (∀ , d), open and closed balls are termed as open and closed discs respectively.

A metric space (X, d) is said to be complete if every sequence in X converges to a point of X, R and ∀ are examples of complete metric spaces.

If G is an open set in ∀ and (X, d) is complete metric space then the set of all continuous functions from G to X is denoted by C(G, X).

The set C(G, X) is always non empty as it contains the constant functions. However it is possible that C(G, X) contains only the constant functions. For example, suppose that G is connected and X = N = {1, 2, 3, 4,..}. If f ∈ C(G, X) then f(G) must be connected in X and hence, must be singleton as the only connected subsets of N are singleton sets.

In this section we shall be mainly concern with the case when X is either $\forall$ or $\forall_\infty$.

To put a metric on C(G, X), we need the following result.

3.2. Theorem : If G is open in $\forall$ then there is a sequence $\{K_n\}$ of compact subsets of G such that $G = \bigcup_{n=1}^{\infty} K_n$. Moreover, the sets K_n can be chosen to satisfy the following conditions :

(a) $K_n \subset \text{int } K_{n+1}$;

(b) $K \subset G$ and K compact implied $K \subset K_n$ for some n.

Now we define a metric on C(G, X).

Since G is open set in $\forall$, we have $G = \bigcup_{n=1}^{\infty} K_n$ where each K_n is compact and $K_n \subset$ int K_{n+1}. For $n \in N$, we define

$$\rho_n(f, g) = \sup\{d(f(z), g(z)) : z \in K_n\}$$

for all functions f and g in C(G, X).

Also if we define

$$\rho(f, g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f,g)}{1+\rho_n(f,g)} \qquad \text{for all } f, g \in C(G, X)$$

then (C(G, X), ρ) is a metric space. In fact (C(G, X), ρ) is a complete metric space.

3.3. Definitions : A set $\Phi \subset C(G, X)$ is normal if each sequence in Φ has a subsequence which converges to a function f in C(G, X).

A set $\Phi \subset C(G, X)$ is normal iff its closure is compact.

A set $\Phi \subset C(G, X)$ is called **equicontinuous** at a point z_0 in G iff for every $\in > 0$ there is a $\delta > 0$ such that for $|z - z_0| < \delta$,

$$d(f(z), f(z_0)) < \in$$

for every f in Φ.

Φ is said to be **equicontinuous** over a set $E \subset G$ if for every $\in > 0$ there is a $\delta > 0$ such that for z and z′ in E and $|z - z'| < \delta$, we have

$$d(f(z), f(z')) < \in \qquad \text{for all f in } \Phi$$

Notice that if Φ consists of a single function f then the statement that Φ is equicontinuous at z_0 is only the statement that f is continuous at z_0. The important thing about equicontinuity is that the same δ will work for all the functions in Φ. Also for $\Phi = \{f\}$ to be equicontinuous over E is equivalent to the uniform continuity of f on E.

Further, suppose Φ ⊂ C(G, X) is equicontinuous at each point of G then Φ is equicontinuous over each compact subset of G.

3.4 Arzela-Ascoli Theorem : A set $\Phi \subset C(G, X)$ is normal iff the following two conditions are satisfied :

(a) For each z in G, $\{f(z) ; f \in F\}$ has compact closure in X.

(b) F is equicontinuous at each point of G.

Let G be an open subset of the complex plane and H(G) be the collection of holomorphic (analytic) functions on G.

The following theorem shows that H(G) is a closed subset of C(G, ∀).

3.5. Theorem. If $\{f_n\}$ is a sequence in H(G) and f belongs to $C(G, \forall)$ such that $f_n \to f$ then f is analytic and $f_n^{(k)} \to f^{(k)}$ for each integer $k \geq 1$.

Proof. To show f is analytic on G, we shall use the following form of Morera's theorem which states

"Let G be a region and let $f : G \to \forall$ be a continuous function such that $\int_T f = 0$ for every triangular path T in G ; then f is analytic in G".

Let T be a triangle contained inside a disk $D \subset G$. Since T is compact, $\{f_n\}$ converges to f uniformly over T. Hence $\int_T f = \lim \int_T f_n = 0$, since each f_n is analytic. Thus f must be analytic in every disk $D \subset G$. This gives that f is analytic in G.

To show that $f_n^{(k)} \to f^{(k)}$, let D denote the closure of B(a, r) contained in G. Then there is a number R > r such that $\overline{B}(a\,;R) \subset G$.

If γ is the circle $|z - a| = R$ then by Cauchy's integral formula,

$$f_n^{(k)}(z) - f^{(k)}(z) = \frac{k!}{2\pi i}\int_\gamma \frac{f_n(w) - f(w)}{(w-z)^{k+1}}\,dw \qquad \text{for z in D.}$$

Let $M_n = \sup\{\,|f_n(w) - f(w)| : |w - a| = R\}$. Then by Cauchy's estimate, we have

$$|f_n^{(k)}z - f^{(k)}(z)| \leq \frac{k!M_nR}{(R-r)^{k+1}} \qquad \text{for } |\,z - a| \leq r \qquad \ldots(1)$$

Since $f_n \to f$, $\lim M_n = 0$

Thus, it follows from (1) that $f_n^{(k)} \to f^{(k)}$ uniformly on $\overline{B}(a\,;R)$. Now let K be an arbitrary compact subset of G and $0 < r < d(K, \partial G)$ then there are $a_1, a_2, \ldots, a_n$ in K such that

$$K \subset \bigcup_{j=1}^{n} B(a_j; r)$$

Since $f_n^{(k)} \to f^{(k)}$ uniformly on each $B(a_j\,;r)$, it follows that $f_n^{(k)} \to f^{(k)}$ uniformly on K, which completes the proof of the theorem.

Cor. 1 : H(G) is a complete metric space.

Proof. Since $C(G, \forall)$ is a complete metric space and H(G) is a closed subset of $C(G, \forall)$, we get that H(G) is also complete using "Let (X, d) be a complete metric space and $Y \subset X$. Then (Y, d) is complete iff Y is closed in X".

Cor. 2 : If $f_n : G \to \forall$ is analytic and $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on compact sets to f(z) then

$$f^{(k)}z = \sum_{n=1}^{\infty} f_n^{(k)}(z)$$

3.6. Hurwitz's Theorem. Let G be a region and suppose the sequence $\{f_n\}$ in H(G) converges to f. If $f \neq 0$, $\overline{B}(a\,;R) \subset G$ and $f(z) \neq 0$ for $|z - a| = R$ then there is an integer N such that for $n \geq N$, f and f_n have the same number of zeros in B(a ; R).

Proof : Let $\delta = \inf\{|f(z)| : |z - a| = R\}$

Since $f(z) \neq 0$ for $|z - a| = R$, we have $\delta > 0$.

Now $f_n \to f$ uniformly on $\{z : |z - a| = R\}$ so there is an integer N such that if $n \geq N$ and $|z - a| = R$ then

$$|f(z) = f_n(z)| < \frac{\delta}{2} < |f(z)|$$

Hence by Rouche's theorem, f and f_n have the same number of zeros in B(a; R).

Cor : If $\{f_n\} \subset H(G)$ converges to f in H(G) and each f_n never vanishes on G then either $f \equiv 0$ or f never vanishes.

3.7. Remark : Another form of Hurwitz's theorem is "Let $\{f_n(z)\}$ be a sequence of functions, each analytic in a region D bounded by a simple closed contour and let $f_n(z) \to f(z)$ uniformly in D. Suppose that f(z) is not identically zero. Let z_0 be an interior point of D. Then z_0 is a zero of f(z) if and only if it is a limit point of the set of zeros of the functions $f_n(z)$, points which are zeros of $f_n(z)$ for an infinity of values of n being counted as limit points."

Proof. Let z_0 be any point of D and let γ be a circle with centre z_0 and radius ρ so small that γ lies entirely in D. Suppose γ neither contains nor has on it any zero of f(z) except possibly for the point z_0 itself. Then $|f(z)|$ has a strictly positive lower bound on the circle $|z - z_0| = \rho$, say $|f(z)| \geq K > 0$ …(1)

Having fixed ρ and K, we can choose N so large that, on the circle,

$$|f_n(z) - f(z)| < K \text{ for all } n > N \qquad \ldots(2)$$

From (1) and (2), we get

$$|f_n(z) - f(z)| < |f(z)|$$

Thus, if we set $g(z) = f_n(z) - f(z)$, then on the circle $|z - z_0| = \rho$, $|g(z)| < |f(z)|$

Hence by Rouche's theorem, for $n > N$, $g(z) + f(z)$ i.e. $f_n(z)$ has the same number of zeros as f(z) inside the circle γ . Thus if $f(z) = 0$, then $f_n(z)$ has exactly one zero inside γ for $n > N$. Therefore, z_0 is the limit point of the zeros of $f_n(z)$. If $f(z_0) \neq 0$, then $f_n(z_0) \neq 0$ inside γ for $n > N$ which completes the proof.

3.8. Definition : A set $\Phi \subset H(G)$ is called locally bounded if for each point a in G there are constants M and $r > 0$ such that for all f in Φ ,

$$|f(z)| \leq M \quad \text{for } |z - a| < r.$$

Alternately, Φ is locally bounded if there is an $r > 0$ such that

$$\sup \{|f(z)| : |z - a| < r, f \in \Phi\} < \infty.$$

That is, Φ is locally bounded if about each point a in G there is a disk on which Φ is uniformly bounded.

3.9. Lemma : A set Φ in H(G) is locally bounded iff for each compact set $K \subset G$ there is a constant M such that

$$|f(z)| \leq M$$

for all f in Φ and z in K.

3.10. Montel's Theorem : A family Φ in H(G) is normal iff Φ is locally bounded.

Proof : Suppose Φ is normal. We have to show Φ is locally bounded. Let, if possible, suppose that Φ is not locally bounded. Then there is a compact set $K \subset G$ such that

$$\sup \{|f(z)| : z \in K, f \in \Phi\} = \infty$$

That is, there is a sequence $\{f_n\}$ in Φ such that

$$\sup\{|f_n(z) : z \in K\} \geq n.$$

Since Φ is normal there is a function f in H(G) and a subsequence $\{ f_{n_k} \}$ such that $f_{n_k} \to f$.

This gives $\lim\limits_{k\to\infty} \sup(| f_{n_k}(z) - f(z) : z \in K\} = 0$

Let $|f(z)| \leq M$ for z in K. Then

$$\begin{aligned} n_k &\leq \sup\{ | f_{n_k}(z)| : z \in K\} \\ &= \sup\{ | f_{n_k}(z) - f(z) + f(z) | : z \in K\} \\ &\leq \sup\{ | f_{n_k}(z) - f(z)| : z \in K\} + \sup\{ | f(z) | : z \in K\} \end{aligned}$$

$\Rightarrow$ $$n_k \leq \sup\{| f_{n_k}(z) - f(z)| : z \in K\} + M$$

$\Rightarrow$ $$\lim_{k\to\infty} n_k \leq M .$$

A contradiction since $<n_k>$ is strictly monotonically increasing sequence. Hence our supposition is wrong. So Φ must be locally bounded.

Conversely suppose that Φ is locally bounded. Then for each z in G, $\{f(z) : f \,\varepsilon\, \Phi\}$ has compact closure. We now show that Φ is equicontinuous at each point of G. Let a be any fixed point of G and $\in > 0$. By hypothesis, there is an $r > 0$ and $M > 0$ such that $\overline{B}(a ; r) \subset G$ and

$$|f(z)| \leq M \qquad \text{for all z in } \overline{B}(a ; r) \text{ and for all f in } \Phi .$$

Let $|z - a| < \frac{1}{2} r$ and $f \,\varepsilon\, \Phi$. Then by Cauchy's formula, with $r(t) = a + re^{it}$, $0 \leq t \leq 2\pi$,

$$\begin{aligned} |f(a) - f(z)| &\leq \frac{1}{2\pi}\left|\int_\gamma \frac{f(w)(a-z)}{(w-a)(w-z)}dw\right| \\ &\leq \frac{4M}{r}|a - z| \end{aligned}$$

Choose δ straight line $0 < \delta$, min $\left\{\frac{r}{2}, \frac{r}{4M} \in\right\}$

Then $$|a - z| < \delta \text{ gives}$$

$$|f(a) - f(z)| < \in \qquad \text{for all f in } \Phi .$$

$\therefore$ F is equicontinuous at a ε G.

Hence by Ascoli-Arzela theorem. Φ is normal.

Cor. A set $\Phi \subset H(G)$ is compact iff it is closed and locally bounded.

3.11. Definition : A region G_1 is called conformally equivalent to G_2 if there is an analytic function $f : G_1 \to \forall$ such that f is one one and $f(G_1) = G_2$.

It is immediate that $\forall$ is not equivalent to any bounded region by Liouville's theorem. Also it follows from the definition that if G_1 is simply connected and G_1 is equivalent to G_2 then G_2 must be simply connected.

We now prove Riemann mapping theorem which states that every simply connected region G in the plane (other than the plane itself) is conformally equivalent to the open unit disc D. We shall use the following results.

Theorem (1) : Let G be an open connected subset of $\forall$. Then the following are equivalent.

(a) G is simply connected.

(b) For any f in H(G) such that $f(z) \neq 0$ for all z in G, there is a function g in H(G) such that $f(z) = [g(z)]^2$.

Theorem (2) : Let $f : D \to D$ be a one one analytic function of D onto itself and suppose $f(a) = 0$. Then there is a complex number c with $|c| = 1$ such that

$$f = c\phi_a \quad \text{where } \phi_a(z) = \frac{z-a}{1-\bar{a}z} \text{ with } |a| < 1.$$

Theorem (3) (Open mapping theorem) : Let G be a region and suppose that f is a non-constant analytic function on G. Then for any open set U in G, f(U) is open.

3.12. Riemann mapping theorem. Let G be a simply connected region which is not the whole plane and let $a \in G$. Then there is a unique analytic function $f : G \to \forall$ having the properties :

(a) $f(a) = 0$ and $f'(a) > 0$

(b) f is one one

(c) $f(G) = \{z : |z| < 1\}$

Proof : First we show f is unique.

Let g be another analytic function on $\forall$ such that $g(a) = 0$, $g'(a) > 0$ g is one one and $g(G) = \{z : |z| < 1\} = D$.

Then $f_0 g^{-1} : D \to D$ is analytic, one one and onto

Also $f_0 g^{-1}(0) = f(a) = 0$. So there is a constant c with $|c| = 1$

and $f_0 g^{-1}(z) = cz$ for all z. [Applying theorem (2) with a = 0]

But then $f(z) = c\, g(z)$ gives that $0 < f'(a) = cg'(a)$.

Since $g'(a) > 0$, it follows that $c = 1$. Hence $f = g$ and so f is unique.

Now let $\Phi = \{f \in H(G) : f \text{ is one one}, f(a) = 0, f'(a) > 0, f(G) \subset D\}$

We first show $\Phi \neq \phi$.

Since $G \neq \forall$ so there exists $b \in \forall$ such that $b \notin G$

Also G is simply connected so there exists an analytic function g on G such that $[g(z)]^2 = z - b$.

Then g is one-one

For this let $z_1, z_2 \in G$ such that $g(z_1) = g(z_2)$

Then $[g(z_1)]^2 = [g(z_2)]^2$

$\Rightarrow$ $z_1 - b = z_2 - b$

$\Rightarrow$ $z_1 = z_2$

$\Rightarrow$ g is one-one.

So by open mapping theorem, there is a positive number r such that

$$\mathbf{B(g(a)\,;\, r) \subset g(G)} \qquad \ldots(1)$$

Let z be a point in G such that $g(z) \in B(-g(a)\,;\, r)$

Then $|g(z) + g(a)| < r$

$\Rightarrow$ $|-g(z) - g(a)| < r$

$\Rightarrow$ $-g(z) \in B(g(a)\,;\, r)$

$\Rightarrow$ $-g(z) \in g(G)$ [using (1)]

So $\exists$ some $w \in G$ such that

$-g(z) = g(w)$

$\Rightarrow$ $[g(z)]^2 = [g(w)]^2$

$\Rightarrow$ $z - b = w - b$

$\Rightarrow$ $z = w$

Thus, we get,

$-g(z) = g(z)$

$\Rightarrow$ $g(z) = 0$

But then $z - b = [g(z)]^2 = 0$ implies $b = z \in G$, a contradiction.
Hence $g(G) \cap B(-g(a)\,;\, r) = \phi$
Let $U = B(-g(a)\,;\, r)$. There is a Mobius transformation T such that

$T(C_\infty - \overline{U}) = D$

Let $g_1 = T_0 g$ then g_1 is analytic and $g_1(G) \subset D$.

Consider $g_2(z) = \dfrac{g_1(z) - \alpha}{1 - \overline{\alpha} g_1(z)}$ where $\alpha = g_1(a)$.

Then g_2 is analytic, $g_2(G) \subset D$ and $g_2(a) = 0$
Choose a complex number c, $|c| = 1$, such that

$g_3(z) = c\, g_2(z)$ and $g_3'(a) > 0$

Now $g_3 \in \Phi$ hence $\Phi \neq \phi$.

Next we assume that $\overline{\Phi} = \Phi \cup \{0\}$ …(2)
Since $f(G) \subset D$, sup $\{|f(z) : z \in G\} \le 1$ for f in Φ. So by Montel's theorem, Φ is normal.

This gives $\overline{\Phi}$ is compact.

Consider the function $\phi : H(G) \to C$
as $\phi(f) = f'(a)$

Then ϕ is continuous function. Since $\overline{\Phi}$ is compact, there is an f in $\overline{\Phi}$ such that $f'(a) \ge g'(a)$ for all $g \in \Phi$.

As $\Phi \neq \phi$, (2) implies that $f \in \Phi$. We show that $f(G) = D$. Suppose $w \in D$ such that $w \notin f(G)$. Then the function

$$\frac{f(z) - w}{1 - \overline{w} f(z)}$$

is analytic in G and never vanishes. Since G is simply connected, there is an analytic function $h : G \to \forall$ such that

$$[h(z)]^2 = \frac{f(z) - w}{1 - \overline{w} f(z)} \qquad \dots(3)$$

Since the Mobius transformation $T_\xi = \dfrac{\xi - w}{1 - \overline{w}\xi}$ maps D onto D,

we have $h(G) \subset D$.

Define $g : G \to \forall$ as

$$g(z) = \frac{|h'(a)|}{h'(a)} \cdot \frac{h(z) - h(a)}{1 - \overline{h(a)}h(z)}$$

Then $g(G) \subset D$, $g(a) = 0$ and g is one-one.

Also $$g'(a) = \frac{|h'(a)|}{h'(a)} \cdot \frac{h'(a)[1 - |h(a)|^2]}{[1 - |h(a)|^2]^2} = \frac{|h'(a)|}{1 - |h(a)|^2}$$

But $$|h(a)|^2 = \left|\frac{f(a) - w}{1 - \overline{w} f(a)}\right| = |-w| = |w| \qquad [\because f(a) = 0]$$

Differentiating (3), we get

$$2\, h(a)\, h'(a) = f'(a)\, [1 - |w|^2]$$

$$\Rightarrow \qquad h'(a) = \frac{f'(a)(1 - |w|^2)}{2h(a)} = \frac{f'(a)(1 - |w|^2)}{2\sqrt{|w|}}$$

$\therefore \qquad g'(a) = \frac{f'(a)(1-|w|^2)}{2\sqrt{|w|}} \cdot \frac{1}{(1-|w|)} = \frac{f'(a)(1+|w|)}{2\sqrt{|w|}} > f'(a)$

Thus g ε Φ. A contradiction to the choice of f .

Hence we must have f(G) = D.

Next we prove $\bar{\Phi} = \Phi \cup \{0\}$.

Suppose $\{f_n\}$ is a sequence in Φ and $f_n \to f$ in H(G).

Then $\qquad f(a) = \lim_{n\to\infty} f_n(a) = 0 \qquad$ Also $f_n'(a) \to f'(a)$ so $f'(a) \geq 0$

Let z_1 be an arbitrary element of G and let $w = f(z_1)$. Let $w_n = f_n(z_1)$. Let $z_2 \in G$, $z_2 \neq z_1$ and K be a closed disk centred at z_2 such that $z_1 \equiv K$.

Then $f_n(z) - w_n$ never vanishes on K since f is one one But $f_n(z) - w_n$ converges uniformly to $f(z) - w$ on K as K is compact. So Hurwitz's theorem gives that $f(z) - w$ never vanishes on K or $f(z) = w$.

If $f(z) \equiv w$ on K then f is constant function throughout G and since f(a) = 0, we have $f(z) \equiv 0$. Otherwise we have f is one. So f ′ can never vanish. This gives

$$f'(a) > 0 \qquad\qquad [\because f'(a) \geq 0]$$

and so $\qquad f \ \varepsilon\ \Phi$.

Hence $\bar{\Phi} = \Phi \cup \{0\}$ which completes the proof of the theorem.

4. Factorization of an Integral Function

We know that a function which is regular in every finite region of the z-plane is called an integral function or entire function. In other words, integral function is an analytic function which has no singularity except at infinity.

e.g. $\qquad e^z = 1 + z + \frac{z^2}{\lfloor 2} + \ldots$

The simplest integral functions are polynomials. We know that a polynomial can be uniquely expressed as the product of linear factors in the form :

$$f(z) = f(0)\left(1-\frac{z}{z_1}\right)\left(1-\frac{z}{z_2}\right)\ldots\left(1-\frac{z}{z_n}\right)$$

where $z_1, z_2, \ldots, z_n$ are the zeros of the polynomial.

An integral function which is not a polynomial may have an infinity of zeros z_n and the product $\pi\left(1-\frac{z}{z_n}\right)$ taken over these zeros may be divergent.

So, a integral function cannot be always factorized in the same way as a polynomial and thus we have to consider less simple factors than $\left(1-\frac{z}{z_n}\right)$. We observe that

(a) An integral function may have no zero e.g. e^z.
(b) An integral function may have finite number of zeroes e.g. polynomials of finite degree.
(c) An integral function may have infinite number of zeroes. e.g. sin z, cos z.

4.1 Theorem : The most general integral function with no zero is the form $e^{g(z)}$, where g(z) is itself an integral function.

Proof : Let f(z) be an integral function with no zero, then $f'(z)$ is also an integral function and so is $\frac{f'(z)}{f(z)}$.

Let $F(z) = \int_{z_0}^{z} \frac{f'(w)dw}{f(w)}$, where the integral is taken along any path from fixed point z_0 to a point z.

Thus $f(z) = [\log f(w)]_{z_0}^{z} = \log f(z) - \log f(z_0)$

$\Rightarrow$ $\log f(z) = F(z) + \log f(z_0)$

$\Rightarrow$ $f(z) = \exp[\log f(z_0) + F(z)]$

$= e^{g(z)}$, where $g(z) = \log f(z_0) + F(z)$ is itself an integral function.

Hence the result.

4.2. Construction of an Integral Function with Given Zeros. If f(z) is an integral function with only a finite number of zeros, say $z_1, z_1, .., z_n$, then the function

$$\frac{f(z)}{(z-z_1)(z-z_2)...(z-z_n)}$$

is an integral function with no zeros. Also we know that the most general form of an integral function is $e^{g(z)}$, where g(z) is an integral function. Thus, we put

$$\frac{f(z)}{(z-z_1)(z-z_2)...(z-z_n)} = e^{g(z)}$$

$$\Rightarrow \quad f(z) = (z - z_1)(z - z_2)\ldots(z - z_n)\, e^{g(z)}$$

If, however, f(z) is an integral function with an infinite number of zeros, then the only limit point of the sequence of zeros, $z_1, z_2, \ldots, z_n, \ldots$ is the point at infinity. To determine an integral function f(z) with an infinity of zeros, we have an important theorem due to Weierstrass.

4.3. Weierstrass Primary Factors. The expressions

$$E_0(z) = 1 - z$$

$$E_p(z) = (1 - z)\exp\left(z + \frac{z^2}{2} + ... + \frac{z^p}{p}\right), \; p \geq 1$$

are called Weierstrass primary factors. Each primary factor is an integral function which has only a simple zero at z = 1. Thus, $E_p(z/a)$ has a simple zero at z = a and no other zero.

The behaviour of $E_p(z)$ as $z \to 0$, depends upon p, since for $|z| < 1$, we have

$$E_p(z) = \exp\left[\log(1-z) + \left(z + \frac{z^2}{2} + ... + \frac{z^p}{p}\right)\right]$$

$$= \exp\left[\left(-z - \frac{z^2}{2} - .. - \frac{z^p}{p} - \frac{z^{p+1}}{p+1} - \frac{z^{p+2}}{p+2} ..\right) + \left(z + \frac{z^2}{2} + .. + \frac{z^p}{p}\right)\right]$$

$$= \exp\left[-\frac{z^{p+1}}{p+1} - \frac{z^{p+2}}{p+2} ...\right] = \exp\left(-\sum_{n=p+1}^{\infty} \frac{z^n}{n}\right)$$

$$\Rightarrow \quad \log E_p(z) = -\frac{z^{p+1}}{p+1} - \frac{z^{p+2}}{p+2} \ldots..$$

Hence if K > 1 and $|z| \leq \frac{1}{K}$, then

$$|\log E_p(z)| \leq |z|^{p+1} + |z|^{p+2} + \ldots$$

$$= |z|^{p+1}(1 + |z| + |z|^2 + \ldots)$$
$$= |z|^{p+1}\left(1 + \frac{1}{K} + \frac{1}{K^2} + \ldots\right)$$
$$= \frac{K}{K-1}|z|^{p+1}$$

This inequality helps in determining the convergence of a product of primary factors. In particular, when $|z| \le \frac{1}{2}$, then

$$|\log E_p(z)| \le 2|z|^{p+1} \qquad \ldots(1)$$

4.4. Theorem : If $z_1, z_2, \ldots, z_n, \ldots$ be any sequence of numbers whose only limit point is the point at infinity, then it is possible to construct an integral function which vanishes at each of the points z_n and no where else.

Proof : Let the given zeros $z_1, z_2, \ldots, z_n, \ldots$ be arranged in order of non-decreasing modulus i.e. $|z_1| \le |z_2| \le \ldots \le |z_n| \le \ldots$

Let $\quad |z_n| = r_n, \quad n = 1, 2, \ldots$

and let $p_1, p_2, \ldots$ be a sequence of positive integers such that the series $\sum_{n=1}^{\infty}\left(\frac{r}{r_n}\right)^{p_n}$ is convergent for all values of r. It is always possible to find such a sequence since $r_n = |z_n|$ is increasing definitely with n and we can take $p_n = n$, since $\left(\frac{r}{r_n}\right)^n < \frac{1}{2^n}$ for $r_n > 2r$ and hence the series is convergent.

Now, let

$$f(z) = \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right)$$

This integral function is found to have the required property according to the specifications of the theorem. To prove this, we observe that if $|z_n| > 2|z|$, then by the inequality (1) for $|\log E_p(z)|$, we have

$$\left|\log E_{p_n}\left(\frac{z}{z_n}\right)\right| \le 2\left|\frac{z}{z_n}\right|^{p_{n+1}} = 2\left(\frac{r}{r_n}\right)^{p_{n+1}}$$

and hence by Weierstrass's test, the series $\sum_{n=1}^{\infty}\log E_{p_n}\left(\frac{z}{z_n}\right)$ is uniformly convergent for $|z_n| > 2R$, $|z| \le R$ and also by Weierstrass's test for the uniform convergence of an infinite product to be convergent, so is the product

$$\prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right), \quad |z_n| > 2R, \qquad |z| \le R$$

Hence f(z) is regular for $|z| \le R$ and its only zeros in this region are those of $\prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right)$, $|z_n| > 2R$ i.e. the points $z_1, z_2, \ldots, z_n, \ldots$ Since R can be taken as large as we please, therefore we conclude that f(z) is the integral function which vanishes only at $z_1, z_2, .. z_n, \ldots$ and nowhere else. Hence the result.

Remarks : (i) The function f(z) is not uniquely determined, since we have a wide choice of the sequence of positive integers $p_1, p_2, \ldots, p_n, \ldots$

(ii) The most general integral function with the points $z_1, z_2, \ldots z_n$ as its only zeros is $f(z)\, e^{g(z)}$, where f(z) is the integral function constructed above and g(z) is an arbitrary integral function.

4.5. Weierstrass Factorization Theorem. Let f(z) be an entire function and let $\{z_n\}$ be the non-zero. Zeros of f(z) repeated according to multiplicity, suppose f(z) has a zero at z = 0 of order $m \geq 0$ (a zero of order m = 0 at z = 0 means $f(0) \neq 0$). Then there is an entire function g(z) and a sequence of integers $\{p_n\}$ such that

$$f(z) = z^m\, e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right)$$

Proof : According to the preceding theorem $\{p_n\}$ can be chosen such that

$$h(z) = z^m \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right)$$

has the same zeros as f(z) with the same multiplicity. It follows that $\frac{f(z)}{h(z)}$ has removable singularity at $z = 0, z_1, z_2, \ldots$ Thus f/h is an entire function and, furthermore, has no zeros. Since $\forall$ is simply connected, there is an entire function g(z) such that

$$\frac{f(z)}{h(z)} = e^{g(z)}$$

i.e.

$$f(z) = z^m\, e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right).$$

4.6. Theorem : If f(z) is an integral function and $f(0) \neq 0$, then $f(z) = f(0)\, P(z)\, e^{g(z)}$, where P(z) is the product of primary factors and g(z) is an integral function.

Proof : We form P(z) from the zeros of f(z). Let

$$\phi(z) = \frac{f'(z)}{f(z)} - \frac{P'(z)}{P(z)}$$

Then $\phi(z)$ is an integral function, since the poles of one term are cancelled by those of the other. Hence

$$g(z) = \int_0^z \phi(t)dt = [\log f(t) - \log P(t)]_0^z$$

$$= \log f(z) - \log f(0) - \log P(z) + \log P(0)$$

$$\Rightarrow \quad \log f(z) = g(z) + \log f(0) + \log P(z)$$

$$\Rightarrow \quad f(z) = f(0)\, P(z)\, e^{g(z)}$$

Hence the result.

Remark : This factorization is not unique.

5. The Gamma Function

Here, we shall construct a function, called gamma function or Euler's gamma function which is meromorphic with pole at non-positive integers i.e. $z = 0, -1, -2, \ldots$.

There are two natural approaches to construct the gamma function. One is via the Weierstrass product and the other is via a Mellin integral. Certain properties are clear from one definition but not from the other, although the two definitions give the same function. We start with the former approach which involves more algebraic properties of

the gamma function. For this, we introduce functions which have only negative zeros. The simplest function of this kind is

$$G(z) = \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n} \qquad \ldots(1)$$

Obviously, the function $G(-z)$ has only positive zeros and moreover, from (1), we note that

$$zG(z)\,G(-z) = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) = \frac{\sin \pi z}{\pi} \qquad \ldots(2)$$

Now, zeros of

$$G(z-1) = \prod_{n=1}^{\infty}\left(1+\frac{z-1}{n}\right)e^{-(z-1)/n} \qquad \ldots(3)$$

are given by $z = 1 - n$, $n = 1, 2, \ldots$

Thus $G(z - 1)$ has the same zeros as $G(z)$, and in addition, a simple zero at $z = 0$. By Weierstrass factorization theorem, we can write

$$G(z-1) = z^{g(z)}\,G(z) \qquad \ldots(4)$$

where $g(z)$ is an integral function.
Using (1) and (2) in (3), we get

$$\prod_{n=1}^{\infty}\left(\frac{z-1+n}{n}\right)e^{-(z-1)/n} = z\,e^{g(z)}\prod_{n=1}^{\infty}\left(\frac{z+n}{n}\right)e^{-z/n} \qquad \ldots(5)$$

For determining $g(z)$, we take logarithmic derivative of both sides of (5). This gives the equation

$$\sum_{n=1}^{\infty}\left(\frac{1}{z-1+n}-\frac{1}{n}\right) = \frac{1}{z}+g'(z)+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right) \qquad \ldots(6)$$

Replacing n by n + 1, the series on L.H.S. of (6) can be written as

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n+1}\right) &= \frac{1}{z}-1+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n+1}\right)\\ &= \frac{1}{z}-1+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right)+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)\\ &= \frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right), \text{where } \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = 1\end{aligned}$$

Using this in (6), we conclude that $g'(z) = 0$ and thus $g(z)$ is constant which we denote by γ. Hence (4) reduces to

$$G(z-1) = z\,e^{\gamma}\,G(z) \qquad \ldots(7)$$

To determine γ, we put $z = 1$ in (7) to have

$$G(0) = e^{\gamma}G(1)$$

But from (1), $\quad G(0) = 1, \quad G(1) = \prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)e^{-1/n}$

Hence, $$e^{-\gamma} = \prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)e^{-1/n} \qquad \ldots(8)$$

The nth partial product of this infinite product can be written in the form.

$$(n+1)\,e^{-\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)}$$

and we therefore deduce from (8) that

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}-\log n\right)$$

This constant γ is called Euler's constant, and its approximate value is 0.577216.

5.1. Definition : The gamma function is defined by

$$\Gamma(z) = \frac{1}{ze^{\gamma z}G(z)} \qquad \ldots(9)$$

$$= \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1} e^{z/n} \qquad \ldots(10)$$

We observe that Γ(z) is well-defined in the whole complex plane except for z = 0, −1, −2,… which are simple poles of the function. Hence Γ(z) is meromorphic with these poles, but no zeros.

5.2. Properties of Gamma Function. The following are the simple properties of gamma function.

(i) Using (7) in (9), we get

$$\Gamma(z-1) = \frac{e^{-(z-1)\gamma}}{(z-1)G(z-1)} = \frac{e^{-(z-1)\gamma}}{(z-1)ze^{\gamma}G(z)}$$

$$= \frac{1}{z-1}\left(\frac{e^{-\gamma z}}{zG(z)}\right) = \frac{\Gamma(z)}{z-1}$$

i.e. $\Gamma(z) = (z-1)\,\Gamma(z-1)$

or $\boldsymbol{\Gamma(z+1) = z\,\Gamma(z)}$ **…(11)**

(ii) Using (2), (9) and (11), we get

$$\Gamma(1-z)\,\Gamma(z) = -z\Gamma(-z)\,\Gamma(z)$$

$$= \frac{1}{z}\frac{1}{e^{-\gamma z}G(-z)}\cdot\frac{1}{e^{\gamma z}G(z)}$$

$$= \frac{1}{zG(z)G(-z)} = \frac{\pi}{\sin\pi z} \qquad \ldots(12)$$

(iii) Putting $z = \frac{1}{2}$ in (12), we get $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Then putting $z = -\frac{1}{2}$ in (11), we find

$$\Gamma\left(\frac{1}{2}\right) = -\frac{1}{2}\,\Gamma\left(-\frac{1}{2}\right) \;\Rightarrow\; \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}.$$

(iv) For z = 1, (8) and (10) result in

$$\Gamma(1) = e^{-\gamma}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-1} e^{1/n} = 1.$$

Further, if n is a positive integer, then using (11) repeatedly, we get

$$\Gamma(2) = 1.\,\Gamma(1) = 1,\;\; \Gamma(3) = 2\,\Gamma(2) = 2 = 1.\,2$$

$$\Gamma(4) = 3\Gamma(3) = 3.2 = 1.2.3 \text{ and so on.}$$

Thus, finally, we get

$$\Gamma(n+1) = \lfloor\underline{n}$$

Thus gamma function can be considered as a generalization of the factorial function.

(v) Taking logarithm of both sides of (10) and then differentiating, we get

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\left(-\frac{1}{z+n} + \frac{1}{n}\right)$$

Differentiating again, we get

$$\frac{d}{dz}\left[\frac{\Gamma'(z)}{\Gamma(z)}\right] = \frac{1}{z^2} + \sum_{n=1}^{\infty}\frac{1}{(z+n)^2} = \sum_{n=0}^{\infty}\frac{1}{(z+n)^2}$$

Similarly, replacing z by 2z in (10), we find that

$$\frac{d}{dz}\left[\frac{\Gamma'(2z)}{\Gamma(2z)}\right] = 2\sum_{n=0}^{\infty}\frac{1}{(2z+n)^2}$$

Thus, we have

$$\frac{d}{dz}\left[\frac{\Gamma'(z)}{\Gamma(z)}\right] + \frac{d}{dz}\left[\frac{\Gamma'\left(z+\frac{1}{2}\right)}{\Gamma\left(z+\frac{1}{2}\right)}\right]$$

$$= \sum_{n=0}^{\infty}\frac{1}{(z+n)^2} + \sum_{n=0}^{\infty}\frac{1}{\left(z+n+\frac{1}{2}\right)^2}$$

$$= 4\left[\sum_{n=0}^{\infty}\frac{1}{(2z+2n)^2} + \sum_{n=0}^{\infty}\frac{1}{(2z+2n+1)^2}\right]$$

$$= 4\sum_{m=0}^{\infty}\frac{1}{(2z+m)^2} \qquad |\text{Even } (2n) + \text{odd } (2n+1)\,|$$

$$= 2\frac{d}{dz}\left[\frac{\Gamma'(2z)}{\Gamma(2z)}\right]$$

Thus $$\frac{d}{dz}\left[\frac{\Gamma'(z)}{\Gamma(z)} + \frac{\Gamma'\left(z+\frac{1}{2}\right)}{\Gamma\left(z+\frac{1}{2}\right)}\right] = 2\frac{d}{dz}\left[\frac{\Gamma'(2z)}{\Gamma 2z)}\right] + z$$

which on integration gives

$$\frac{\Gamma'(z)}{\Gamma(z)} + \frac{\Gamma'\left(z+\frac{1}{2}\right)}{\Gamma\left(z+\frac{1}{2}\right)} = 2\frac{\Gamma'(2z)}{\Gamma(2z)} + a$$

Integrating again, we get

$$\log \Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right) = \log \Gamma(2z) + az + b$$

where a and b are arbitrary constants.

Thus $$\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right) = e^{az+b}\,\Gamma(2z) \qquad \ldots(13)$$

To determine a and b, we substitute $z = \frac{1}{2}$ and z = 1 so that we make use of the results $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, $\Gamma(1) = 1$ to have

$$\Gamma\left(\frac{1}{2}\right)\Gamma(1) = e^{(a/2)+b}\,\Gamma(1),\;\; \Gamma(1)\,\Gamma\left(\frac{3}{2}\right) = e^{a+b}\,\Gamma(2)$$

or $\quad e^{(a/2)+b} = \sqrt{\pi},\;\; e^{a+b} = \frac{1}{2}\sqrt{\pi} \qquad [\;\Gamma(z+1) = z\Gamma(z)$

$\Rightarrow \quad \frac{a}{2} + b = \frac{1}{2}\log\pi, \qquad \Rightarrow \Gamma\left(\frac{3}{2}\right) = \Gamma\left(1+\frac{1}{2}\right)$

$\quad a+b = \frac{1}{2}\log\pi - \log 2 \qquad = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}\;\;]$

Solving, these equations, we obtain

$$a = -2\log 2, \qquad b = \frac{1}{2}\log\pi + \log 2$$

Using the values of a and b in (13), we get

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right) \qquad \ldots(14)$$

Formula (14) is known as Lengendre's duplication formula.

(vi) Residue of the gamma function at $z = -n$, $n = 0, 1, ..$ is $\frac{(-1)^n}{\lfloor n}$.

To prove this, we know that the function $\Gamma(z)$ has the simple poles at $z = 0, -1, -2, \ldots, -n, \ldots$

Also, we have $\quad \mathbf{\Gamma(z+1) = z\,\Gamma(z)}$

i.e. $\quad \Gamma(z) = \frac{\Gamma(z+1)}{z}$

By repeated application of this formula, we can write

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{z(z+1)} = \frac{\Gamma(z+3)}{z(z+1)(z+2)} = \ldots = \frac{\Gamma(z+n+1)}{z(z+1)..(z+n)}$$

Hence, the residue at $z = -n$ is given by

$$\begin{aligned}
\text{Res}(z=-n) &= \lim_{z\to -n}(z+n)\,\Gamma(z)\\
&= \lim_{z\to -n}(z+n)\,\frac{\Gamma(z+n+1)}{z(z+1)(z+2)\ldots(z+n)}\\
&= \lim_{z\to -n}\frac{\Gamma(z+n+1)}{z(z+1)(z+2)\ldots(z+n-1)}\\
&= \frac{\Gamma(1)}{-n(-n+1)(-n+2)\ldots(-2)(-1)}\\
&= \frac{1}{(-1)^n\,n(n-1)(n-2)\ldots 2.1} = \frac{(-1)^n}{\lfloor n}.
\end{aligned}$$

5.3. Integral Representation of $\mathbf{\Gamma(z)}$. For Re $z > 0$, we define

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt \qquad \ldots(1)$$

Integral (1) is also called Mellin integral or Mellin transform. We shall show that this function is well defined for Re z = > 0 and the integral (1) is convergent.

For this, let $0 < C < \infty$. If the principle value of t^{z-1} is taken, then

$$\Gamma(z) = \int_0^C e^{-t} t^{z-1} dt + \int_C^\infty e^{-t} t^{z-1} dt \qquad \ldots(2)$$

We note that

$$| e^{-t} t^{z-1} | = \left| \frac{e^{-\frac{t}{2}} t^{z-1}}{e^{\frac{t}{2}}} \right| \le \left| \frac{t^{z-1}}{e^{\frac{t}{2}}} \right| \to 0 \qquad \text{as } t \to \infty$$

Hence, for large values of C, the second integral in (2) converges, since

$$\left| \int_C^\infty e^{-t} t^{z-1} dt \right| \le \int_C^\infty e^{-t/2} dt = 2e^{-C/2}$$

For the first integral, we find that

$$\left| \int_0^C e^{-t} t^{z-1} dt \right| \le \int_0^C | t^{z-1} | \, dt \le \int_0^C t^{\operatorname{Re} z-1} dt < \infty, \ \operatorname{Re} z > 0.$$

Thus we conclude that $\Gamma(z)$ is well defined for Re $z > 0$.
Moreover $\Gamma(z)$ is analytic in this domain.

We have another integral form of gamma function which is obtained by substituting $t = x^2$ in (1). Thus

$$\Gamma(z) = 2 \int_0^\infty e^{-x^2} x^{2z-1} \, dx, \qquad \operatorname{Re} z > 0 \qquad \ldots(3)$$

Remark : The integral (1) diverges for Re $z \le 0$ which causes limitation of the gamma function as an analytic function in the domain Re $z > 0$. However, we may extend this domain of definition to the entire complex plane except the point $z = 0, -1, -2, \ldots$

Using integration by parts, we get from (1)

$$\Gamma(z + 1) = \int_0^\infty e^{-t} t^z \, dt = z \int_0^\infty e^{-t} t^{z-1} \, dt = z \, \Gamma(z)$$

or

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}, \qquad \operatorname{Re}(z + 1) > 0$$

The repeated application of this formula gives

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)(z+2)...(z+n)}, \ n = 1, 2, 3, .. \qquad \ldots(4)$$

If n is chosen such that Re (z + n + 1) > 0, then (4) represents an analytic function for all z except 0, –1, –2,…

5.4. Analytic Continuation of Γ(z). By definition, we have

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt \qquad (\operatorname{Re} z > 0) \qquad \ldots(1)$$

From this we are not able to infer anything about $\Gamma(z)$ on the imaginary axis or to the left on it. Now, let us consider

$$F(z) = \int_C e^{-w} \, (-w)^{z-1} \, dw \qquad \ldots(2)$$

where C is the contour consisting of the real axis from ∞ to δ ($\delta > 0$), the circle $| w | = \delta$ described in the positive direction and the real axis from δ to ∞ again. The many-valued function $([w)^{z-1} = e^{(z-1)\log(-w)}$ is made definite by taking log $(-w)$ to be real at $w = -\delta$. The contour integral (2) is then uniformly convergent in any finite region of the z-plane, since the question of convergence now need to be settled only at infinity. But since $|w^{z-1}| = w^{x-1}$, where $z = x + iy$, the integral is uniformly convergent for z in any finite region throughout which $\operatorname{Re} z \geq a > 0$. Thus F(z) is regular for all finite values of z.

Now, if $w = \rho\, e^{i\phi}$, then

$\log([-w) = \log \rho = i(\phi - \pi)$, for $w \in C$ i.e. on the contour. The integral along the real axis therefore gives

$$\int_{\delta}^{\infty} \{-e^{-\rho+(z-1)(\log \rho - i\pi)} + e^{-\rho(z-1)(\log \rho + i\pi)}\}\, d\phi$$

$$= -2i \sin \pi z \int_{\delta}^{\infty} e^{-\rho}\, \rho^{z-1}\, d\rho$$

On the circle of radius δ,

$$\begin{aligned} |(-w)^{z-1}| &= |e^{(z-1)\log \delta + i(\phi - \pi)}| && \mid O(1) \text{ means a function which} \to 0 \\ &= e^{(x-1)\log \delta - y(\phi - \pi)} && \mid O(1) \text{ means a bounded function} \\ &= O(\delta^{x-1}) && \mid O(x) \text{ means a function of order x.} \end{aligned}$$

The integral round the circle of radius δ is thus $O(\delta^x) = O(1)$ as $\delta \to 0$ if $x > 0$. Hence, letting $\delta \to 0$, we get

$$F(z) = -2i \sin \pi z \int_{0}^{\infty} e^{-\rho}\, \rho^{z-1}\, d\rho = -2i \sin \pi z\, \Gamma(z), \operatorname{Re} z > 0$$

i.e.
$$\Gamma(z) = \frac{F(z)}{-2i \sin \pi z} = \frac{1}{2}\, i\, F(z) \operatorname{cosec} \pi z, \quad \operatorname{Re} z > 0.$$

Now, $\frac{1}{2}\, i\, F(z) \operatorname{cosec} \pi z$ is an analytic function of z except possibly at the poles of cosec πz i.e. $z = 0, \pm 1, \pm 2, \ldots$ is equal to Γ(z) for Re z >0. Thus it is the analytic continuation of Γ(z) for the domain defined by the points on the imaginary axis and points to the left of it i.e. this function can be taken as analytic continuation of Γ(z) over the whole z-plane. But we already know that Γ(z) is regular at $z = 1, 2, \ldots$ Hence the only possible poles are $z = 0, -1, -2, \ldots$

5.5. Stirling Formula. In most cases where the gamma function can be applied, it is essential to have some information on the behaviour of Γ(z) for very large values of z. Such asymptotic character of Γ(z) is given by Stirling formula. There are many proofs of this formula. We use the well known method of comparing a sum $\Sigma\phi(n)$ with the corresponding integral $\int \phi(t)\, dt$.

By definition, we have

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}$$

$$\Rightarrow \quad \log \Gamma(z) = \sum_{n=1}^{\infty} \left\{\frac{z}{n} - \log\left(1 + \frac{z}{n}\right)\right\} - \gamma z - \log z \qquad \ldots(1)$$

where we take the principal value of each logarithm. If [u] denotes the greatest integer not greater than u, then we have

$$\int_0^N \frac{[u] - u + 1/2}{u + z} du = \sum_{n=0}^{N-1} \int_n^{n+1} \frac{n - u + (1/2)}{u + z} du$$

$$= \sum_{n=0}^{N-1} \int_n^{n+1} \left(\frac{n+\frac{1}{2}+z}{u+z} - 1 \right) du = \sum_{n=0}^{N-1} \left[\left(n+\frac{1}{2}+z \right) \log(u+z) - u \right]_n^{n+1}$$

$$= \sum_{n=0}^{N-1} \left[\left(n+\frac{1}{z}+z \right) \log(n+1+z) - \left(n+\frac{1}{2}+z \right) \log(n+z) - 1 \right]^n$$

$$= \sum_{n=0}^{N-1} \left(n+\frac{1}{2}+z \right) \log(n+1+z) - \sum_{n=0}^{N-1} \left(n+\frac{1}{2}+z \right) \log(n+z) - N$$

$$= \sum_{n=1}^{N} \left(n-\frac{1}{2}+z \right) \log(n+z) - \sum_{n=0}^{N-1} \left(n+\frac{1}{2}+z \right) \log(n+z) - N$$

(Replacing n + 1 by n in 1st series)

$$= \sum_{n=1}^{N-1} \left(n-\frac{1}{2}+z \right) \log(n+z) - \left(N-\frac{1}{2}+z \right) \log(N+z)$$

$$- \left(z+\frac{1}{2} \right) \log z - \sum_{n=1}^{N-1} \left(n+\frac{1}{2}+z \right) \log(n+z) - N$$

$$= - \sum_{n=1}^{N-1} \log(n+z) - \left(z+\frac{1}{2} \right) \log z + \left(N-\frac{1}{2}+z \right) \log(N+z) - N \qquad \ldots(2)$$

Now, $-\sum_{n=1}^{N-1} \log(n+z) = -\sum_{n=1}^{N-1} \left\{ \log n + \log\left(1+\frac{z}{n}\right) \right\} = -\sum_{n=1}^{N-1} \log\left(1+\frac{z}{n}\right) - \log \lfloor\!\underline{N-1}$

$$= \sum_{n=1}^{N-1} \left\{ \frac{z}{n} - \log\left(1+\frac{z}{n}\right) \right\} - \log \lfloor\!\underline{N-1} - z\left(1+\frac{1}{2}+\ldots+\frac{1}{N-1}\right)$$

Thus (2) becomes

$$\int_0^N \frac{[u]-u+\frac{1}{2}}{u+z} du = \sum_{n=1}^{N-1} \left\{ \frac{z}{n} - \log\left(1+\frac{z}{n}\right) \right\} - \log \lfloor\!\underline{N-1}$$

$$- z\left(1+\frac{1}{2}+\ldots+\frac{1}{N-1}\right) - \left(z+\frac{1}{2}\right) \log z$$

$$+ \left(N-\frac{1}{2}+z\right) \log (N+z) - N \qquad \ldots(3)$$

Now, we shall show that if n is a positive integer, then

$$\log \Gamma(n) = \log \lfloor\!\underline{n-1} = \left(n-\frac{1}{2}\right) \log n - n + C + O(1) \qquad \ldots(4)$$

where C is a constant.

To prove this, we observe that

$$\log (\lfloor\!\underline{n-1}) = \sum_{v=1}^{n-1} \log v$$

Now $$\int_{v-\frac{1}{2}}^{v+\frac{1}{2}} \log t \, dt = \int_0^{1/2} \{\log(v+t) + \log(v-t)\} \, dt$$

$$= \int_0^{1/2} \left\{\log v^2 + \log\left(1 - \frac{t^2}{v^2}\right)\right\} dt = \log v + C_v$$

i.e. $$\log v = \int_{v-\frac{1}{2}}^{v+\frac{1}{2}} \log t \, dt - C_v, \text{ where } C_v = O(1/v^2).$$

Hence $$\log \Gamma(n) = \log(\underline{|n-1}) = \int_{1/2}^{n-\frac{1}{2}} \log t \, dt - \sum_{v=1}^{n-1} C_v$$

$$= \left(n - \frac{1}{2}\right) \log\left(n - \frac{1}{2}\right) - \left(n - \frac{1}{2}\right) - \frac{1}{2}\log\frac{1}{2}$$

$$+ \frac{1}{2} - \sum_{v=1}^{\infty} C_v + 0(1) \qquad (\textstyle\int \log x \, dx = x \log x - x\}$$

$$= \left(n - \frac{1}{2}\right)\log n - n + C + 0(1), \text{ where C is constant.}$$

$$\log\left(\left(n - \frac{1}{2}\right) = \log n \left(1 - \frac{1}{2n}\right) = \log n + \text{kig}\left(1 - \frac{1}{2n}\right)\right)$$

Now, using (4) and the relations

$$1 + \frac{1}{2} + ... + \frac{1}{N-1} = \log N + \gamma + 0(1)$$

and $$\log (N + z) = \log N + \frac{z}{N} + O\left(\frac{1}{N^2}\right)$$

in equation (3) and then making $N \to \infty$, we obtain from (1) that

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log 2\pi + \int_0^{\infty} \frac{[u] - u + \frac{1}{2}}{u + z} du \qquad \dots(5)$$

where the integral is known as the error term. Writing

$$\phi(u) = \int_0^u \left([w] - w + \frac{1}{2}\right) dw,$$

we find that $\phi(u)$ is bounded since

$\phi(n + 1) = \phi(n)$ for integer values of n.

Thus,

$$\int_0^{\infty} \frac{[u] - u + \frac{1}{2}}{u + z} du = \int_0^{\infty} \frac{\phi'(u)}{u + z} du = \int_0^{\infty} \frac{\phi(u)}{(u + z)^2} du \qquad \text{(Integ. by parts)}$$

$$= O\left\{\int_0^{\infty} \frac{du}{u^2 + r^2 - 2ur\cos\delta}\right\} = O\left(\frac{1}{r}\right),$$

where r = | z | .

The usefulness of the error term is that it tends to zero uniformly in every sector of complex numbers $z = re^{i\theta}$ such that $-\pi + \delta \le \theta \le \pi - \delta$, $0 < \delta < \pi$.

Hence (5) assumes the familiar form of Stirling formula for complex values of z, as

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log 2\pi + O\left(\frac{1}{|z|}\right), \text{ as } |z| \to \infty$$

which can also be expressed in the form

$$\Gamma(z) \sim z^{z-(1/2)}\, e^{-z}\, \sqrt{2\pi}, \qquad |z| \to \infty.$$

5.6. Riemann's Zeta-function. Riemann's Zeta-function $\zeta(z)$ is defined by series $\sum_{n=1}^{\infty} n^{-z}$ i.e.

$$\begin{aligned}\zeta(z) &= 1^{-z} + 2^{-z} + 3^{-z} + \ldots + n^{-z} + \ldots \\ &= \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \ldots + \frac{1}{n^z} + \ldots\end{aligned}$$

which is known to converge uniformly and absolutely in any bounded closed domain to the right of the line Re $z = 1$. Hence the function $\zeta(z)$ is an analytic function, regular when Re $z > 1$.

We proceed to show that analyticity of ζ (z) can be extended to the region where Re z > 0, by the function

$$(1 - 2^{1-z})\,\zeta(z) \qquad = 1^{-z} - 2^{-z} + 3^{-z} - 4^{-z} + \ldots$$

From this we shall be led to the result that the only singularity of ζ (z) in the right half plane Re $z > 0$ is a simple pole at the point $z = 1$ and residue of ζ (z) at this pole is 1.

For Re $z > 1$, we have

$$\begin{aligned}(1 - 2^{1-z})\,\zeta(z) &= \sum_{n=1}^{\infty} n^{-z}(1 - 2^{1-z}) = \sum_{n=1}^{\infty} n^{-z} - 2\sum_{n=1}^{\infty}(2n)^{-z} \\ &= 1^{-z} - 2^{-z} + 3^{-z} - 4^{-z} + \ldots\end{aligned}$$

the re-ordering of the terms being justified by absolute convergence.

It can be shown that this last mentioned series in uniformly convergent in any bounded closed domain D in which Re $z \geq \delta > 0$. For this we use the following criterion (Hardy's Test) for the uniform convergence of a series of terms which are functions of a complex variable. According to this criterion the series $\Sigma a_n(z)\, u_n(z)$ is uniformly convergent in a bounded closed domain D, if in D

(i) $\Sigma a_n(z)$ has uniformly bounded partial sums

(ii) $\Sigma|u_n(z) - u_{n+1}(z)|$ is uniformly convergent

(iii) $u_n(z) \to 0$ uniformly, as $n \to \infty$.

Now, let us take

$$a_n(z) = (-1)^n, \qquad u_n(z) = (n+1)^{-z}$$

Then, the partial sums of $\Sigma a_n(z)$ are alternately 1 and 0, so that condition (i) of Hardy's test is satisfied. Condition (iii) is also satisfied, since

$$|u_n(z)| = (n+1)^{-\text{Re } z} \leq (n+1)^{-\delta},$$

so that $u_n(z) \to 0$ uniformly in the bounded closed domain D in which Re $z \geq \delta > 0$.

For the condition (ii), we observe that

$$\begin{aligned}u_n(z) - u_{n+1}(z) &= (n+1)^{-z} - (n+2)^{-z} \\ &= z\int_{n+1}^{n+2} t^{-z-1}\, dt\end{aligned}$$

Hence

$$|u_n(z) - u_{n+1}(z)| \leq |z| \int_{n+1}^{n+2} |t^{-z-1}|\, dt$$

$$\leq |z| \int_{n+1}^{n+2} t^{-\delta-1}\, dt$$
$$< |z|\,(n+1)^{-\delta-1}$$

Since z is bounded in D, by Weierstrass's M-test, it follows that $\Sigma|u_n(z) - u_{n+1}(z)|$ converges uniformly. Thus it follows finally that the series

$$1^{-z} - 2^{-z} + 3^{-z} - 4^{-z} + \ldots$$

converges uniformly in D. Hence its sum function n(z) is a regular function in the region defined by Re z > 0. But for Re z > 1, we have

$$\eta(z) = (1 - 2^{1-z})\,\zeta(z) \qquad \ldots(1)$$

Equation (1) provides the analytic continuation of ζ (z) into the region $0 < \text{Re } z \leq 1$. Now $1 - 2^{1-z}$ has simple zeros at the points z given by $(1 - z) \log 2 = 2K\pi i$, where K is any integer or zero.

From (1), $$\zeta(z) = \frac{\eta(z)}{1-2^{1-z}}$$

and thus it follows that a point of the above mentioned set is a simple pole of $\zeta(z)$, provided $\eta(z)$ does not vanish there and that $\zeta(z)$ has no singularity in the right half plane Re z > 0. Since $\eta(1) = \log 2 \neq 0$, the point z = 1 is a simple pole of $\zeta(z)$. The residue at this pole is

$$\begin{aligned}
\text{Res}(z = 1) &= \lim_{z\to1} (z-1)\,\zeta(z) \\
&= \lim_{z\to1} \frac{(z-1)\eta(z)}{1-2^{1-z}} \\
&= \lim_{z\to1} \frac{\eta(z)}{\dfrac{2^{1-z}-1}{1-z}} = \frac{\eta(1)}{\log 2} = 1
\end{aligned}$$

It is observed that no other zero if $1 - 2^{1-z}$ is a pole of $\zeta(z)$. Hence we conclude that the only singularity of $\zeta(z)$ in Re z > 0 is a simple pole at z = 1 and residue at this pole is 1.

5.7. Analytic Continuation of ζ (z). We observe that

$$\begin{aligned}
\int_0^\infty \frac{w^{z-1}}{e^w-1}\,dw &= \int_0^\infty \{\Sigma w^{z-1} e^{-nw}\}\,dw \\
&= \Sigma \int_0^\infty w^{z-1} e^{-nw}\,dw = \Sigma\, n^{-z} \int_0^\infty v^{z-1} e^{-v}\,dv \\
&= \Sigma n^{-z}\,\Gamma(z) = \Gamma(z)\,\zeta(z)
\end{aligned}$$

the operation being justified for Re z > 1.
Hence, we get

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{w^{z-1}}{e^w-1}\,dw, \ \text{Re } z > 1 \qquad \ldots(1)$$

In the same manner, it can be shown that if Re z > 1,

$$\zeta(z) = \frac{-1}{2i\sin\pi z\,\Gamma(z)} \int_C \frac{(-w)^{z-1}}{e^w-1}\,dw, \qquad \ldots(2)$$

where the contour C starts at infinity on the positive real axis, encircles the origin once in the positive direction, but excludes all the poles of $\dfrac{1}{(e^w-1)}$ other than w = 0 i.e. the points $w = \pm 2i\pi, \pm 4i\pi, \ldots$.

Now, using the functional equation of the gamma function, i.e.

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

we may write (2) in the form

$$\zeta(z) = \frac{i\Gamma(1-z)}{2\pi}\int_C \frac{(-w)^{z-1}}{e^w - 1}\,dw \qquad \ldots(3)$$

As in case of the gamma function, this contour integral is an integral function of z. The formula (3) therefore provides the analytic continuation of $\zeta(z)$ over the whole plane. The only possible singularities of $\zeta(z)$ are the poles of $\Gamma(1-z)$ i.e. $z = 1, 2, \ldots$ But we know that $\zeta(z)$ is regular at $z = 2, 3, \ldots$ Hence the only possible pole is at $z = 1$. The residue at this simple pole is 1, since for $z = 1$, the contour integral in (3) is

$\int_C \frac{dw}{e^w - 1} = 2\pi i$, by theorem of residues, and $\Gamma(1-z)$ has a simple pole at $z = 1$, with residue -1.

5.8. Riemann's Functional Equation. The function $\zeta(z)$ satisfied the functional equation

$$\zeta(1-z) = 2^{1-z}\,\pi^{-z}\cos\frac{1}{2}\pi z\,\Gamma(z)\,\zeta(z)$$

To prove this, we consider the formula

$$\zeta(z) = \frac{-1}{2i\sin\pi z\Gamma(z)}\int_C \frac{(-w)^{z-1}}{e^w - 1}\,dw \qquad \ldots(1)$$

and let z have any value and we deform C into the contour C_n consisting of the square with centre at the origin and sides parallel to the axis, of length $(4n+2)\pi$, together with the positive real axis from $(2n+1)\pi$ to ∞. In this process, we pass over the poles of the integrand at the points $w = 2i\pi, 4i\pi, \ldots, 2ni\pi$ and $-2i\pi, -4i\pi, \ldots, -2ni\pi$. The residue at $2Ki\pi(K > 0)$ is

$$e^{(z-1)(\log 2K\pi - i\pi/2)} = (2K\pi)^{z-1}\, i\, e^{-\frac{1}{2}i\pi z}$$

and the residue at $-2Ki\pi$ $(K > 0)$ is

$$e^{(z-1)(\log 2K\pi + i\pi/2)} = -(2K\pi)^{z-1}\, i\, e^{\frac{1}{2}i\pi z}$$

The sum of these two residues is

$$(2K\pi)^{z-1} 2\sin\frac{1}{2}\pi z.$$

Hence formula (1) gives

$$\sin \pi z\Gamma(z)\, g(z) = -\frac{1}{2i}\int_{C_n} \frac{(-w)^{z--1}}{e^w - 1} + 2\pi\sin\frac{1}{2}\pi z\sum_{K=1}^{n}(2K\pi)^{z-1} \qquad \ldots(2)$$

$$\left(\int_C = \int_{C_n} - \int_{\text{poles}} = \int_{C_n} - 2\pi i\sum \text{Res}\right)$$

Now, let Re $z < 0$. On the square

$$|(-w)^{z-1}| = e^{(x-1)}\log|w| - y\arg(-w) = O(n^{x-1})$$

and $\quad |(e^w - 1)^{-1}| \le K$, a positive finite constant.

Moreover, the length of the square is $O(n)$, as $n \to \infty$. Hence, this part of the integral is $O(n^x)$, and hence it tends to zero, since $x < 0$. The remaining part of the integral obviously tends to zero. Hence letting $n \to \infty$, we have from (2)

$$\sin \pi z\Gamma(z)\, \zeta(z) = 2\pi \sin \frac{1}{2}\pi z(2\pi)^{z-1} \sum_{K=1}^{\infty} K^{z-1}$$

$$\Rightarrow \quad 2 \sin \frac{1}{2}\, \pi z \cos \frac{1}{2}\pi z\, \Gamma(z)\, \zeta(z) = 2\pi \sin \frac{1}{2}\pi z\, (2\pi)^{z-1}\, \zeta(1-z)$$

$$\Rightarrow \quad \cos \frac{1}{2}\pi z\, \Gamma(z)\, \zeta(z) = \pi(2\pi)^{z-1}\, \zeta(1-z)$$

$$\Rightarrow \quad \zeta(1-z) = 2^{1-z}\, \pi^{-z} \cos \frac{1}{2}\pi z\, \Gamma(z)\, \zeta(z) \qquad \ldots(3)$$

which is the required function equation and provides a relationship between $\zeta(z)$ and $\zeta(1-z)$.

We have proved the functional equation for Re z < 0. By analytic continuation, it holds for all z.

Remark : An equivalent form of the functional equation (3) is obtained by using the identity of the gamma function i.e. $\Gamma(z)\,\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$ in (3) to get

$$\zeta(z) = 2^{z}\, \pi^{z-1} \sin \frac{1}{2}\pi z\, \Gamma(1-z)\, \zeta(1-z)$$

6. Runge's Theorem

Before proving the theorem we state some basic concepts which will be used in sequel.

6.1. Definition : A metric space (X, d) is called **connected** if the only subsets of X which are both open and closed are ϕ and X.

A maximal connected subset of (X, d) is called a **component** of X. We note that

(i) If A and B are connected subsets of (X, d) such that $A \cap B \neq \phi$ then $A \cup B$ is also connected.

(ii) If A is a component of X then A is closed in X.

(iii) If G is open in ∀ then components of G are open.

(iv) A subset A of a metric space is open iff int (A) = A.

Definition : If A is subset of a metric space and $x \in X$ then the distance from x to the set A, denoted by d(x, A) is defined as

$$d(x, A) = \inf\{d(x, a) : a \in A\}$$

6.2. Definition : Let A be a subset of a metric space (X, d). Then the boundary of A denoted by ∂A, is defined as the set of all points of X which are neither in the interior of A nor in the interior of (X – A). We note that

(i) $\partial A = \overline{A} \cap \overline{X - A}$

(ii) $\text{int}(A) = \overline{A} - \partial A$

6.3. The extended plane. The set $\forall \cup \{\infty\} \equiv \forall_\infty$ is called extended plane. If we define

$$d(z, z') = \frac{2\,|\,z - z'\,|}{[(1 + |\,z\,|^2)(1 + |\,z'\,|^2)]^{1/2}} \qquad \text{for } z, z' \in \forall$$

and

$$d(z, \infty) = \frac{2}{(1 + |\,z\,|^2)^{1/2}} \text{ for } z \in \forall$$

then $(\forall_\infty, d)$ is a metric space.

6.4. Definition : A path in a region $G \subset \forall$ is a continuous function $r : [a, b] \to G$ for some interval $[a, b]$ in R. If $r : [a, b] \to \forall$ is a path then the set $\{r(t) : a \le t \le b\}$ is called the trace of γ and is denoted by $\{\gamma\}$.

Note that trace of a path is always a compact set. We have the following two useful results :

Lemma (a). Let K be a compact subset of the region G ; then there are straight line segments $\gamma_1, \gamma_2, \ldots, \gamma_n$ in G–K such that for every function f in H(G),

$$f(z) = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(w)}{w - z} dw$$

for all z in K.

The line segments form a finite number of closed polygons.

Lemma (b). Let γ be a rectifiable curve and let K be a compact set such that $K \cap \{\gamma\} = \phi$. If f is a continuous function on $\{\gamma\}$ and $\in > 0$ then there is a rational function R(z) having all its poles on $\{\gamma\}$ and such that

$$\left| \int_{\gamma} \frac{f(x)}{w - z} - R(z) \right| < \in \quad \text{for all z in K.}$$

6.5. Theorem (Runge Theorem). Let K be a compact subset of $\forall$ and let E be a subset of $\forall_\infty - K$ that meets each component of $\forall_\infty - K$. If f is analytic in an open set containing K and $\in > 0$ then there is a rational function R(z) whose only poles lie in E and such that

$$|f(z) - R(z)| < \in \quad \textbf{for all z in K.}$$

Proof : For f, $g \in C(K, \forall)$, define

$$\rho(f, g) = \sup\{|f(z) - g(z)| : z \,\varepsilon\, K\}$$

Then ρ is a metric on $C(K, \forall)$. Also $\rho(f_n, f) \to 0$ iff $<f_n>$ converges uniformly to f on K. So Runge's theorem says that if f is analytic on a neighbourhood of K and $\in > 0$ then there is a rational function R(z) with poles in E such that $\rho(f, R) < \in$. By taking $\in = \frac{1}{n}$, it is seen that we want to find a sequence of rational functions $\{R_n(z)\}$ with poles in E such that $\rho(f, R_n) \to 0$, that is, R_n converge uniformly to f on K.

Let B(E) be the collection of all functions f in $C(K, \forall)$ such that there is a sequence $\{R_n\}$ of rational functions with poles in E such that $\{R_n\}$ converges uniformly to f on K. Then B(E) is a closed subalgebra of $C(K, \forall)$ that contains every rational function with a pole in E. To say B(E) is an algebra we mean that if f and g are in B(E) and $\alpha \,\varepsilon\, \forall$ then αf, $f + g$, $fg \,\varepsilon\, B(E)$. We now prove two **lemmas.**

Lemma 1. Let V and U be open subsets of $\forall$ with $V \subset U$ and $\partial V \cap U = \phi$. If H is a component of U and $H \cap V \ne \phi$ then $H \subset V$.

Proof. Let $a \,\varepsilon\, H \cap V$ and let G be the component of V such that $a \,\varepsilon\, G$. Then $H \cup G$ is connected since $H \cap G \ne \phi$ and $H \cup G \subset U$. Since H is a component of U so $H \cup G \subseteq H$. This gives $G \subset H$. But $\partial G \subset \partial V$ as $G \subset V$. So $\partial G \cap H = \phi$ as $\partial G \cap H \subset \partial G \cap U \subset \partial V \cap U = \phi$. Now V is open in $\forall$ and G is component of V so G is open

$\therefore \quad G^\circ = G$

Also $\quad G^\circ = \overline{G} - \partial G$

$\therefore \quad G = \overline{G} - \partial G$

Taking complements,

$$\forall - G = \forall - (\overline{G} - \partial G)$$
$$= \forall - [(\overline{G} \cap (\forall - \partial G)]$$
$$= (\forall - \overline{G}) \cup \partial G.$$

$\therefore$ $$H - G = H \cap (\forall - G)$$
$$= H \cap [(\forall - \overline{G}) \cup \partial G]$$
$$= H \cap [(\forall - \overline{G}) \qquad [\because H \cap \partial G = \phi)$$

As $\forall - \overline{G}$ is open in $\forall$ so $H \cap (\forall - \overline{G})$ is open in H that is, $H - G$ is open in H.

Also G is open in $\forall$ so $\forall - G$ is closed in $\forall$.

$\therefore$ $H \cap (\forall - G)$ is closed in H

$\Rightarrow$ **$H - G$ is closed in H.**

Now $H - G$ is open as well closed in H. Since H is connected and $G \neq \phi$, we must have $H - G = \phi$. That is, H = G as $G \subset V$, we have $H \subset V$.

Lemma (2). If a ε $\forall - K$ then $(z - a)^{-1}$ ε B(E).

Proof. Case 1. When $\infty \notin E$.

Let $U = \forall - K$ and let $V = \{a \; \varepsilon \; \forall : (z - a)^{-1} \; \varepsilon \; B(E)\}$

Then $E \subset V \subset U$.

We first show "if a ε V and $|b - a| < d(a, K)$ then b ε V". …(1)

Since $|b - a| < d(a, K)$ and $d(a, K) = mf\{|a - k| : k \; \varepsilon \; K|\}$, there exists a number r, $0 < r < 1$, such that

$|b - a| < r\,|z - a|$ for all z in K i.e. $|b - a|\,|z - a|^{-1} < r \;\forall\; z \;\varepsilon\; K$.

But

$$(z - b)^{-1} = [(z - a) - (b - a)]^{-1} = (z - a)^{-1}\left[1 - \frac{b-a}{z-a}\right]^{-1} \qquad \ldots(2)$$

Hence $|b - a|\,|z - a|^{-1} < r < 1$ for all z ε K gives that

$$\left(1 - \frac{b-a}{z-a}\right)^{-1} = \sum_{n=0}^{\infty}\left(\frac{b-a}{z-a}\right)^{n} \qquad \ldots(3)$$

converges uniformly on K by Weierstrass M-test.

Let $Q_n(z) = \sum_{k=0}^{n}\left(\frac{b-a}{z-a}\right)^{k}$ then $(z - a)^{-1} Q_n(z)$ ε B(E) since G ε V and B(E) is an algebra. Since B(E) is closed, equation (1) and the uniform convergence of (3) imply that $(z - b)^{-1}$ ε B(E). So b ε V.

Now (1) implies $B(a ; \delta) \subset V$ where $\delta = d(a, K)$.

$\Rightarrow$ a is interior point of V. But a is any arbitrary point of V So V is open.

We claim that $\partial V \cap U = \phi$.

Let b ε ∂V, then there exists a sequence $\{a_n\}$ in V such that $b = \lim a_n$. Since V is open so $\partial V \cap V = \phi$. Thus $b \notin V$. So (1) implies

$$|b - a_n| \geq d(a_n, K)$$

Letting $n \to \infty$, we have $0 \geq d(b, K)$ i.e. $d(b, K) = 0$ or b ε K. Thus $\partial V \cap U = \phi$.

$[\because K = \forall - U]$

Let H be a component of $U = \forall - K$. Then by hypothesis,

$$H \cap E \neq \phi$$

So $\qquad H \cap V \neq \phi \qquad [\because E \subset V]$

BY Lemma (1), H ⊂ V. But H is arbitrary so U ⊂ V.

$\therefore \qquad V = U$

Case 2. $\infty \in E$.

Let d be the metric on $\forall_\infty$. Choose a_0 in the unbounded component of $\forall - K$ such that $d(a_0, \infty) \leq \frac{1}{2} d(\infty, K)$ and

$$|a_0| > 2 \max \{|z| : z \in K\}$$

Let $\quad E_0 = (E - \{\infty\}) \cup \{a_0\}$.

Then E_0 meets each component of $\forall_\infty - K$. If $a \in \forall - K$. Case 1 gives that $(z - a)^{-1} \in B(E_0)$. If we show that $(z - a_0)^{-1} \in B(E)$ then it will follow that $B(E_0) \subset B(E)$ and so $(z - a)^{-1} \in B(E)$ for each a in $\forall - K$. Now $\left|\frac{z}{a_0}\right| \leq \frac{1}{2}$ for all z in K so

$$\frac{1}{z - a_0} = -\frac{1}{a_0\left(1 - \frac{z}{a_0}\right)} = -\frac{1}{a_0}\sum_{n=0}^{\infty}\left(\frac{z}{a_0}\right)^n$$

Converges uniformly on K. So $Q_n(z) = -a_0^{-1} \sum_{k=0}^{n}\left(\frac{z}{a_0}\right)^k$ is a polynomial and Q_n converges uniformly to $(z - a_0)^{-1}$ on K. Since Q_n has its only pole at ∞, $Q_n \in B(E)$. Thus $(z - a_0)^{-1} \in B(E)$.

Proof of main theorem. If f is analytic on an open set G and $K \subset G$ then for each $\in > 0$, there exists a rational function R(z) with poles in $\forall - K$ such that

$$|f(z) - R(z)| < \in \qquad \text{for all z in K, by lemma (a) and (b).}$$

Since B(E) is an algebra, lemma (2) implies that $R \in B(E)$.

Hence the result.

Corollary. Let G be an open subset of the plane and let E be a subset of $\forall_\infty - G$ such that E meets every component of $\forall_\infty - G$. Let R(G, E) be the set of rational functions with poles in E and consider R(G, E) as a subspace of H(G). If $f \in H(G)$ then there is a sequence $\{R_n\}$ in R(G, E) such that $f = \lim R_n$. That is, R(G, E) is dense in H(G).

7. Mittag Leffler's Theorem

Let G be an open subset of $\forall$ and let $\{a_k\}$ be a sequence of distinct points in G such that $\{a_k\}$ has no limit point in G. Let $\{S_k(z)\}$ be the sequence of rational functions given by

$$S_k(z) = \sum_{j=1}^{m_k} \frac{A_{jk}}{(z - a_k)^j}$$

where m_k is some positive integer and $A_{1k}, A_{2k}, \ldots, A_{m_k k}$ are arbitrary complex coefficients. Then there is a meromorphic function f on G whose poles are exactly the points $\{a_k\}$ and such that the singular part of f at a_k is $S_k(z)$.

Proof : Since G is open in C there is a sequence $\{K_n\}$ of compact subsets of G such that $G = \bigcup_{n=1}^{\infty} K_n$, $K_n \subset \text{int}(K_{n+1})$ and each component of $\forall_\infty - K_n$ contains a component of $\forall_\infty - G$. Since each K_n is compact and $\{a_k\}$ has no limit point in G, there are only a finite number of points a_k in each K_n.

Define the sets of integers I_n as follows :

$$I_1 = \{k : a_k \ \varepsilon \ K_1\}$$
$$I_n = \{k : a_k \ \varepsilon \ K_n - K_{n-1}\} \text{ for } n \geq 2.$$

Define functions f_n by

$$f_n(z) = \sum_{k \in I_n} S_k(z) \qquad \text{for } n \geq 1.$$

Then f_n is rational and its poles are the points

$$\{a_k : k \ \varepsilon \ I_n\} \subset K_n - K_{n-1}$$

Note that if I_n is empty then let $f_n = 0$.

Since f_n has no poles in K_{n-1} (for $n \geq 2$), it is analytic in a neighbourhood of K_{n-1}. By Runge's theorem, there is a rational function $R_n(z)$ with its poles in $\forall_\infty - G$ such that

$$|f_n(z) - R_n(z)| < \left(\frac{1}{2}\right)^n \qquad \text{for all z in } K_{n-1}.$$

Let

$$f(z) = f_1(z) + \sum_{n=2}^{\infty} [f_n(z) - R_n(z)] \qquad \ldots(1)$$

We claim that f is required meromorphic function.

Let K be a compact subset of $G - \{a_k : k \geq 1\}$. Then K is a compact subset of G. So there is an integer N such that $K \subset K_N$ as $G = \bigcup_{n=1}^{\infty} K_n$.

If $\quad n \geq N$ then

$$|f_n(z) - R_n(z)| < \left(\frac{1}{2}\right)^n \qquad \text{for all z in K.}$$

So by Weierstrass M-test, the series (1) converges uniformly to f on K. Thus f is analytic on $G\{a_k : k \geq 1\}$.

It remains to show that each a_k is a pole of f with singular part $S_k(z)$. For this consider a fixed integer $k \geq 1$. Then there is a number $R > 0$ such that

$$|a_j - a_k| > R \qquad \text{for } j \neq k.$$

Thus

$$f(z) = S_k(z) + g(z) \ \text{ for } 0 < |z - a_k| < R$$

where g is analytic in $B(a_k ; R)$. Hence $z = a_k$ is a pole of f and $S_k(z)$ is its singular part.

UNIT-IV

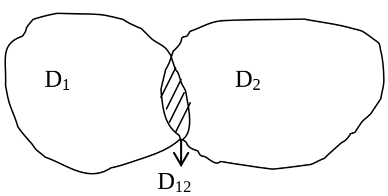
1. Analytic Continuation

From the results regarding zeros of an analytic function, it follows that if two functions are regular in a domain D and if they coincide in a neighbourhood, however small, of any point a of D, or only along a path-segment, however small, terminating in a point a of D, or only at an infinite number of distinct points with a limit-point a in D, then the two functions are identically the same in D. Thus it emerges that a regular function defined in a domain D is completely determined by its values over any such sets of points. This is a very great restraint in the behaviour of analytic functions. One of the remarkable consequences of this feature of analytic functions, which is extremely helpful in studying them, is know as analytic continuation. Analytic continuation is a process of extending the definition of a domain of an analytic function in which it is originally defined i.e. it is a concept which is utilized for making the domain of definition of an analytic function as large as possible.

Let us suppose that two functions $f_1(z)$ and $f_2(z)$ are given, such that $f_1(z)$ is analytic in the domain D_1 and $f_2(z)$ in a domain D_2 We further assume that D_1 and D_2 have a common part D_{12} ($D_1 \cap D_2$). If $f_1(z) = f_2(z)$ in the common part D_{12}, then we say that $f_2(z)$ is the direct analytic continuation of $f_1(z)$ from D_1 into D_2 via D_{12}. Conversely, $f_1(z)$ is the direct analytic continuation of $f_2(z)$ from D_2 into D_1 via D_{12}. Indeed $f_1(z)$ and $f_2(z)$ are analytic continuations of each other. Both $f_1(z)$ and $f_2(z)$ may be regarded as partial representations or elements of one and the same function F(z) which is analytic in the domain $D_1 \cup D_2$, and is defined as

$$F(z) = \begin{cases} f_1(z) \text{ for all } z \in D_1 \\ f_2(z) \text{ for all } z \in D_2 \end{cases}$$

under the condition that $f_1(z) = f_2(z)$ at an infinite set of points with a limit-point in D_{12}



It is observed that for the purpose of analytic continuation, it is sufficient that the domains D_1 and D_2 have only a small arc in common.

1.1. Definition. An analytic function $f(z)$ with its domain of definition D is called a function element and is denoted by (f, D). If $z \in D$, then (f, D) is called a function element of z. Using this notation, we may say that (f_1, D_1) and (f_2, D_2) are direct analytic continuations of each other iff $D_1 \cap D_2 \neq \phi$ and $f_1(z) = f_2(z)$ for all $z \in D_1 \cap D_2$.

Remark. We use the word 'direct' because later on we shall deal with continuation along a curve. i.e. just to distinguish between the two.

1.2. Analytic continuation along a chain of Domain. Suppose we have a chain of function elements (f_1, D_1), $(f_2, D_2), \ldots, (f_k, D_K), \ldots, (f_n, D_n)$ such that D_1 and D_2 have the part D_{12} in common, D_2 and D_3 have the part D_{23} is common and so on. If $f_1(z) = f_2(z)$ in D_{12}, $f_2(z) = f_3(z)$ in D_{23} and so on, then we say that (f_k, D_K) is direct analytic continuation of (f_{K-1}, D_{K-1}). In this way, (f_n, D_n) is analytic continuation of (f_1, D_1) along a chain of domains $D_1, D_2, \ldots, D_n$. Without loss of generality, we may take these domains as open circular discs. Since (f_{K-1}, D_{K-1}) and (f_K, D_K) are direct analytic continuations of each other, thus we have defined an equivalence relation and the equivalence classes are called global analytic functions.

1.3. Complete Analytic Function

Suppose that $f(z)$ is analytic in a domain D. Let us form all possible analytic continuations of (f, D) and then all possible analytic continuations (f_1, D_1), $(f_2, D_2), \ldots, (f_n, D_n)$ of these continuations such that

$$F(z) = \begin{cases} f_1(z) \text{ if } z \in D_1 \\ f_2(z) \text{ if } z \in D_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ f_n(z) \text{ if } z \in D_n \end{cases}$$

Such a function F(z) is called complete analytic function. In this process of continuation, we may arrive at a closed curve beyond which it is not possible to take analytic continuation. Such a closed curve is known as the **natural boundary** of the complete analytic function. A point lying outside the natural boundary is known as the singularity of the complete analytic function. If no analytic continuation of $f(z)$ is possible to a point z_0, then z_0 is a singularity of $f(z)$. Obviously, the singularity of $f(z)$ is also a singularity of the corresponding complete analytic function F(z).

1.4. Theorem (Uniqueness of Direct Analytic Continuation). There cannot be two different direct analytic continuations of a function.

Proof. Let $f_1(z)$ be an analytic function regular in the domain D_1 and let $f_2(z)$ and $g_2(z)$ be two direct analytic continuations of $f_1(z)$ from D_1 into the domain D_2 via D_{12} which is the domain common to both D_1 and D_2. Then by definition of analytic continuation, $f_2(z)$ and $g_2(z)$ are two functions analytic in D_2 such that

$$f_1(z) = f_2(z) \text{ and } f_1(z) = g_2(z)$$

at all points z in D_{12} i.e. $f_2(z) = g_2(z)$ in D_{12}. Thus $f_2(z)$ and $g_2(z)$ are two functions analytic in the domain D_2 such that they coincide in a part D_{12} of D_2. It follows from the **well known result** that they coincide throughout D_2. i.e. $f_2(z) = g_2(z)$ throughout D_2. Hence the result.

1.5. Example. Given the identity $\sin^2 z + \cos^2 z = 1$ holds for real values of z, prove that it also holds for all complex values of z.

Solution. Let $f(z) = \sin^2 z + \cos^2 z - 1$ and let D be a region of the z-plane containing a portion of x-axis (real axis). Since sin z and cos z are analytic in D so $f(z)$ is also analytic in D. Also $f(z) = 0$ on the x-axis. Hence by the well known result, it follows that $f(z) = 0$ identically in D, which shows that $\sin^2 z + \cos^2 z = 1$ for all z in D. Since D is arbitrary, the result holds for all values of z.

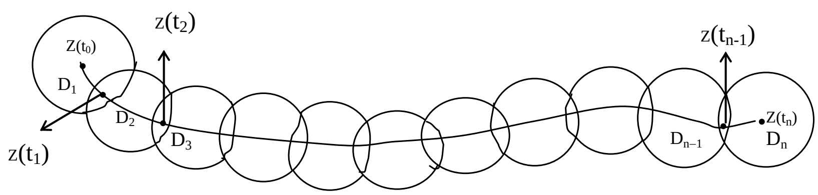
Remark. This method is useful in proving for complex values many of the results true for real values.

1.6. Analytic continuation along a curve Let γ be a curve in the complex pane having equation

$$z = z(t) = x(t) + iy(t), a \le t \le b.$$

We take the path along γ to be continuous. Let $a = t_0 \le t_1 \le \ldots \le t_n = b$ be the portion of the interval. If there is a chain $(f_1, D_1), (f_2, D_2), \ldots, (f_n, D_n)$ of function elements such that (f_{K+1}, D_{K+1}) is a direct analytic continuation of (f_K, D_K) for $K = 1, 2\ldots., n-1$ and $z(t) \in D_K$ for $t_{K-1} \le t \le t_K$, $K = 1, 2, \ldots, n$ then (f_n, D_n) is said to be analytic continuation of (f_1, D_1) along the curve γ.

Thus we shall obtain a well defined analytic function in a nbd. of the end point of the path, which is called the analytic continuation of (f_1, D_1) along the path γ. Here, D_K may be taken as discs containing $z(t_{K-1})$ as shown in the figure.



Further, we say that the sequence $\{D_1, D_2, \ldots, D_n\}$ is connected by the curve γ along the partition if the image $z([t_{K-1}, t_K])$ is contained in D_K.

1.7. Theorem (Uniqueness of Analytic Continuation along a Curve). Analytic continuation of a given function element along a given curve is unique. In other words, if (f_n, D_n) and (g_m, E_m) are two analytic continuations of (f_1, D_1) along the same curve γ defined by

$$z = z(t) = x(t) + iy(t), a \le t \le b.$$

Then $f_n = g_m$ on $D_n \cap E_m$

Proof. Suppose there are two analytic continuations of (f_1, D_1) along the curve γ, namely

$$(f_1, D_1), (f_2, D_2), \ldots, (f_n, D_n)$$

and

$$(g_1, \in_1), (g_2, \in_2), \ldots, (g_m, \in_m)$$

where $g_1 = f_1$ and $E_1 = D_1$

Then there exist partitions

$$a = t_0 \le t_1 \le \ldots\ldots\ldots.. \le t_n = b$$

and

$$a = s_0 \le s_1 \le \ldots\ldots\ldots. < s_m = b$$

such that $z(t) \in D_i$ for $t_{i-1} \le t \le t_i$, $i = 1, 2, \ldots, n$ and $z(t) \in E_j$ for $s_{j-1} \le t \;\; s_j$, $j = 1, 2, \ldots, m$. We claim that if $1 \le i \le n$, $1 \le j \le m$ and

$$[t_{i-1}, t_i) \cap [s_{j-1}, s_j] \ne \phi$$

then (f_i, D_i) and (g_i, E_j) are direct analytic continuations of each other. This is certainly true when $i = j = 1$, since $g_1 = f_1$ and $E_1 = D_1$. If it is not true for all i and j, then we may pick from all (i, j), for which the statement is false, a pair such that i + j is minimal. Suppose that $t_{i-1} \ge s_{j-1}$, where $i \ge 2$. Since $[t_{i-1}, t_i] \cap [s_{j-1}, s_j] \ne \phi$ and $s_{j-1} \le t_{i-1}$, we must have $t_{i-1} \le s_j$. Thus $s_{j-1} \le t_{i-1} \le s_j$. It follows that $z(t_{i-1}) \in D_{i-1} \cap E_i \cap E_j$. In particular, this intersection is non-empty. None (f_i, D_1) is a direct analytic continuation of (f_{i-1}, D_{i-1}). Moreover, (f_{i-1}, D_{i-1}) is a direct analytic continuation of (g_j, E_j) since $i + j$ is minimal, where we observe that $t_{i-1} \in [t_{i-2}, t_{i-1}] \cap [s_{j-1}, s_j]$ so that the hypothesis of the claim is satisfied. Since $D_{i-1} \cap D_i \cap E_j \neq \phi$, (f_i, D_1) must be direct analytic continuation of (g_j, E_j) which is contradiction. Hence our claim holds for all i and j. In particular, it holds for $i = n$, $j = m$ which proves the theorem.

1.8. Power series Method of Analytic continuation. Here we consider the problem of continuing analytically a function $f(z)$ defined initially as the sum function of a power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ whose circle of convergence C_0 has a finite non-zero radius. Thus, we shall use only circular domain and Taylor's expansion in such domain.

The first thing to observe here is that, when the continuation has been carried out, there must be at least one singularity of the complete analytic function on the circle of convergence C_0. For if there were not, we would construct, by analytic continuation, an analytic function which is equal to $f(z)$ within C_0 but is regular in a larger concentric circle C_0'. The expansion of this function as a Taylor's series in powers of $z - z_0$ would then converge everywhere within C_0', which is, however, impossible since the series would necessarily be the original series, whose circle of convergence is C_0.

To carry out the analytic continuation, we take any fixed point z_1 within C_0, and calculate the values of $f(z)$ and its successive derivatives at that point from the given power series by repeated term-by-term differentiation. We then form the Taylor's series

$$\sum_{0}^{\infty} \frac{f^{n}(z_1)}{\lfloor n}(z - z_1)^n \qquad (1)$$

whose circle of convergence is C_1, say. Let γ_1 denote the circle with centre z_1 which touches C_0 internally. By Taylor's theorem, this new power series is certainly convergent within γ_1 and has the sum $f(z)$ there. Hence the radius of C_1 cannot be less than that of γ_1. There are now three possibilities

(i) C_1 may have a larger radius than γ_1. In this case C_1 lies partially outside C_0 and the new power series (1) provides an analytic continuation of $f(z)$. We can then take a point z_2 within C_1 and outside C_0 and repeat the process as far as possible.

(ii) C_0 may be a natural boundary of $f(z)$. In this case, we cannot continue $f(z)$ outside C_0 and the circle C_1 touches C_0 internally, no matter what point z_1 within C_0 was chosen.

(iii) C_1 may touch C_0 internally even if C_0 is not a natural boundary of $f(z)$. The point of contact of C_0 and C_1 is then a singularity of the complete analytic function obtained by the analytic continuation of the sum function of the original power series, since there is necessarily one singularity of the complete analytic function on C_1 and this cannot be within C_0.

Thus, if C_0 is not a natural boundary for the function $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, this process of forming the new power series of the type (1) provides a simple method for the analytic continuation of $f(z)$, know as power series method.

Remark. Power series method is also called standard method of analytic continuation.

1.9. Example. Explain how it is possible to continue analytically the function

$$f(z) = 1 + z + z^2 + \ldots + z^n + \ldots$$

outside the circle of convergence of the power series.

Solution. The circle of convergence of the given power series is $|z| = 1$. Denoting it by C_0, we observe that within C_0 the sum function $f(z) = (1-z)^{-1}$ is regular. Further, this function is regular in any domain which does not contain the point $z = 1$. We carry out the analytic continuation by means of power series. If a is any point inside C_0 such that a is not real and positive, then

$$|1 - a| > 1 - |a| \tag{1}$$

Now, the Taylor's expansion of $f(z)$ about the point $z = a$ is given by

$$\sum_{n=0}^{\infty} \frac{f^n(a)}{\lfloor n} (z-a)^n \tag{2}$$

But $\dfrac{f^n(a)}{\lfloor n} = \dfrac{1}{(1-a)^{n+1}}$ and thus (2) becomes

$$\sum_{n=0}^{\infty} \frac{(z-a)^n}{(1-a)^{n+1}} = \frac{1}{1-a} \sum_{n=0}^{\infty} \left(\frac{z-a}{1-a}\right)^n \tag{3}$$

Clearly, (3) converges for $\left|\dfrac{z-a}{1-a}\right| < 1$

and its circle of convergence C_1 is given by $|z - a| = |1 - a|$. It follows from the inequality (1) that C_1 goes beyond C_0 and hence (3) provides an analytic continuation of $f(z)$ outside C_0, since we note that the sum function of (3) is also $(1-z)^{-1}$.

On the other hand, if a be a real point inside C_0 such that $0 < a < 1$, C_1 touches C_0 at $z = 1$, which is, therefore, a singularity of the complete analytic function obtained by analytic continuation of $f(z)$.

1.10. Example. Show that the function

$$f(z) = \frac{1}{a} + \frac{z}{a^2} + \frac{z^2}{a^3} + \ldots$$

can be continued analytically.

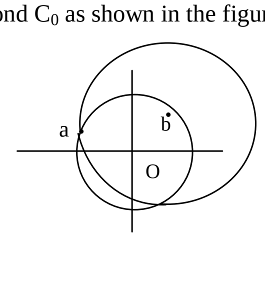
Solution. We have

$$f(z) = \frac{1}{a} + \frac{z}{a^2} + \frac{z^2}{a^3} + \ldots \tag{1}$$

This series converges within the circle C_0 defined by $|z| = |a|$ and has the sum

$$f(z) = \frac{1/a}{1 - z/a} = \frac{1}{a-z}$$

The only singularity of $f(z)$ on C_0 is at $z = a$. Hence the analytic continuation of $f(z)$ beyond C_0 is possible. For this purpose we take a point $z = b$ not lying on the line segment joining $z = 0$ and $z = a$. We draw a circle C_1 with centre b and radius $|a - b|$ i.e. C_1 is $|z - b| = |a-b|$ This new circle C_1 clearly extends beyond C_0 as shown in the figure



Now we reconstruct the series (1) in powers of z – b in the form

$$\sum_{n=0}^{\infty} \frac{(z-b)^n}{(a-b)^{n+1}}, \text{ where } \frac{f^n(b)}{\lfloor n} = \frac{1}{(a-b)^{n+1}} \tag{2}$$

This power series has circle of convergence C_1 and has the sum function $\frac{1}{a-z}$. Thus the power series (1) and (2) represent the same function in the region common to the interior of C_0 and C_1 Hence the series (2) represents an analytic continuation of series (1).

1.11. Example. Show that the circle of convergence of the power series

$$f(z) = 1 + z + z^2 + z^4 + z^8 + \ldots\ldots$$

is a natural boundary of its sum function

Solution. We have

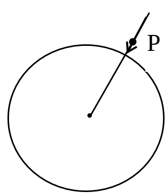
$$f(z) = 1 + \sum_{n=0}^{\infty} z^{2^n}$$

Evidently, $|z| = 1$ is the circle of convergence of the power series. We write

$$f(z) = 1 + \sum_{0}^{q} z^{2^n} + \sum_{q+1}^{\infty} z^{2^n} = f_1(z) + f_2(z), \text{ say.}$$

Let P be a point at $z = r\, e^{2\pi ip/2^q}$ lying outside the circle of convergence, where p and q are integers and $r > 1$.

We examine the behaviour of $f(z)$ as P approaches the circle of convergence through radius vector.



Now, $z^{2^n} = r^{2^n} e^{2\pi ip 2^n/2^q} = r^{2^n} e^{\pi ip 2^{n-q+1}}$

$$\therefore \qquad f_1(z) = 1 + \sum_{n=0}^{q} r^{2^n} e^{\pi ip 2^{n+1-q}}$$

which is a polynomial of degree 2^q and tends to a finite limit as $r \to 1$

Also $\quad f_2(z) = \sum_{q+1}^{\infty} r^{2^n} e^{\pi ip 2^{n+1-q}}$

Here, $n > q$ so 2^{n+1-q} is an even integer and thus

$$e^{\pi ip 2^{n+1-q}} = 1$$

$$\therefore \qquad f_2(z) = \sum_{q+1}^{\infty} r^{2^n} \to \infty \text{ as } r \to 1$$

Thus $f(z) = f_1(z) + f_2(z) \to \infty$, when $z = e^{2\pi ip/2^q}$.

Hence the point $z = e^{2\pi ip/2^q}$ is a singularity of $f(z)$. This point lies on the boundary of the circle $|z| = 1$. But any arc of $|z| = 1$, however small, contains a point of the form $e^{2\pi ip/2^q}$, where p and q are integers. Thus the singularities of $f(z)$ are everywhere dense on $|z| = 1$and consequently $|z| = 1$ constitutes the natural boundary for the sum function of the given power series.

1.12. Example. Show that the power series $\sum_{n=0}^{\infty} z^{3n}$ cannot be continued analytically beyond the circle $|z| = 1$

Solution. Here $|u_n(z)|^{1/n} = |z^{3n}|^{1/n} = |z^3| = |z|^3$
So the series is convergent if $|z| < 1$

$\therefore$ Circle of convergence is $|z| = 1$ Now take the point P at $z = r\ e^{2\pi ip/3q}$, $r > 1$ and then proceeds as in the above two examples.

1.13. Example. Show that the power series

$$z - \frac{z^2}{2} + \frac{z^3}{3} ....$$

may be continued analytically to a wider region by means of the series

$$\log 2 - \frac{(1-z)}{2} - \frac{(1-z)^2}{2\cdot 2^2} - \frac{(1-z)^3}{3\cdot 2^3}$$

Solution. The first series converges within the circle C_1 given by $|z| = 1$ and has the sum function $f_1(z) = \log(1 + z)$. The second series has the sum function

$$f_2(z) = \log 2 + \left[-\left(\frac{1-z}{2}\right) - \frac{1}{2}\left(\frac{1-z}{2}\right)^2 - \frac{1}{3}\left(\frac{1-z}{2}\right)^3 ....\right]$$

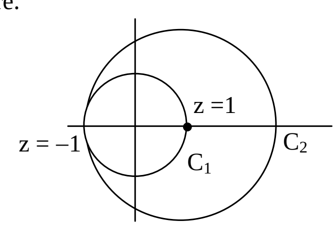
$$= \log 2 + \log\left[1 - \left(\frac{1-z}{2}\right)\right] \qquad \left|\frac{1-z}{2} < 1\right.$$

$$= \log 2 + \log\left(\frac{1+z}{2}\right) = \log(1 + z)$$

and thus is convergent within the circle C_2 given by $\left|\frac{1-z}{2}\right| = 1$ i.e. $|z-1| = 2$ thus we observe that

(i) $f_1(z)$ is analytic within C_1
(ii) $f_2(z)$ is analytic within C_2
(iii) $f_1(z) = f_2(z)$ in the region common to C_1 and C_2.

Hence the second series is an analytic continuation of the first series to circle C_2 which evidently extends beyond the circle C_1, as shown in the figure.



Remark. The circles C_1 and C_2 touch internally at $z = -1$ which is a singularity for both $f_1(z)$ and $f_2(z)$ i.e. $z = -1$ is a singularity of the complete analytic function whose two representations (members) are $f_1(z)$ & $f_2(z)$.

1.14. Example. Show that the functions defined by the series

$$1 + az + a^2 z^2 + \ldots..$$

and $$\frac{1}{1-z} - \frac{(1-a)z}{(1-z)^2} + \frac{(1-a)^2 z^2}{(1-z)^3}$$

are analytic continuations of each other.

Solution. The first power series represents the function $f_1(z) = \frac{1}{1-az}$ and has the circle of convergence C_1 given by $|az| = 1$ i.e. $|z| = \frac{1}{|a|}$ The only singularity is at the point $z = \frac{1}{a}(a > 0)$ on the boundary of the circle. The second series has the sum function

$$f_2(z) = \frac{1}{1-z} - \frac{(1-a)z}{(1-z)^2} + \frac{(1-a)^2 z^2}{(1-z)^3}\ldots..$$

$$= \frac{1}{1-z}\left[1 - \left(\frac{1-a}{1-z}\right)z + \left(\frac{1-a}{1-z}\right)^2 z^2 \ldots\ldots\ldots\right]$$

$$= \frac{1}{1-z} \frac{1}{1+\left(\frac{1-a}{1-z}\right)z} \qquad \left|\left(\frac{1-a}{1-z}\right)z < 1\right.$$

$$= \frac{1}{1-z} . \frac{1-z}{1-az} = \frac{1}{1-az}$$

and has the circle of convergence C_2 given by

$$\left|\frac{z(1-a)}{1-z}\right| = 1 \text{ i.e. } |z(1-a)| = |1-z|$$

i.e. $|z(1-a)|^2 = |1-z|^2$

i.e. $z\bar{z}\ (1-a)^2 = (1-z)\,(\overline{1-z})$, where a is assumed to be real and $a > 0$

i.e. $z\bar{z}\,(1-a)^2 = 1 - (z + \bar{z}) + z\bar{z}$

i.e. $(x^2 + y^2)\,(1+a^2 - 2a) = 1 - 2x + x^2 + y^2$, $z = x + iy$

i.e. $(x^2 + y^2)\ a\,(a-2) = 1 - 2x$

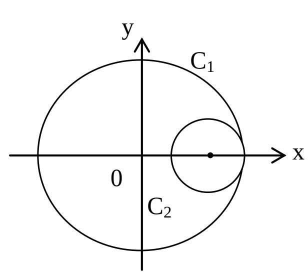
i.e. $$x^2 + y^2 - \frac{2x}{a(2-a)} + \frac{1}{a(2-a)} = 0$$

i.e. $$\left(x - \frac{1}{a(2-a)}\right)^2 + (y-0)^2 = \left(\frac{1-a}{a(2-a)}\right)^2$$

Thus the circle C_2 has the centre $\left(\frac{1}{a(2-a)}, 0\right)$ and radius $\frac{1-a}{a(2-a)}$.

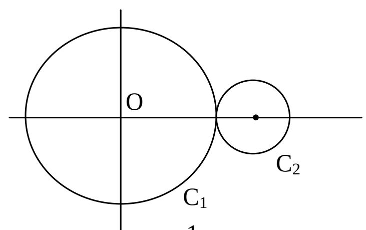
Since the two circles depend upon a, where we shall assume that a > 0, we have the following cases

Case I. Let $0 < a < 1$. In this case, the two circles C_1 and C_2 touch internally, since the distance between their centres is equal to the difference of their radii. Thus the first series represent the analytic continuation of the second from C_2 to C_1



Case II. If a = 1, then the second series reduces to $\frac{1}{1-z}$ and the first series is $1 + z + z^2 + \ldots$ which has the sum function $\frac{1}{1-z}$.

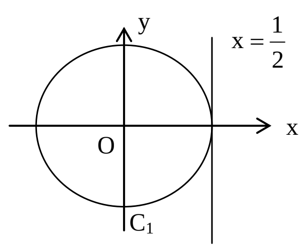
Case III. If 1 < a < 2. In this case, the two circles touch externally at $z = \frac{1}{a}$ so that the two series have no common region of convergence. Nevertheless they are analytic continuations of the same function $\frac{1}{1-az}$



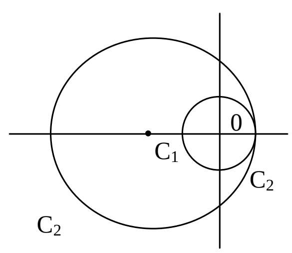
Case IV. If a = 2, then the first series represents the function $\frac{1}{1-2z}$ within C_1 given by $|z| = \frac{1}{2}$ and the second series defines the sum function $\frac{1}{1-2z}$ in the region $\left|\frac{z}{1-z}\right| < 1$

i.e. $z\bar{z} < (1-z)\overline{(1-z)}$, i.e. $x < \frac{1}{2}$.

Thus the second series represents the function $\frac{1}{1-2z}$ in the half plane $x < \frac{1}{2}$. We note that the line $x = \frac{1}{2}$ touches the circle $|z| = \frac{1}{2}$ as shown in the figure. Hence in this case, the second series represents the analytic continuation of the first series from the region $|z| < \frac{1}{2}$ to the half plane $x < \frac{1}{2}$.



Case V. Let a > 2. In this case C_1 and C_2 touch internally, where C_1 being the inner circle, as shown in the figure. Hence the second series represents an analytic continuation of the first series from C_1 to C_2



1.15. Example. Show that the function defined by

$$f_1(z) = \int_0^\infty t^3 e^{-zt}\, dt$$

is analytic at all points z for which Re(z) > 0. Find also a function which is analytic continuation of $f_1(z)$.

Solution. $$f_1(z) = \int_0^\infty t^3 e^{-zt}\, dt$$

$$= \left[t^3 \frac{e^{-zt}}{-z} - 3t^2 \frac{e^{-zt}}{z^2} + 6t \frac{e^{-zt}}{-z^3} - 6 \frac{e^{-zt}}{z^4} \right]_0^\infty \qquad \text{(Integ. by parts)}$$

$$= -6 \left(0 - \frac{1}{z^4} \right) = \frac{6}{z^4}, \text{ if } \operatorname{Re}(z) > 0$$

Let $\quad f_2(z) = \dfrac{6}{z^4}$

Then $f_1(z) = f_2(z)$ for $\operatorname{Re}(z) > 0$

The function $f_2(z)$ is analytic throughout the complex plane except at $z = 0$ and $f_1(z) = f_2(z)$ $\forall$ z s. t. $\operatorname{Re}(z) > 0$. Hence $f_2(z)$ is the required analytic continuation of $f_1(z)$.

2. Schwarz's Reflection Principle

We observe that some elementary functions $f(z)$ possess the property that $f(\bar{z}) = \overline{f(z)}$ for all points z in some domain. In other words, if $w = f(z)$, then it may happen that $\bar{w} = f(\bar{z})$ i.e. the reflection of z in the real axis corresponds to one reflection of w in the real axis. For example, the functions

$$z,\ z^2 + 1,\ e^z,\ \sin z \text{ etc}$$

have the above said property, since, when z is replaced by its conjugate, the value of each function changes to the conjugate of its original value. On the other hand, the functions

$$iz,\ z^2 + i,\ e^{iz},\ (1 + i) \sin z \text{ etc}$$

do not have the said property.

2.1. Definition. Let G be a region and $G^* = \{z : \bar{z} \in G\}$ then G is called symmetric region if $G = G^*$

If G is a symmetric region then let $G_+ = \{z \ \varepsilon\ G : I_m\, z > 0\}$ $G_- = \{z \in G : I_m\, z < 0\}$ and $G_0 = \{z \in G : I_m\, z = 0\}$.

2.2. Theorem (Schwarz's Reflection Principle). Let G be a region such that $G = G^*$ if $f : G_+ \cup G_0 \to \forall$ is a continuous function which is analytic on G_+ and $f(x)$ is real for x in G_0 then there is an analytic function $g : G \to \forall$ s.t. $g(z) = f(z)$ for all z in $G_+ \cup G_0$.

Proof. For z in G_-, define $g(z) = \overline{f(\bar{z})}$ and for z in $G_+ \cup G_0$, define $g(z) = f(z)$.

Then $g : G \to \forall$ is continuous. We will show that g is analytic. Clearly g is analytic on $G_+ \cup G_-$.
To show g is analytic on G_0, let x_0 be a fixed point in G_0 and let $R > 0$ be such that

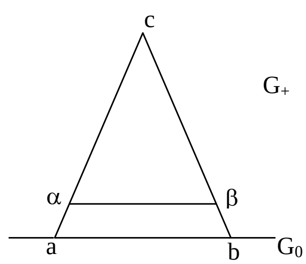
$$B(x_0\,;\,R) \subset G.$$

It is sufficient to show that g is analytic on $B(x_0\,;\,R)$ We shall apply Morera's theorem.

Let $T = [a, b, c, a]$ be a triangle in $B(x_0\,;\,R)$. Assume that $T \subset G_+ \cup G_0$ and $[a, b] \subset G_0$ Let Δ represent T together with its inside. Then $g(z) = f(z)$ for all z in Δ. $[\because T \subset G_+ \cup G_0]$ By hypothesis f is continuous on $G_+ \cup G_0$, so f is uniformly continuous on Δ. So given $\in > 0$, there is a $\delta > 0$ s.t. $z, z' \in \Delta$ implies

$$|f(z) - f(z')| < \in \text{ whenever } |z - z'| < \delta.$$

Choose α and β on the line segments [c, a] and [b, c] respectively so that $|\alpha - a| < \delta$ and $|\beta - b| < \delta$. Let $T_1 = [\alpha, \beta, c, \alpha]$ and $Q = [a, b, \beta, \alpha, a]$. Then $\int_T f = \int_{T_1} f + \int_Q f$



But T_1 and its inside are contained in G_+ and f is analytic there.

So $\int_{T_1} f = 0$

$$\therefore \quad \int_T f = \int_Q f$$

By if $0 \le t \le 1$, then

$$|[t\beta + (1-t)\alpha] - [tb + (1-t)a]| < \delta$$

so that

$$|f(t\beta + (1-t)\alpha) - f(tb + (1-t)a)| < \in.$$

Let M = max. $\{1\, f(z)\,| : z \in \Delta\}$ and l be the perimeter of T then

$$\left|\int_{[a,b]} f + \int_{[\beta,\alpha]} f\right| = \left|(b-a)\int_0^1 f(tb + (1-t)a)dt - (\beta-\alpha)\int_0^1 f(t\beta + (1-t)\alpha)\,dt\right|$$

$$\le |b-a|\left|\int_0^1 [f(tb + (1-t)a) - f(t\beta + (1-t)\alpha)]\,dt\right|$$

$$+ |b-a) - (\beta-\alpha)|\left|\int_0^1 f(t\beta + (1-t)\alpha)\,dt\right|$$

$$\le \in|b-a| + M\,|(b-\beta) + (\alpha - a)|$$

$$\le \in l + 2M\delta.$$

Also $\left|\int_{[\alpha,a]} f\right| \le M\,|a-\alpha| \le M\,\delta$

and $\left|\int_{[b,\beta]} f\right| \le M\delta.$

$$\therefore \quad \left|\int_T f\right| = \left|\int_{[a,b]} f + \int_{[\beta,\alpha]} f + \int_{[\alpha,a]} f + \int_{[b,\beta]} f\right| \le \left|\int_{[a,b]} f + \int_{[\beta,\alpha]} f\right| + \left|\int_{[\alpha,a]} f\right| + \left|\int_{[b,\beta]} f\right|$$

$$\le \in l + 4M\delta$$

Choosing $\delta > 0$ s. t. $\delta < \in$. Then

$\left|\int_T f\right| < \in(l + 4M)$. Since $\in$ is arbitrary it follows that $\int_T f = 0$. Hence f must be analytic.

3. Monodromy Theorem and its Consequences

We first give some definitions.

3.1. Definition. Let $\gamma_0, \gamma_1 : [0, 1] \to G$ be two closed rectifiable curves in a region G then γ_0 is homotopic to γ_1 in G if there is a continuous function

$$F : [0, 1] \times [0, 1] \to G$$

such that

$$F(s, 0) = \gamma_0(s)$$
$$F(s, 1) = \gamma_1(s) \qquad (0 \le s < 1)$$
$$F(0, t) = F(1, t) \qquad (0 \le t \le 1)$$

3.2. Definition. $\gamma_0, \gamma_1 : [0, 1]\to G$ are two rectifiable curves in G such that $\gamma_0(0) = \gamma_1(0) = a$ and $\gamma_0(1) = \gamma_1(1) = b$ then γ_0 and γ_1 are **fixed-end-point homotopic** (FEP homotopic) if there is a continuous map $F : [0, 1] \times [0, 1]\to G$ s.t.

$$F(s, 0) = \gamma_0(s), F(s, 1) = \gamma_1(s)$$
$$\mathbf{F(0, t) = a, F(1, t) = b \qquad for\ 0 \le s, t \le 1.}$$

We note that the relation of FEP homotopic is an equivalence relation on the curves from one given point to another.

3.3. Definition. An open set G is called simply connected if G is connected and every closed curve in G is homotopic to zero.

3.4. Definition. A function element is a pair (f, G) where G is a region and f is an analytic function on G.

For a given function element (f ,G) define the germ of f at a to be the collection of all function elements (g, D) such that $a\in D$ and $f(z) = g(z)$ for all z in a neighbourhood of a. The germ of f at 'a' is denoted by $[f]_a$.
Notice that $[f]_a$ is a collection of function elements.

3.5. Definition. Let $\gamma : [0, 1] \to \mathbb{C}$ be a path and suppose that for each t in [0, 1] there is a function element $(f_t. D_t)$ such that

(i) $\gamma(t) \in D_t$;

(ii) for each t in [0, 1] there is a $\delta > 0$ such that $| s - t| < \delta$ implies $\gamma(s) \in D_t$ and $[f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}$

Then (f_1, D_1) is called analytic continuation of (f_0, D_0) along the path γ.

Remark. Since γ is a continuous function and $\gamma(t)$ is in the open set D_t so there is a $\delta > 0$ such that $\gamma(s) \varepsilon D_t$ for $| s-t| < \delta$
So part (ii) of above definition implies

$$f_s(z) = f_t(z) \text{ for all } z \in D_s \cap D_t,$$

whenever $|s - t| < \delta$

3.6. Theorem. Let $\gamma : [0, 1]\to \forall$ be a path from a to b and let $\{(f_t, D_t) : 0 \le t \le 1\}$ and $\{(g_t, B_t) : 0 \le t \le 1\}$ be analytic continuations along γ such that $[f_0]_a = [g_0]_a$. Then $[f_1]_b = [g_1]_b$

Proof. Consider the set

$$T = \{t \in [0, 1] : [f_t]_{\gamma(t)} = [g_t]_{\gamma(t)}\}$$

Since $[f_0]_a = [g_0]_a$ so $0 \in T$. Thus $T \ne \phi$.

We shall show that T is both open and closed. To show T is open, let t be a fixed point of T st $t \ne 0$. By definition of analytic continuation, there is a $\delta > 0$ such that for $|s - t| < \delta$,

$$\gamma(s) \in D_t \cap B_t \text{ and}$$
$$[f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}$$
$$[g_s]_{\gamma(s)} = [g_t]_{\gamma(s)}$$

But $t \in T$ implies

$$f_t(z) = g_t(z) \text{ for all z in } D_t \cap B_t.$$

Hence $\qquad [f_t]_{\gamma(s)} = [g_t]_{\gamma(s)}$ for all $\gamma(s)$ in $D_t \cap B_t$.
So $\qquad [f_s]_{\gamma(s)} = [g_s]_{\gamma(s)}$ whenever $|s-t| < \delta$.
That is, $s \in T$ whenever $|s - t| < \delta$
or $\qquad$ **$(t - \delta, t + \delta) \subset T$.**
If $t = 0$ then the above argument shows that $[a, a + \delta) \subset T$ for some $\delta > 0$. Hence T is open.

To show T is closed let t be a limit point of T. Again by definition of analytic continuation there is a $\delta > 0$ s. t. for $|s - t| < \delta$, $\gamma(s) \in D_t \cap B_t$ and

$$[f_s]_{\gamma(s)} = (f_t]_{\gamma(s)}$$
$$[g_s]_{\gamma(s)} = [g_t]_{\gamma(s)} \tag{1}$$

Since t is a limit point of T there is a point s in T s.t. $|s - t| < \delta$. Let $G = D_t \cap B_t \cap D_s \cap B_s$. Then $\gamma(s) \varepsilon G$. So G is non-empty open set thus by definition of T,

$$f_s(z) = g_s(z) \text{ for all z in G.}$$

But (1) implies

$$f_s(z) = f_t(z) \text{ and } g_s(z) = g_t(z) \text{ for all z in G.}$$
$$\therefore \quad f_t(z) = g_t(z) \text{ for all z in G.}$$

Since G has a limit point in $D_t \cap B_t$, this gives $[f_t]_{\gamma(t)} = [g_t]_{\gamma(t)}$
Thus t ε T and so T is closed.

Now T is a non-empty subset of [0, 1] s.t. T is both open and closed. So connectedness of [0, 1] implies T = [0, 1].

Thus $1 \varepsilon T$ and hence $[f_1]_{\gamma(1)} = [g_1]_{\gamma(1)}$ i.e. $[f_1]_b = [g_1]_b$ as $\gamma(1) = b$.

3.7. Definition. If $\gamma : [0, 1] \to \forall$ is a path from a to b and $\{(f_t, D_t) : a \le t \le 1\}$ is an analytic continuation along γ then the germ $[f_1]_b$ is the analytic continuation of $[f_0]_a$ along γ.

Remark. Suppose a and b are two complex numbers and let γ and σ be two paths from a to b. Suppose $\{(f_t, D_t)\}$ and $\{(g_t, D_t)\}$ are analytic continuations along γ and σ respectively s.t. $[f_0]_a = [g_0]_a$. Now the question is "Does it follow that $[f_1]_b = [g_1]_b$"?. If γ and σ are the same path then above result gives an affirmative answer. However if γ and σ are distinct then the answer can be no.

3.8. Lemma. Let $\gamma : [0\ 1] \to \forall$ be a path and let $\{f_t, D_t) : 0 \le t \le 1\}$ be an analytic continuation along γ. For $0 \le t \le 1$ let R(t) be the radius of convergence of the power series expansion of f_t about $z = \gamma(t)$. Then either $R(t) \equiv \infty$ or $R : [0,1] \to (0, \infty)$ is continuous.

Proof. Suppose $R(t) = \infty$ for some value of t.
Then f_t can be extended to an entire function.

It follows that $f_s(z) = f_t(z)$ for all z in D_s so that $R(s) = \infty$ for all s in [0, 1] ; that is $R(s) \equiv \infty$. Now suppose that $R(t) < \infty$ for all t. Let t be a fixed number in [0, 1] and let $a = \gamma(t)$.

Let $\qquad f_t(z) = \sum_{n=0}^{\infty} a_n (z - a)^n$

be the power series expansion of f_t about a.
Now let $\delta_1 > 0$ be such that $|s-t| < \delta_1$ implies that

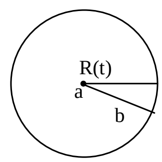
$$\gamma(s) \in D_t \cap B(a ; R(t)) \text{ and } [f_s]_{\gamma(s)} = [f_t]_{\gamma(s} \text{ Fix s with } |s-t| < \delta_1 \text{ and let } b = \gamma(s).$$

Now f_t can be extended to an analytic function on B(a; R(t)) Since f_s agrees with f_t on a neighbourhood of f_s can be extended so that it is also analytic on $B(a ; R(t)) \cup D_s$. If f_s has power series expansion

$$f_s(z) = \sum_{n=0}^{\infty} b_n (z - b)^n \text{ about } z = b.$$

Then the radius of convergence R(s) must be at least as big as the distance from b to the circle $|z - a| = R(t)$; that is,

$$R(s) \geq d(b, \{z : |z - a| = R(t)\})$$
$$\geq R(t) - |a - b|$$



This implies $R(t) - R(s) \leq |a - b|$

i.e. $\quad R(t) - R(s) \leq |\gamma(t) - \gamma(s)|$

Similarly, we can show

$$R(s) - R(t) \leq |\gamma(t) - \gamma(s)|$$

$\therefore \ |R(s) - R(t)| \leq |\gamma(t) - \gamma(s)|$ for $|s - t| < \delta_1$.

Since $\gamma : [0, 1] \rightarrow \forall$ is continuous so given $\in > 0$, $\exists\ \delta_2 > 0$ s.t.

$$|\gamma(t) - \gamma(s)| < \in \text{ for } |s - t| < \delta_2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$ and

$$|R(s) - R(t)| , \in \text{ for } |s - t| < \delta.$$

Hence R is continuous at t.

3.9. Lemma. Let $\gamma : [0, 1] \rightarrow \forall$ be a path from a to b and let $\{(f_t, D_t) : 0 \leq t \leq 1\}$ be an analytic continuation along γ. There is a number $\in > 0$ s.t. if $\sigma : [0, 1] \rightarrow \forall$ is any path from a to b with $|\gamma(t) - \sigma(t)| < \in$ for all t and if $\{(g_t, B_t) : 0 \leq t \leq 1\}$ is any continuation along σ with $[g_0]_a = [f_0]_a$; then $[g_1]_b = [f_1]_b$.

Proof. For $0 \leq t \leq 1$, let R(t) be the radius of convergence of the power series expansion of f_t about $z = \gamma(t)$.

If $R(t) \equiv \infty$ then any value of $\in$ will be sufficient.

So suppose $R(t) < \infty$ for all t.

Since R is a continuous function and $R(t) > 0$ for all t, R has a positive minimum value.

Let $0 < \in < \dfrac{1}{2} \min\{R(t) : 0 \leq t \leq 1\}$

Suppose $\sigma : [0, 1] \rightarrow \forall$ is any path from a to b with $|\gamma(t) - \sigma(t)| < \in$ for all t and $\{(g_t, B_t) : 0 \leq t \leq 1\}$ is any continuation along σ with $[g_0]_a = [f_0]_a$.

Suppose D_t is a disk of radius R(t) about $\gamma(t)$.

Since $|\sigma(t) - \gamma(t)| < \in < R(t)$, $\sigma(t)\ \varepsilon\ B_t \cap D_t$ for all t.

Define the set $T = \{t\ \varepsilon\ [0, 1] : f_t(z) = g_t(z) \text{ for all z in } B_t \cap D_t\}$

Then $0\ \varepsilon\ T$ since $[g_0]_a = [f_0]_a$. So $T \neq \phi$.

We will show $1\ \varepsilon\ T$. For this it is sufficient to show that T is both open and closed subset of [0, 1].

To show T is open, let t be any fixed point of T.

Choose $\delta > 0$ such that

$$\left.\begin{array}{ll} |\gamma(s) - \gamma(t)| < \in, & [f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}, \\ |\sigma(s) - \sigma(t)| < \in, & [g_s]_{\sigma(s)} = [g_t]_{\sigma(s)} \end{array}\right\} \qquad (1)$$

and $\sigma(s)\ \varepsilon\ B_t$ whenever $|s - t| < \delta$.

We now show that $B_s \cap B_t \cap D_s \cap D_t \neq \phi$ for $|s - t| < \delta$. For this we will show $\sigma(s)\ \varepsilon\ B_s \cap B_t \cap D_s \cap D_t$ for $|s - t| < \delta$. If $|s - t| < \delta$ then

$$|\sigma(s) - \gamma(s)| < \in < R(s)$$

so that $\sigma(s) \in D_s$.

Also $|\sigma(s) - \gamma(t)| = |\sigma(s) - \gamma(s) + \gamma(s) - \gamma(t)|$

$$\leq |\sigma(s) - \gamma(s)| + |\gamma(s) - \gamma(t)| < 2\in < R(t)$$

$\therefore$ $\sigma(s) \varepsilon D_t$.

Since we already have $\sigma(s) \varepsilon B_s \cap B_t$ by (1), so

$$\sigma(s) \varepsilon B_s \cap B_t \cap D_s \cap D_t = G.$$

Since t ε T, it follows that $f_t(z) = g_t(z)$ for all z in G.

Also (1) implies $f_s(z) = f_t(z)$ and $g_s(z) = g_t(z)$ for all z in G.

Thus $f_s(z) = g_s(z)$ for all z in G.

But since G has a limit point in $B_s \cap D_s$, we must have $s \in T$.

That is, $(t - \delta, t + \delta) \subset T$

$\therefore$ T is open.

Similarly we can show that T is closed.

$\therefore$ T is non-empty open and closed subset of [0, 1]. As [0, 1] is connected, we have [0, 1] = T.

Thus 1 ε T and the result is proved.

3.10. Definition. Let (f, D) be a function element and let G be a region which contains D; then (f, D) admits unrestricted analytic continuation in G if for any path γ in G with initial point in D there is an analytic continuation of (f, D) along γ.

3.11. Theorem (Monodromy Theorem). Let (f, D) be a function element and let G be a region containing D such that (f, D) admits unrestricted continuation in G. Let a ε D, b ε G and let γ_0 and γ_1 be paths in G from a to b; let $\{(f_t, D_t) : 0 < t \leq 1\}$ and $\{(g_t, D_t) : 0 \leq t \leq 1\}$ be analytic continuations of (f, D) along γ_0 and γ_1 respectively. If γ_0 and γ_1 are FEP homotopic in G then

$$[f_1]_b = [g_1]_b.$$

Proof. Since γ_0 and γ_1 are fixed end point homotopic in G there is a continuous function $F : [0, 1] \times [0, 1] \rightarrow G$ such that

$$F(t, 0) = \gamma_0(t), \quad F(t, 1) = \gamma_1(t)$$
$$F(0, u) = a, \quad F(1, u) = b$$

For all t and u in [0, 1]

Let u be a fixed point of [0, 1]. Consider the path γ_u, defined by

$$\gamma_u(t) = F(t, u) \quad \text{for all } t \,\varepsilon\, [0, 1].$$

Then $\gamma_u(0) = F(0, u) = a, \gamma_u(1) = F(1, u) = b$

$\therefore$ γ_u is a path from a to b.

By hypothesis there is an analytic continuation

$$\{(h_{t,u}, D_{t,u}) : 0 \leq t \leq 1\}$$

of (f, D) along γ_u.

Now $\{(h_{t,0}, D_{t,0}) : 0 \leq t \leq 1\}$ and $\{(f_t, D_t) : 0 \leq t \leq 1\}$ are analytic continuations along γ_0 so by theorem 3.6, we have

$$[f_1]_b = [h_{1,0}]_b$$

Similarly, $[g_1]_b = [h_{1,1}]_b$

To prove the theorem, it is sufficient to show

$$[h_{1,0}]_b = [h_{1,1}]_b$$

Consider the set

$$U = \{u \,\varepsilon\, [0, 1] : [h_{1,u}]_b = [h_{1,0}]_b. \text{ We will show } 1 \,\varepsilon\, U.$$

Now, $0 \in U$. So $U \neq \phi$.

We claim U is both open and closed subset of [0, 1].
Let u be any point in [0, 1].
We assert that there is $\delta > 0$ such that if $|u - v| < \delta$
then $$[h_{1,u}]_b = [h_{1,v}]_b \quad (2)$$
By lemma 3.9, there an $\in > 0$ such that if σ is any path from a to b with $|\gamma_u(t) - \sigma(t)| < \in$ for all t and if $\{(k_t, E_t)\}$ is any continuation of (f, D) along σ, then
$$[h_{1,u}]_b = [k_1]_b \quad (3)$$
Now F is uniformly continuous function so there is $\delta > 0$ s.t.
$$|F(t, u) - F(t, v)| < \in \quad \text{whenever } |u - v| < \delta$$
i.e. $$|\gamma_u(t) - \gamma_v(t)| < \in \quad \text{whenever } |u - v| < \delta.$$
So for $|u - v| < \delta$, γ_v is a path from a to b with
$$|\gamma_u(t) - \gamma_v(t)| < \in \text{ for all t and } \{(h_{t,v}, D_{t,v})\} \text{ is continuation of } (f, D) \text{ along } \gamma_v,$$
so by (3),
$$[h_{1,u}]_b = [h_{1,v}]_b$$
Suppose $u \in U$ sp that $[h_{1,u}]_b = [h_{1,0}]_b$. Then as proved above, there is a $\delta > 0$ s.t. $|u - v| < \delta$
implies $[h_{1,u}]_b = [h_{1,v}]_b$
i.e. $v \in (u - \delta, u + \delta)$ implies $[h_{1,v}]_b = [h_{1,0}]_b$
i.e. $v \in (u - \delta, u + \delta)$ implies $v \in U$
i.e. $(u - \delta, u + \delta) \subset U$.
Hence U is open.

To show U is closed, we show $\bar{U} = U$. Let $u \in \bar{U}$ and δ be the +ve number satisfying (2). Then there is a $v \in U$ such that
$$|u - v| < \delta$$
So by (2), $[h_{1,u}]_b = [h_{1,v}]_b$.
Since $v \in U$, $[h_{1,v}]_b = [h_{1,0}]_b$ Therefore $[h_{1,u}]_b = [h_{1,0}]_b$. so that $u \in U$.
Thus U is closed as $\bar{U} = U$.
Now U is a non-empty open and closed subset of [0, 1] and [0, 1] is connected.
$\therefore \quad [0, 1] = U$
So $1 \in U$ and the result is proved.
The following corollary is the main consequence of the Monodromy theorem.

3.12. Corollary. Let (f, D) be a function element which admits unrestricted continuation in the simply connected region G. Then there is an analytic function $F : G \to \mathbb{C}$ such that $F(z) = f(z)$ for all z in D.

Proof. Let a be a fixed point in D and z is any point in G. If γ is a path in G from a to z and $\{(f_t, D_t) : 0 \leq t \leq 1\}$ is an analytic continuation of (f, D) along γ then let $F(z, \gamma) = f_1(z)$ since G is simply connected,

$F(z, \gamma) = F(z, \sigma)$ for any two paths γ and σ in G from a to z. Thus $F(z) = F(z, \gamma)$ is a well defined function from G to C. To show that F is analytic let $z \in G$. Let γ be a path in G from a to z and $\{(f_t, D_t)\}$ be the analytic continuation of (f, D) along γ. Then $F(\omega) = f_1(\omega)$ for all ω in a neighbourhood of z. Hence F must be analytic.

4. Harmonic Functions on a Disk

If G is an open subset of $\forall$ then a function $u : G \to R$ is called **harmonic** if u has continuous second partial derivatives and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

We recall the following facts about harmonic functions.

(i) A function f on a region G is analytic iff Ref = u and Imf = v are harmonic functions which satisfy the Cauchy-Riemann equations.

(ii) A region G is simply connected iff for each harmonic function u on G there is a harmonic function v on G such that f = u + iv is analytic on G.

If $f : G \to \forall$ is an analytic function then u = Re f and v = Im f are called **harmonic** conjugates.

With this terminology, (ii) implies that every harmonic function on a simply connected region has a harmonic conjugate. If u is a harmonic function on G and D is a disk s.t. D $\subset$ G then there is a harmonic function v on D s.t. u + iv is analytic on D.

4.1. Theorem. If $u : G \to \forall$ is harmonic then u is infinitely differentiable.

Proof. Let $z_0 = x_0 + iy_0$ be a fixed point in G and $\delta > 0$ be choosen s.t.

$$B(z_0 ; \delta) \subset G.$$

Then u has a harmonic conjugate v on $B(z_0 ; \delta)$, that is, f = u + iv is analytic and hence infinitely differentiable on $B(z_0 ; \delta)$. Thus it follows that u is infinitely differentiable.

The next result gives a property that harmonic functions share with analytic functions.

4.2. Mean Value Theorem. Let $u : G \to R$ be a harmonic function and let $\bar{B}(a : r)$ be a closed disk contained in G. If γ is the circle $|z - a| = r$

then $$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})\, d\theta.$$

Proof. Let D be a disk such that $\bar{B}(a ; r) \subset D \subset G$ and let f be an analytic function on D such that u = Re f. Then by Cauchy's Integral formula,

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz.$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{i\theta})\, d\theta \qquad \text{where } z - a = re^{i\theta}$$

$$\therefore \quad u(a) + i\,v(a) = \frac{1}{2\pi}\int_0^{2\pi} [u(a + re^{i\theta}) + i\,v(a + re^{i\theta})]\, d\theta \text{ where } v = \text{Im } f.$$

Equating real parts on both sides, we get

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})\, d\theta$$

4.3. Definition. A continuous function $u : G \to R$ has the Mean Value Property (MVP) if whenever $\bar{B}(a ; r) \subset G$,

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})\, d\theta.$$

4.4. Maximum Principle. Let G be a region and suppose that u is a continuous real valued function on G with the MVP. If there is a point a in G such that $u(a) \geq u(z)$ for all z in G then u is a constant function.

4.5. Definition. The function

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$$

for $0 \le r < 1$ and $-\infty < \theta < \infty$, is called the Poisson kernel.
Let $z = re^{i\theta}$ where $0 \le r < 1$.

Then
$$\frac{1+re^{i\theta}}{1-re^{i\theta}} = \frac{1+z}{1-z}$$
$$= (1 + z)(1-z)^{-1}$$
$$= (1+z)(1 + z + z^2 + z^3 + \ldots)$$
$$= 1 + 2\sum_{n=1}^{\infty} z^n$$
$$= 1 + 2\sum_{n=1}^{\infty} r^n e^{in\theta} = 1 + 2\sum_{n=1}^{\infty} r^n (\cos n\theta + i \sin n\theta)$$

$$\therefore \qquad \mathrm{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) = 1 + 2\sum_{n=1}^{\infty} r^n \cos n\theta$$
$$= 1 \sum_{n=1}^{\infty} r^n (e^{in\theta} + e^{-in\theta})$$
$$= P_r(\theta)$$

Also
$$\frac{1+re^{i\theta}}{1-re^{i\theta}} = \left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right)\left(\frac{1-re^{-i\theta}}{1-re^{-i\theta}}\right) = \frac{1+re^{i\theta}-re^{-i\theta}-r^2}{1-re^{i\theta}-re^{-i\theta}+r^2}$$
$$= \frac{(1-r^2)+2ir\sin\theta}{1-2r\cos\theta+r^2}$$

so that
$$P_r(\theta) = \mathrm{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) = \frac{1-r^2}{1-2r\cos\theta+r^2}$$

4.6. Proposition. The Poisson kernel satisfies the following:

(i) $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\, d\theta = 1$;

(ii) $P_r(\theta) > 0$ for all θ,
$P_r(-\theta) = P_r(\theta)$
and P_r is periodic in θ with period 2π

(iii) $P_r(\theta) < P_r(\delta)$ if $0 < \delta < |\theta| \le \pi$;

(iv) for each $\delta > 0$, $\lim_{r\to 1-} P_r(\theta) = 0$ uniformly in θ for $\delta \le |\theta| \le \pi$.

Proof. (i) For a fixed value of r, $0 \le r < 1$, the series $\sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$ converges uniformly in θ t0 $P_r(\theta)$. So applying term by term integration, we have

$$\int_{-\pi}^{\pi} P_r(\theta)d\theta = \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$$
$$= \sum_{n=-\infty}^{\infty} r^{|n|} \int_{-\pi}^{\pi} e^{in\theta}\, d\theta = 2\pi + 2\sum_{n=1}^{\infty} r^n \int_{-\pi}^{\pi} \cos n\theta\, d\theta = 2\pi.$$

$$\therefore \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\, d\theta = 1$$

(ii) We have $P_r(\theta) = \dfrac{1-r^2}{|1-re^{i\theta}|^2} > 0$ since r < 1.

Also $$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta+r^2}$$

$$\therefore \quad P_r(-\theta) = \frac{1-r^2}{1-2r\cos(-\theta)+r^2} = \frac{1-r^2}{1-2r\cos\theta+r^2} = P_r(\theta)$$

Further since cos θ is periodic with period 2π, it follows from

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta+r^2}$$

that $P_r(\theta)$ is also periodic in θ with period 2π.

(iii) Let $0 < \delta < \theta \le \pi$. Define $f : [\delta, \theta] \to R$

by $$f(t) = P_r(t) \text{ for all t in } [\delta, \theta].$$

Then $f'(t) = P'_r(t) = \dfrac{-2(1-r^2)r\sin\theta}{(1-2r\cos\theta+r^2)^2} < 0$

$$\therefore \quad f(\delta) > f(\theta) \text{ and so } P_r(\theta) < P_r(\delta)$$

We have $$0 < P_r(\theta) \le P_r(\delta) \qquad \text{if } \delta \le |\theta| \le \pi \tag{1}$$

Also $$P_r(\delta) = \frac{1-r^2}{1-2r\cos\delta+r^2}$$

$$\therefore \quad \lim_{r\to1-} P_r(\delta) = \lim_{r\to1-} \frac{1-r^2}{1-2r\cos\delta+r^2} = 0$$

Thus (1) implies $\lim\limits_{r\to1-} P_r(\theta) = 0$

Hence $\lim\limits_{r\to1-} P_r(\theta) = 0$ uniformly in θ for $\delta \le |\theta| \le \pi$.

5. The Dirichlet's Problem

The Dirichlet's Problem consists in determining all regions G such that for any continuous function $f : \partial G \to R$ there is a continuous function $u : \bar{G} \to R$ such that

$$u(z) = f(z) \text{ for z in } \partial G$$

and u is harmonic in G.

The next theorem states that the Dirichlet's Problem can be solved for the unit disk.

5.1. Theorem. Let D = {z : |z| < 1} and suppose that $f : \partial D \to R$ is a continuous function. Then there is a continuous function

$$u : \bar{D} \to R$$

such that

(a) $u(z) = f(z)$ for z in ∂D ;
(b) u is harmonic in D.

Moreover u is unique and is defined by the formula

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f(e^{it})\, dt$$

for $0 \le r < 1$, $0 \le \theta \le 2\pi$.

Proof. Define $u : \bar{D} \to R$

as $$u(r\,e^{i\theta}) = \begin{cases} \dfrac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\, f(e^{it})dt & \text{for } 0 \le r < 1 \\ f(r\,e^{i\theta}) & \text{for } r = 1 \end{cases}$$

Then $u(e^{i\theta}) = f(e^{i\theta})$

$\Rightarrow$ $u(z) = f(z)$ for z in ∂D.

It remains to show that u is continuous on $\bar{D}$ and harmonic in D.

(i) u is harmonic in D.

If $0 \le r < 1$ then

$$u(r\,e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\, f(e^{it})dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{Re}\left[\frac{1 + r\,e^{i(\theta - t)}}{1 - r\,e^{i(\theta - t)}}\right] f(e^{it})dt$$

$$= \mathrm{Re}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})\left[\frac{1 + r\,e^{i(\theta - t)}}{1 - r\,e^{i(\theta - t)}}\right]dt\right\}$$

$$= \mathrm{Re}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})\left[\frac{e^{it} + r\,e^{i\theta}}{e^{it} - r\,e^{i\theta}}\right]dt\right\}$$

Define $g : D \to \forall$ by

$$g(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})\left[\frac{e^{it} + z}{e^{it} - z}\right]dt$$

Since u = Re g, then g is analytic it follows that u is harmonic in D.

(ii) u is continuous on $\bar{D}$.

Since u is harmonic on D, it only remains to show that u is continuous at each point of the boundary of D. For this we prove the following.

Given α in $[-\pi, \pi]$ and $\in > 0$ there is a ρ, $0 < \rho < 1$ and an arc A of ∂D about $e^{i\alpha}$ such that for $\rho < r < 1$ and $e^{i\theta}$ in A,

$$|u(r\,e^{i\theta}) - f(e^{i\alpha})| < \in$$

We prove the result by taking $\alpha = 0$.

Since f is continuous at z = 1, there is a $\delta > 0$ such that

$$|f(e^{i\theta}) - f(1)| < \frac{1}{3}\,\varepsilon \text{ if } |\theta| < \delta. \qquad (1)$$

Let M = max. $\{1\, f(e^{i\theta}) : |\theta| \le \pi\}$

Since for each $\delta > 0$,

$$\lim_{r \to 1-} P_r(\theta) = 0 \text{ uniformly in } \theta, \text{ for } \delta \le |\theta| \le \pi,$$

there is a number ρ, $0 < \rho < 1$, such that

$$P_r(\theta) < \frac{\in}{3M} \qquad (2)$$

for $\rho < r < 1$ and $|\theta| \ge \dfrac{\delta}{2}$

Let A be the arc $\{e^{i\theta} : |\theta| < \frac{1}{2}\delta\}$. Then if $e^{i\theta} \in A$ and $\rho < r < 1$,

$$u(r\,e^{i\theta}) - f(1) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)\, f(e^{it})dt - f(1)$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)\, [f(e^{it}) - f(1)]dt$$
$$= \frac{1}{2\pi}\int_{|t|<\delta} P_r(\theta-t)\, [f(e^{it}) - f(1)]dt + \frac{1}{2\pi}\int_{|t|\geq\delta} P_r(\theta-t)[f(e^{it}) - f(1)]dt$$

If $|t| \geq \delta$ and $|\theta| \leq \frac{1}{2}\delta$ then $|t-\theta| \geq |t| - |\theta| \geq \delta - \frac{\delta}{2} = \frac{\delta}{2}$

So from (1) & (2), it follows that

$$|u\,(r\,e^{i\theta}) - f(1)| \leq \frac{1}{3}\in + 2M\left(\frac{\in}{3M}\right) = \in$$

i.e. $\quad |u(r\,e^{i\theta}) - f(e^{i\alpha})| < \in$ for $\alpha = 0$

Since f is continuous function, it follows that u is continuous at $e^{i\alpha}$.

Finally to show u is unique, suppose that v is a continuous function an $\overline{D}$ such that v is harmonic on D and $v(e^{i\theta}) = f(e^{i\theta})$ for all θ. Then $u - v$ is harmonic in D and $(u - v)(z) = 0$ for all z in ∂D.

So $u - v \equiv 0$ is $u \equiv v$.

Cor. (a) : If $u : \overline{D} \to R$ is a continuous function that is harmonic in D

then $\quad u(r\,e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)\, u\,(e^{it})\, dt$

for $0 \leq r < 1$ and all θ.

Moreover, u is the real part of the analytic function

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\, u(e^{it})dt$$

Cor. (b): Let $a \ \varepsilon\ \forall$, $\rho > 0$ and suppose h is a continuous real valued function on $\{z : |z-a| = \rho\}$; then there is a unique continuous function $w : \overline{B}(a;\rho) \to R$ such that w is harmonic on $B(a : \rho)$ and $w(z) = h(z)$ for $|z-a| = \rho$.

Proof. Consider $f(e^{i\theta}) = h(a + \rho e^{i\theta})$

Then f is continuous on ∂D.

So $\exists$ a continuous function $u : \overline{D} \to R$ such that

u is harmonic in D and $u(e^{i\theta}) = f(e^{i\theta})$.

Define $w : \overline{B}(a\,;\rho) \to R$

as $w(z) = u\left(\frac{z-a}{\rho}\right)$ for $z \ \varepsilon\ \overline{B}(a\,;\rho)$.

Then w is harmonic on $B(a\,;\rho)$ since u is harmonic on D.

Also for $z = a + \rho e^{i\theta}$,

$$w(z) = u(e^{i\theta}) = f(e^{i\theta}) = h(a + \rho e^{i\theta}) = h(z).$$

Hence $w(z) = h(z)$ for $|z-a| = \rho$.

We shall use the following result in proving next theorem.

5.2. Lemma. Let G be a bounded region and suppose that $w : \overline{G} \to R$ is a continuous function that satisfies the MVP on G. If $w(z) = 0$ for all z in ∂G then $w(z) = 0$ for all z in G.

5.3. Theorem. If $u : G \to R$ is a continuous function which has the MVP then u is harmonic.

Proof. Let a ε G and choose ρ s.t. $\bar{B}(a ; \rho) \subset G$.
It is sufficient to show u is harmonic on B(a ; ρ)
By last cor., there is a continuous function

$$w : \bar{B}(a ; \rho) \to R$$

which is harmonic in B(a ; ρ) and $w(a + \rho e^{i\theta}) = u(a + \rho e^{i\theta})$ for all θ. Since u−w satisfies the MVP and (u − w) z = 0 for |z−a| = ρ, it follows that u ≡ w in B(a, ρ). Since w is harmonic on B(a, ρ); we have u must be harmonic.

5.4 Harnack's Inequality. If u : $\bar{B}(a ; R) \to \mathbf{R}$ is continuous harmonic in B(a ; R) and u ≥ 0, then for 0 ≤ r < R and all θ,

$$\frac{R-r}{R+r}u(a) \le u(a + r e^{i\theta}) \le \frac{R+r}{R-r}u(a).$$

Proof. Define w : $\bar{D} \to \mathbf{R}$ as

$$w(\rho e^{i\theta}) = u(a + \rho R e^{i\theta}) \text{ for } 0 \le \rho \le 1$$

Then w is continuous function on $\bar{D}$ s.t. w is harmonic in D. So by cor. (a) to theorem 5.1,

$$w(\rho e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_\rho(\theta - t)\, w(e^{it})\, dt \text{ for } 0 \le \rho < 1 \text{ and all } \theta.$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-\rho^2}{1-2\rho\cos(\theta-t)+\rho^2} w(e^{it})dt \qquad (1)$$

Since 0 ≤ r < R so $0 \le \frac{r}{R} < 1$. $\left[\because P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}\right.$

Replacing ρ by $\frac{r}{R}$ in (1), we get

$$w\left(\frac{r}{R}e^{i\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-\frac{r^2}{R^2}}{1-\frac{2r}{R}\cos(\theta-t)+\frac{r^2}{R^2}} w(e^{it})dt$$

or

$$u(a + r e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R^2-r^2}{R^2-2rR\cos(\theta-t)+r^2}.u(a + R e^{it})\, dt \qquad (2)$$

Now $R - r \le | Re^{it} - r e^{i\theta}| \le R + r$

$$\Rightarrow \quad (R-r)^2 \le |R e^{it} - r e^{i\theta}|^2 \le (R + r)^2$$

$$\Rightarrow \quad \frac{1}{(R+r)^2} \le \frac{1}{|Re^{it} - re^{i\theta}|^2} \le \frac{1}{(R-r)^2}$$

Multiplying by $(R^2 - r^2)$, we get

$$\frac{R-r}{R+r} \le \frac{R^2-r^2}{|Re^{it}-re^{i\theta}|^2} \le \frac{R+r}{R-r}$$

$$\Rightarrow \quad \frac{R-r}{R+r} \le \frac{R^2-r^2}{R^2-2rR\cos(\theta-t)+r^2} \le \frac{R+r}{R-r}$$

Multiplying by $\frac{u(a+Re^{it})}{2\pi}$ and integrating w.r.t t between the limits −π to π, we get

$$\frac{1}{2\pi}\left(\frac{R-r}{R+r}\right)\int_{-\pi}^{\pi} u(a+Re^{it})dt \le u(a + re^{i\theta}) \le \frac{1}{2\pi}\left(\frac{R+r}{R-r}\right)\int_{-\pi}^{\pi} u(a + Re^{it})dt$$

[using(2)]

that is $\left(\frac{R-r}{R+r}\right)u(a) \le u(a + r\, e^{i\theta}) \le \left(\frac{R+r}{R-r}\right) u(a)$ since $u(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(a + R\, e^{it})dt$

Hence the result.

5.5. Definition. If G is an open subset of $\forall$ then Har(G) is the space of all harmonic functions on G.

5.6. Remark. Since Har(G) $\subset$ C(G, R) , Har(G) is given the metric that it inherits from C(G, R)

We now prove Harnack's theorem. The following results will be used.

(i) Let (X, d) be a complete metric space and Y $\subset$ X. Then Y is complete iff Y is closed.

(ii) Let (X, d) be a metric space. Then a set A $\subset$ X is closed iff for every sequence $<x_n>$ in A with $x_n \to x$, we have $x \in A$.

(iii) A metric space is connected iff it is not the union of two non-empty disjoint open sets.

5.7. Harnack's Theorem. Let G be a region

(a) The metric space Har(G) is complete

(b) If $\{u_n\}$ is a sequence in Har(G) such that $u_1 \le u_2 \le \ldots$ then either $u_n(z) \to \infty$ uniformly on compact subsets of G or $\{u_n\}$ converges in Har(G) to a harmonic function.

Proof. (a) We know C(G, R) is complete metric space and Har(G) $\subset$ C(G, R) so to show Har(G) is complete, it is sufficient to show that Har(G) is a closed subspace of C(G, R).

Let $\{u_n\}$ be a sequence in Har(G) such that $u_n \to u$ in C(G, R). Then $\{u_n\}$ converges uniformly to u in C(G, R).

$$\therefore \qquad \int u = \lim_{n\to\infty} \int u_n \qquad (1)$$

Let $\bar{B}(a\,;r)$ be a closed disk contained in G. Then

$$u_n(a) = \frac{1}{2\pi}\int_0^{2\pi} u_n(a + r\, e^{i\theta})\, d\theta \qquad (2)$$

as u_n is a harmonic function.

Now, $\{u_n(a)\}$ converges to u(a) so using (1) and we have

$$u(a) = \lim_{n\to\infty} u_n(a) = \frac{1}{2\pi}\lim_{n\to\infty}\int_0^{2\pi} u_n(a + r^{i\theta})\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} u\,(a + re^{i\theta})\, d\theta.$$

This shows that u has the MVP. Now $u : G \to R$ is a continuous function having MVP. So u is harmonic i.e. $u \in$ Har(G). Thus Har(G) is closed and so complete.

(b) Assume that $u_1 \ge 0$ because, otherwise, we may replace u_n by $u_n - u_1$. Let $u(z) = \sup\{u_n(z) : n \ge 1\}$ for each z in G. Then for each z in G, we have either $u(z) \equiv \infty$ on $u(z) \in R$ and $u_n(z) \to u(z)$.

Define $\qquad A = \{z \in G : u(z) = \infty\}$

$\qquad\qquad B = \{z \in G : u(z) < \infty\}$

Then $G = A \cup B$ and $A \cap B = \phi$.

We will show that both A and B are open. Let $a \in G$ and R be chosen such that $\bar{B}(a\,;R) \subset G$. Then by Harnack's inequality

$$\frac{R-|z-a|}{R+|z-a|}u_n(a) \le u_n(z) \le \frac{R+|z-a|}{R-|z-a|}u_n(a) \qquad (3)$$

for all z in B(a ; R) and all $n \ge 1$.

If $a \in A$ then $u_n(a) \to \infty$ so that $\frac{R-|z-a|}{R+|z-a|} u_n(a) \le u_n(z)$ implies $u_n(z) \to \infty$ for all z in B(a ; R) that is,

$$B(a : R) \subset A.$$

So, 'a' is interior point of A. But 'a' is arbitrary point of A. So every point of A is its interior point and hence A is open.

If $a \in B$ then $u(a) < \infty$. Using right half of (3), we have

$$u(z) < \infty \text{ for } |z-a| < R$$

i.e. $\quad u(z) < \infty$ for all z in B(a ;R)

$\therefore \quad B(a; R) \subset B$ and so B is open.

Since G is connected, we have either $A = \phi$ or $B = \phi$ that is, either B = G or A = G.
Suppose A = G. Then $u \equiv \infty$

Also if $\bar{B}(a ; R) \subset G$ and $0 < \rho < R$ hen $M = \frac{R-\rho}{R+\rho} > 0$ and so (3) implies.

$$M\, u_n(a) \le u_n(z) \text{ for } |z-a| \le \rho.$$

Hence $u_n(z) \to \infty$ uniformly for z in $\bar{B}$ (a ;ρ). Thus we have shown that for each a in G there is a $\rho > 0$ such that $u_n(z) \to \infty$ uniformly for $|z-a| \le \rho$. So $u_n(z) \to \infty$ uniformly for z in any compact set.

Now suppose B = G. Then u(z) < ∞ for all z in G. If ρ < R then there is a constant N, which depends only on a and ρ such that

$$M\, u_n(a) \le u_n(z) \le N\, u_n(a) \text{ for } |z-a| \le \rho \text{ and all n.}$$

So if $m \le n$, we have

$$\begin{aligned} 0 &\le u_n(z) - u_m(z) \\ &\le N\, u_n(a) - M\, u_n(a) \\ &\le c\,[u_n(a) - u_m(a)] \end{aligned}$$

for some constant c .

Thus $\{u_n(z)\}$ is uniformly Cauchy sequence on $\bar{B}$ (a ; ρ). It follows that $\{u_n\}$ is a Cauchy sequence in Har(G). Har(G) is complete, so $\{u_n\}$ must converge to a harmonic function. Since $u_n(z) \to u(z)$, we have $\{u_n\}$ converges to u is Har(G). This completes the proof of the theorem.

5.8. Subharmonic and Superharmonic Functions. Let G be a region and let $\phi : G \to R$ be a continuous function. Then φ is called **subharmonic** function if whenever $\bar{B}(a ; r) \subset G$,

$$\phi(a) \le \frac{1}{2\pi}\int_0^{2\pi} \phi(a + r\, e^{i\theta})d\theta$$

Also φ is called a **superharmonic** function if whenever $\bar{B}$ (a ; r) ⊂ G,

$$\phi(a) \ge \frac{1}{2\pi}\int_0^{2\pi} \phi(a + r\, e^{i\theta})d\theta$$

Clearly every harmonic function is subharmonic as well as superharmonic. In fact, u is harmonic iff u is both subharmonic and superharmonic. Also observe that φ is superharmonic iff −φ is subharmonic.

5.9. Definition. If $G \subset \forall$ then the boundary of G in $\forall_\infty$ is called extended bounded of G and is denoted by $\partial_\infty G$.

Clearly $\quad \partial_\infty G = \partial G$ if G is bounded and

$$\partial_\infty G = \partial G \cup \{\infty\} \text{ if } G \text{ is unbounded.}$$

5.10. Maximum Principle: Let G be a region and let ϕ and ψ be real valued functions defined on G such that ϕ is subharmonic and ψ is superharmonic. If for each point a in $\partial_\infty G$

$$\limsup_{z \to a} \phi(z) \le \liminf_{z \to a} \psi(z),$$

then either $\phi(z) < \psi(z)$ for all z in G or $\phi = \psi$ and ϕ is harmonic.

5.11. Definition. If G is a region and $f : \partial_\infty G \to R$ is a continuous function then the Perron Family $P(f, G)$, consists of all subharmonic functions

$$\phi : G \to R$$

such that $\limsup_{z \to a} \phi(z) \le f(a)$ for all a in $\partial_\infty G$.

Since f is continuous, there is a constant M such that

$$|f(a)| \le M \text{ for all a in } \partial_\infty G$$

So the constant function $-M$ is in $P(f, G)$ and the Perron Family is never empty.

5.12. Theorem. Let G be a region and $f : \partial_\infty G \to R$ be a continuous function : then

$$u(z) = \sup\{\phi(z) : \phi \in P(f, G)\} \text{ defines a harmonic function u on G.}$$

5.13. Definition. Let G be a region and $f : \partial_\infty G \to R$ be a continuous function then the harmonic function u defined by

$$u(z) = \sup\{\phi(z) : \phi \in P(f, G)\}$$

is called the **Perron Function** associated with f.

5.14. Definition. A region G is called a **Dirichlet's Region** if the Dirichlet's Problem can be solved for G. That is, G is a Dirichlet's Region if for each continuous function $f : \partial_\infty G \to R$ there is a continuous function $u : \overline{G} \to R$ such that u is harmonic in G and

$$u(z) = f(z) \text{ for all z in } \partial_\infty G.$$

By theorem 5.1, it follows that a disk is a Dirichlet's Region, but the punctured disk is not, as shown below.

Let $G = \{z : 0 < |z| < 1\}$, $T = \{z : |z| = 1\}$ so that $\partial G = T \cup \{0\}$

Define $f : \partial G \to R$ by

$f(z) = 0$ if $z \in T$

and $f(0) = 1$.

For $0 < \in < 1$, let $u_\in(z) = \dfrac{\log|z|}{\log \in}$

Then $u_\in$ is harmonic in G, $u_\in(z) > 0$ for z in G,

$$u_\in(z) = 0 \text{ for z in T and } u_\in(z) = 1 \text{ if } |z| = \in$$

Suppose that $v \in P(f, G)$

Since $|f| \le 1$, $|v(z)| \le 1$ for all z in G.

If $R_\in = \{z : \in < |z| < 1\}$ then $\limsup_{z \to a} v(z) \le u_\in(a)$ for all a in $\partial R_\in$. By the maximum principle,

$$v(z) \le u_\in(z) \text{ for all z in } R_\in.$$

Since $\in$ is arbitrary, this gives that for each z in G,

$$v(z) \le \lim_{\in \to 0} u_\in(z) = 0$$

Hence the Perron function associated with f is identically zero function and the Dirichlet's Problem cannot be solved for the punctured disk.

In this section we will see conditions that are sufficient for a region to be a Dirichlet Region.

Notation. For a set G and a point a in $\partial_\infty G$, let $G(a\,;r) = G \cap B(a\,;r)$ for all $r > 0$.

5.15. Definition. Let G be a region and let $a \in \partial_\infty G$. A barrier for G at 'a' is a family $\{\psi_r : r > 0\}$ of functions such that

(a) ψ_r is defined and superharmonic on $G(a\,;r)$ with $0 \le \psi_r(z) \le 1$;

(b) $\lim_{z\to a} \psi_r(z) = 0$

(c) $\lim_{z\to w} \psi_r(z) = 1$ for w in $G \cap \{w : |w-a| = r\}$.

5.16. Theorem. If G is a Dirichlet Region then there is a barrier for G at each point of $\partial_\infty G$.

Proof. Suppose $a \in \partial_\infty G$ s.t. $a \ne \infty$.

Let $$f(z) = \frac{|z-a|}{1+|z-a|} \text{ for } z \ne \infty$$

with $f(\infty) = 1$. Then f is continuous on $\partial_\infty G$.

So there is a continuous function $u : \bar{G} \to R$ such that u is harmonic on G and $u(z) = f(z)$ for z in $\partial_\infty G$. In particular, $u(a) = 0$ and a is the only zero of u in $\bar{G}$.

Let $$c_r = \inf\{u(z) : |z-a| = r, z \in G\}$$
$$= \min\{u(z) : |z-a| = r, z \,\varepsilon\, \bar{G}\} > 0$$

Define $\psi_r : G(a\,;r) \to R$ by
$$\psi_r(z) = \frac{1}{c_r}\min\{u(z), c_r\}.$$

Then $\{\psi_r\}$ is a barrier for G at a.

The next result provides a converse of above theorem.

5.17. Theorem. Let G be a region and let $a \,\varepsilon\, \partial_\infty G$ such that there is a barrier for G at a. If $f : \partial_\infty G \to R$ is continuous and u is the Perron Function associated with f then
$$\lim_{z\to a} u(z) = f(a).$$

Proof. Let $\{\psi_r : r > 0\}$ be a barrier for G at a.
For convenience assume that $a \ne \infty$.

Also by replacing f by $f - f(a)$, if necessary, we can suppose that $f(a) = 0$.
Let $\in > 0$ and choose $\delta > 0$ such that $|f(w)| < \in$ whenever $w \,\varepsilon\, \partial_\infty G$ and $|w-a| < 2\delta$. Let $\psi = \psi_\delta$.
Let $\hat{\psi} : G \to R$ be defined by
$$\hat{\psi}(z) = \psi(z) \text{ for z in } G(a\,;\delta)$$
and $$\hat{\psi}(z) = 1 \text{ for z in } G - B(a\,;\delta).$$
Then $\hat{\psi}$ is superharmonic.

If $|f(w)| \le M$ for all w in $\partial_\infty G$, then $-M\hat{\psi} - \in$ is subharmonic.

We claim that $-M\hat{\psi} - \in. \in P(f, G)$.

If $w \in \partial_\infty G - B(a\,;\delta)$ then
$$\limsup_{z\to w}\, [-M\hat{\psi}(z) - \in] = -M - \in < f(w).$$

Because $\hat{\psi}(z) \ge 0$, it follows that

$$\lim_{z\to w} \sup\, [-M\hat{\psi}(z) - \in] \le -\in \qquad \text{for all w in } \partial_\infty G.$$

In particular, if $w \in \partial_\infty G \cap B(a; \delta)$ then

$$\lim_{z\to w} \sup\, [-M\hat{\psi}(z) - \in] \le -\in < f(w) \text{ by the choice of } \delta.$$

Hence $\qquad -M\hat{\psi}(z) - \in \le u(z) \quad$ for all z in G. $\qquad$ (1)

Similarly

$$\lim_{z\to w} \inf\, [M\hat{\psi}(z) + \in] \ge \lim_{z\to w} \sup \phi(z)$$

for all ϕ in $P(f, G)$ and w in $\partial_\infty G$. By Maximum Principle,

$$\phi(z) \le M\hat{\psi}(z) + \in \text{ for } \phi \text{ in } P(f, G) \text{ and z in G}$$

Hence $\qquad u(z) \le M\hat{\psi}(z) + \in$ for all z in G $\qquad$ (2)

From (1) & (2),

$$-M\hat{\psi}(z) - \in \le u(z) < M\hat{\psi}(z) + \in \text{ for all z in G.}$$

But $\qquad \lim_{z\to a} \hat{\psi}(z) = \lim_{z\to a} \psi(z) = 0$ so

$$-\in \le \lim_{z\to a} u(z) \le \in \text{ for all z in G.}$$

As $\in$ is arbitrary +ve number, we get

$$\lim_{z\to a} u(z) = 0 = f(a).$$

This completes the proof.

Cor. A region G is a Dirichlet Region iff there is a barrier for G at each point of $\partial_\infty G$.

6. Green's Function

Here we introduce Green's function and discuss its existence. Such function plays an important role in differential equations and other fields of analysis.

6.1. Definition Let G be a region in the plane and let $a \in G$. A Green's Function of G with singularity at a is a function $g_a : G \to R$ with the properties.

(a) g_a is harmonic in $G - \{a\}$;
(b) $g(z) = g_a(z) + \log |z-a|$ is harmonic in a disk about a;
(c) $\lim_{z\to w} g_a(z) = 0$ for each w in $\partial_\infty G$.

6.2. Remarks. (1) For a given region G and a point a in G, g_a need not exist. However, if it exists, it is unique.

To prove this suppose h_a is another Green's Function for G with singularity at a. Then $h_a - g_a$ is harmonic in G. But (c) implies $\lim_{z\to w} [h_a(z) - g_a(z)] = 0$ for every w in $\partial_\infty G$. So by Maximum Principle, we have $h_a = g_a$.

(2) A Green's Function is positive In fct, g_a is harmonic in $G - \{a\}$ and $\lim_{z\to a} g(z) = +\infty$ since $g_a^{(z)} \log |z-a|$ is harmonic at z = a. So by Maximum Principle.

$$g_a(z) > 0 \text{ for all z in } G-\{a\}.$$

(3) $\forall$ has no Green's Function with a singularity at zero.

Suppose that g_0 is the Green's Function with singularity at zero. Let $g = -g_0$ so $g(z) < 0$ for all z as g_0 is positive.

We will show that g must be constant function. For this let z_1, z_2 be two complex numbers s.t. $0 \neq z_1 \neq z_2 \neq 0$. Let $\in > 0$ be given. Then there is a $\delta > 0$ such that

$$|g(z) - g(z_1)| < \in \text{ if } |z_1 - z| < \delta.$$

So $$g(z) < g(z_1) + \in \text{ if } |z - z_1| < \delta.$$

Let $r > |z_1 - z_2| > \delta$. Then

$$h_r(z) = \frac{g(z_1) + \in}{\log\left(\frac{\delta}{r}\right)} \log\left|\frac{z - z_1}{r}\right|$$

is harmonic in $\forall - \{z_1\}$.

Also $g(z) \leq h_r(z)$ for z on the boundary of the annulus

$A = \{z : \delta < |z - z_1| < r\}$. By the Maximum Principle,

$$g(z) \leq h_r(z) \text{ for z in A.}$$

In particular, $g(z_2) \leq h_r(z_2)$.

Letting $r \to \infty$, we get

$$g(z_2) \leq \lim_{r\to\infty} h_r(z_2) = g(z_1) + \in$$

Since $\in$ was arbitrary chosen positive number,

$$g(z_2) \leq g(z_1)$$

Interchanging the role of z_1 and z_2, we have

$$g(z_1) \leq g(z_2)$$

$$\therefore \quad g(z_1) = g(z_2)$$

Hence g must be a constant function which is a contradiction. Thus $\forall$ has no Green's function with a singularity at zero

The next theorem shows when do Green's Functions exist.

6.3. Theorem. If G is a bounded Dirichlet Region then for each a in G there is a Green's Function on G with singularity at a.

Proof. Define $f : \partial G \to R$ by

$$f(z) = \log |z - a|$$

and let $u : \overline{G} \to R$ be the unique continuous function which is harmonic on G such that $u(z) = f(z)$ for z in ∂G.

Then $g_a(z) = u(z) - \log |z - a|$ is the required Green's Function on G with singularity at a.

The next result shows that Green's Functions are conformal invariants.

6.4. Theorem. Let G and Ω be regions such that there is a one-one analytic function f of G onto Ω ; let $a \in G$ and $\alpha = f(a)$.

If g_a and γ_α are the Green's Function for G and Ω with singularities a and α respectively, then

$$g_a(z) = \gamma_\alpha (f(z))$$

Proof. Let $\phi : G \to R$ be defined by $\phi = \gamma_\alpha$ of.

We shall show $\phi = g_a$.

For this it is sufficient to show that ϕ has the properties of the Green's Function with singularity at z = a.

Clearly ϕ is harmonic in G–{a}.

Let w ε $\partial_\infty G$ and $\{z_n\}$ be a sequence in G with $z_n \to w$.

Then $\{f(z_n)\}$ is a sequence in Ω. So there is a subsequence $\{z_{nk}\}$ such that $f(z_{nk}) \to f(w)$ in $\bar{\Omega}$.
So $\qquad \gamma_\alpha(f(z_{nk})) \to 0$

Since this happens for any convergent subsequence of $\{f(z_n)\}$, it follows that

$$\lim_{n\to\infty} \phi(z_n) = \lim_{n\to\infty} \gamma_\alpha(f(z_n)) = 0.$$

Hence $\lim\limits_{z\to w} \phi(z) = 0$ for every w in $\partial_\infty G$.

By power series expansion of f about z = a, we have

$$f(z) = f(a) + A_1 (z-a) + A_2 (z-a)^2 + \ldots..$$

or

$$f(z) - \alpha = (z-a)[A_1 + A_2 (z-a) + \ldots.]$$

Hence $\log |f(z) - \alpha| = \log |z-a| + h(z)$ (1)

where $h(z) = \log |A_1 + A_2 (z-a) + \ldots.|$ is harmonic near z = a since $A_1 \neq 0$

Suppose $\gamma_\alpha(w) = \Delta(w) - \log |w-\alpha|$ where Δ is a harmonic function on Ω. Since $f : G \to \Omega$ is onto and $w \in \Omega$ so

$$w = f(z) \text{ for some } z \in G.$$

Thus $\qquad \gamma_\alpha (f(z)) = \Delta(f(z)) - \log |f(z) - \alpha|$

i.e. $\qquad \phi(z) = [\Delta(f(z)) - h(z)] - \log |z-a| \qquad$ [using (1)]

Since $\Delta 0 f - h$ is harmonic near z = a, $\phi(z) + \log |z-a|$ is harmonic near z = a. Therefore ϕ is a Green's Function of G with singularity at a. Hence it follows by uniqueness of Green's Function.

$$g_a = \phi$$

i.e. $\qquad g_a(z) = \phi(z)$

i.e. $\qquad g_a(z) = \gamma_\alpha(f(z))$

Hence the result.

7. Canonical Product

We recall the Weierstrass factorization theorem for entire functions. Let $f(z)$ be an entire function with a zero of multiplicity $m \geq 0$ at z = 0. Let $\{z_n\}$ be the non-zero zeros of $f(z)$, arranged so that a zero of multiplicity K is repeated in this sequence K times. Also suppose that $|z_1| \leq |z_2| \leq \ldots..$. If $\{p_n\}$ is a sequence of integers such that

$$\sum_{n=1}^{\infty} \left(\frac{R}{|z_n|}\right)^{p_n+1} < \infty, \text{ for every } R > 0, \text{ then}$$

$$P(z) = \prod_{n=1}^{\infty} E_{p_n}(z/z_n) \qquad (1)$$

converges uniformly on compact subsets of the plane, where by definition of primary factors, we have

$$E_p(z) = (1-z)\exp\left(z + \frac{z^2}{2} + \ldots + \frac{z^p}{p}\right) \qquad (2)$$

for $p \geq 1$ and $E_0(z) = 1-z$

Then the Weierstrass theorem says that

$$f(z) = z^m e^{g(z)} P(z) \qquad (3)$$

where g(z) is an entire function.

We are interested in the case in which g(z) and P(z) have certain characteristics which result in properties of f(z) and conversely. A convenient assumption for P(z) is that all the integers p_n are equal. Then we see that this is to assume that there is an integer $p \ge 1$ such that

$$\sum_{n=1}^{\infty} |z_n|^{-(p+1)} < \infty \tag{4}$$

i.e. it is an assumption on the growth rate of the zeros of f(z). Further, if we assume that p is the smallest integer for which the series (4) converges, then the product

$$P(z) = \prod_{n=1}^{\infty} E_p(z/z_n) \tag{5}$$

is called the **canonical product** associated with the sequence $\{z_n\}$ of zeros of f(z) and the integer p is called the **genus** of the canonical product. The restriction on g(z), we impose, is that it is a polynomial. Such an assumption must impose a growth condition on $e^{g(z)}$. When g(z) is a polynomial, then we say that f(z) is of finite genus and we define the genus of f(z) to be the degree of this polynomial or to be the genus of the canonical product whichever is greater.

Now we drive Jensen's formula which says that there is a relation between the growth rate of the zeros of f(z) and the growth of $M(r) = \sup \{|f(re^{i\theta})| : 0 \le \theta \le 2\pi\}$ as r increases. For this, we shall use Gauss-Mean Value Theorem which states that if f(z) is analytic in a domain D which contains the disc $|z - z_0| \le \rho$, then

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{i\theta})\, d\theta$$

If u is the real part of f(z), the above result gives Gauss-mean value theorem for harmonic function, as

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + \rho e^{i\theta})\, d\theta$$

7.1. Jensen's Formula. Let f(z) be analytic in the closed disc $|z| \le R$ and let $f(0) \ne 0$, $f(z) \ne 0$ on $|z| = R$. If $z_1, z_2, \ldots, z_n$ are zeros of f(z) in the open disc $|z| < R$ repeated according to their multiplicity, then

$$\log |f(0)| = -\sum_{i=1}^{n} \log\left(\frac{R}{|z_i|}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\phi})|\,d\phi.$$

Proof. Consider the function

$$F(z) = f(z) \prod_{i=1}^{n} \frac{R^2 - \bar{z}_i z}{R(z - z_i)} \tag{1}$$

We observe that F(z) is analytic in any domain in which f(z) is analytic and further $F(z) \ne 0$ for $|z| \le R$. Hence F(z) is analytic and never vanish on an open disc $|z| < \rho$ for some $\rho > R$.

Also $$|F(z)| = |f(z)| \tag{2}$$

on $|z| = R$, since

$$\left|\prod_{i=1}^{n} \frac{R^2 - \bar{z}_i z}{R(z - z_i)}\right| = \prod_{i=1}^{n} \left|\frac{R^2 - \bar{z}_i\, Re^{i\phi}}{R^2 e^{i\phi} - Rz_i}\right|, \; z = Re^{i\phi}$$

$$= \prod_{i=1}^{n} \left| \frac{R(R - \bar{z}_i e^{i\phi})}{Re^{i\phi}(R - z_i e^{-i\phi})} \right|$$

$$= \prod_{i=1}^{n} \left| \frac{R - \bar{z}_i e^{i\phi}}{R - z_i e^{-i\phi}} \right|, \ |e^{i\phi}| = 1$$
$$= 1$$

Since F(z) is analytic and non-zero in |z| < ρ, log F(z) is analytic in z| < ρ and consequently its real part log |F(z)| is harmonic there. Hence using Gauss-Mean value theorem for log |F(z)|, we get

$$\log |F(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log |F(Re^{i\phi})| d\phi \qquad (3)$$

Now, from (1),

$$F(0) = f(0) \prod_{i=1}^{n} \left(\frac{-R}{z_i} \right)$$

so that
$$|F(0)| = |f(0)| \prod_{i=1}^{n} \frac{R}{|z_i|}$$

and thus

$$\log |F(0)| = \log |f(0)| + \sum_{i=1}^{n} \log \frac{R}{|z_i|}$$

Also by (2), $|F(Re^{i\phi})| = |f(Re^{i\phi})|$ on |z| = R.
Therefore (3) becomes

$$\log |f(0)| + \sum_{i=1}^{n} \log \frac{R}{|z_i|} = \frac{1}{2\pi}\int_0^{2\pi} \log |f(Re^{i\phi})| d\phi$$

or

$$\log |f(0)| = -\sum_{i=1}^{n} \log \frac{R}{|z_i|} + \frac{1}{2\pi}\int_0^{2\pi} \log |f(Re^{i\phi})| d\phi$$

7.2. Poisson-Jensen Formula. Let $f(z)$ be analytic in the closed disc |z| ≤ R and let $f(z) \neq 0$ on |z| = R. If $z_1, z_2, \ldots, z_n$ are the zeros of $f(z)$ in the open disc |z| < R repeated according to their multiplicity and $z = re^{i\theta}$, $0 \le r < R$, then

$$\log |f(z)| = -\sum_{i=1}^{n} \log \left| \frac{R^2 - \bar{z}_i z}{R(z - z_i)} \right|$$
$$+ \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\log|f(Re^{i\phi})|}{R^2 - 2Rr\cos(\theta - \phi) + r^2} d\phi.$$

Proof. Consider the function

$$F(z) = f(z) \prod_{i=1}^{n} \frac{R^2 - \bar{z}_i z}{R(z - z_i)} \qquad (1)$$

Clearly F(z) is analytic in any domain in which $f(z)$ is analytic and F(z) ≠ 0 for |z| ≤ R. Hence F(z) is analytic and never vanish on an open disc |z| < ρ for some ρ > R. Also

$$|F(z)| = |f(z)| \text{ on } |z| = R$$

Since F(z) is analytic and non-zero in |z| < ρ, log F(z) is analytic in |z| < ρ and consequently its real part log |F(z)| is harmonic there. Hence using Poisson integral formula (unit-I) for log |F(z)|, we get

$$\log |F(z)| = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\log |F(Re^{i\phi})|}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\, d\phi \qquad (2)$$

Now, $\log |F(Re^{i\phi})| = \log |f(Re^{i\phi})|$ on $|z| = R$.

Also
$$\log |F(z)| = \log |f(z)| \prod_{i=1}^{n} \left|\frac{R^2 - \bar{z}_i z}{R(z - z_i)}\right|$$
$$= \log |f(z)| + \sum_{i=1}^{n} \log \left|\frac{R^2 - \bar{z}_i z}{R(z - z_i)}\right|$$

Therefore (2) becomes

$$\log |f(z)| = -\sum_{i=1}^{n} \log \left|\frac{R^2 - \bar{z}_i z}{R(z - z_i)}\right| + \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\log |f(Re^{i\phi})|}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\, d\phi$$

7.3. The Maximum Modulus of an Entire Function. Let $f(z)$ be a non-constant entire function. Define

$$M(r) = \text{Max}\,\{|f(z)| : |z| \le r\}$$

Since $f(z)$ is entire, by maximum modulus principle, $|f(z)|$ reaches its maximum value $M(r)$ on the circle $|z| = r$ so that

$$M(r) = \max\,\{|f(z)| : |z| = r\}$$

In fact, M(r) is a steadily increasing unbounded function of r. For this, by maximum modulus principle, we have.

$$|f(r_1 e^{i\theta})| < M(r_2)$$

whenever $r_1 < r_2$ and consequently $M(r_1) < M(r_2)$ and if $M(r)$ were bounded, then by Liouville's theorem, $f(z)$ would be constant.

7.4. Hadmard's Three Circle Theorem. Let $f(z)$ be analytic in $r_1 \le |z| \le r_3$ and let $r_1 < r_2 < r_3$. Let M_i be the maximum values of $|f(z)|$ on the circles $|z| = r_i$ ($i = 1, 2, 3$), then

$$M_2^{\log(r_3/r_1)} \le M_1^{\log(r_3/r_2)}.M_3^{\log(r_2/r_1)}$$

Proof. Let $F(z) = z^{\alpha} f(z)$, where α is a real constant to be determined later. Since $f(z)$ is analytic and also z^{α} is analytic, therefore $F(z)$ is analytic in the annulus $r_1 \le |z| \le r_3$.

The function $F(z)$ is not, in general, single-valued. But if we cut the annulus along negative real axis, we obtain a domain in which the principal branch of this function is analytic. By the maximum modulus principle, $|F(z)|$ attains maximum value on the boundary of the cut annulus. If we consider a branch of this function which is analytic in the part of the annulus for which $\pi/2 \le \arg z < 3\pi/2$, we see that the principal value cannot attain its maximum modulus on the cut and so must attain it on one of the boundary circles of the annulus. Thus it is shown that when $r_1 \le z| \le r_3$,

$$|z^{\alpha} f(z)| \le \max.\ \{r_1^{\alpha} M_1,\ r_3^{\alpha} M_3\}$$

Hence if $r_1 < r_2 < r_3$, we must have

$$r_2^{\alpha} M_2 \le \text{Max.}\{r_1^{\alpha} M_1,\ r_3^{\alpha} M_3\} \qquad (1)$$

We choose α so that

$$r_1^{\alpha}\ M_1 = r_3^{\alpha}\, M_3$$

which gives

$$(r_1/r_3)^{\alpha} = M_3/M_1$$

i.e. $$\alpha = \frac{\log(M_1/M_3)}{\log(r_3/r_1)} \tag{2}$$

With this value of α, (1) gives

$$r_2^{\alpha} M_2 \le r_1^{\alpha}\ M_1$$

or $$M_2 \le (r_2/r_1)^{-\alpha}\, M_1$$

$$= (r_2/r_1)^{-\frac{\log(M_1/M_3)}{\log(r_3/r_1)}} . M_1$$

Hence,

$$M_2^{\log(r_3/r_1)} \le (r_2/r_1)^{-\log(M_1/M_3)} . M_1^{\log(r_3/r_1)}$$

$$= (M_1/M_3)^{-\log(r_2/r_1)} . M_1^{\log(r_3/r_1)}$$

$$= M_1^{\log(r_3/r_2)} . M_3^{\log(r_2/r_1)} \tag{3}$$

where we have used the result

$$a^{\log b} = (e^{\log a})^{\log b} = (e^{\log b})^{\log a} = b^{\log a}$$

7.5. Remark. We say that a function $f(x)$ of a real variable x is convex downwards (or simply convex) if the arc

$$y = f(x),\ x_1 < x < x_2$$

lies below the chord joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. Equivalently, the condition may be stated as

$$f(x) \le \frac{x_2 - x}{x_2 - x_1} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2), \quad x_1 < x < x_2 \tag{4}$$

where the chord has the equation

$$y - f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) \text{ (Two point form)}$$

i.e. $$y = \left(\frac{x_2 - x}{x_2 - x_1}\right) f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2)$$

Hadmard's three circle theorem may no be expressed in convexity form by saying that M(r) is a convex function of log r since the inequality (3) may be written as (taking logarithm or both sides)

$$\log\ M(r_2) \le \frac{\log r_3 - \log r_2}{\log r_3 - \log r_1} \log M(r_1) + \frac{\log r_2 - \log r_1}{\log r_3 - \log r_1} \log M(r_3) .$$

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1. Growth and Order of an Entire Function

We recall that a polynomial $P_n(z)$ of degree n has exactly n zeros. Further the rate of growth as $z\to\infty$ of $|P_n(z)|$ increases as its degree n increases. Hence there exists a relationship, via the degree, between the number of zeros and the growth of the polynomial

Again, let $f(z)$ be a non-constant entire function. We define

$$M(r) = \max. \{| f(z)| : |z| \le r\}$$

We have already proved that M(r) is a steadily increasing unbounded function of r and thus M(r) steadily approaches ∞ as r→∞.

It is the growth rate of M(r) which is most easily related to the distribution of zeros of $f(z)$. In fact, considerable information about entire functions is gained by studying how fast M(r) approaches infinity. The technique to be used is to compare M(r) for large r with $\exp(r^\lambda)$ for various λ, where λ is a real constant.

An entire function $f(z)$ is said to be of finite order if there exists a real λ such that

$$M(r) \le \exp(r^\lambda) \text{ for all sufficiently large r} \qquad (1)$$

We also then define the order ρ of $f(z)$ as

$$\rho = \inf \{\lambda \ge 0 : M(r) \le \exp (r^\lambda) \text{ for sufficiently large r}\} \qquad (2)$$

i.e. the lower bound ρ of numbers λ for which (1) is true is called order of the entire function $f(z)$. We write $\rho = +\infty$ if $f(z)$ is not of finite order i.e. the set in (2) is empty. If $f(z)$ is of order ρ, then

$$M(r) \le \exp (r^{\rho+\in})$$

for every positive value of ∈ but not for negative values, provided r is sufficiently large. Functions of finite order are, after polynomials, the simplest integral functions.

1.1. Theorem. Let ρ be the order of an integral function $f(z)$, then

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}$$

where $M(r) = \max |f(z)|$ on $|z| = r$.

Proof. Let $\rho_1 = \inf. \{\lambda \ge 0 : M(r) \le \exp (r^\lambda) \text{ for sufficiently large r}\}$ (1)

Then by definition, ρ_1 is the order of the function $f(z)$. To prove the result, we are to prove that $\rho_1 = \rho$, where

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} \qquad (2)$$

Let $\in > 0$ be arbitrary, then (1) suggests that $M(r) \le \exp\ (r^{\rho_1+\in})$ for sufficiently large r .

Taking logarithm of both sides, we get

$$\log M(r) < r^{\rho_1+\in}$$

Again, taking logarithm, we find

$$\log\log M(r) \le (\rho_1 + \in) \log r$$

or $$\frac{\log\log M(r)}{\log r} < \rho_1 + \in$$

$$\therefore \quad \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} \le \rho_1 + \in$$

$$\Rightarrow \quad \rho \le \rho_1 + \in$$

Since $\in$ is arbitrary, so $\rho \le \rho_1$ (3)

On the other hand if $\in > 0$, then (2) shows that

$$\frac{\log\log M(r)}{\log r} < \rho + \in$$

or $\log\log M(r) < (\rho + \in) \log r = \log r^{(\rho+\in)}$

or $\log M(r) < r^{(\rho+\in)}$

or $M(r) < \exp(r^{\rho+\in})$

It follows that

$$\inf.\{\lambda \ge 0 : M(r) \le \exp(r^{\lambda}) \text{ for sufficiently large } r\} < \rho + \in$$

i.e. $\rho_1 < \rho + \in$

Since $\in$ is arbitrary, we obtain

$$\rho_1 \le \rho \qquad (4)$$

From (3) and (4), we conclude that

$$\rho_1 = \rho$$

Remark. In view of the above theorem, the order ρ of an entire function $f(z)$ is also given by

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} \qquad (5)$$

It should be noted that if $\lim\limits_{r\to\infty} \dfrac{\log\log M(r)}{\log r}$ exists whether finite or infinite, then this limit gives the order of $f(z)$. It is only when this limit does not exist that we find the limit (5) to obtain the order of $f(z)$.

1.2. Example. Find the order of the following functions

(i) $f(z) = a_0 + a_1z + a_2z^2 + \ldots + a_nz^n$, $a_n \ne 0$

(ii) e^{az}, $a \ne 0$, (iii) cos z, (iv) sin z, (v) $\cos\sqrt{z}$,

(vi) $e^{z^{\lambda}}$, which λ is a positive integer

(vii) $e^{e^{z}}$

Solution. (i) Here, $M(r) = |a_n|\, r^n$ for large $|z| = r$

$$\therefore \quad \lim_{r\to\infty} \frac{\log\log M(r)}{\log r} = \lim_{r\to\infty} \frac{\log\log(|a_n|\, r^n)}{\log r} \qquad \left|\frac{\infty}{\infty}\right. \text{ form}$$

$$= \lim_{r\to\infty} \frac{\dfrac{1}{\log(|a_n|\, r^n)}\, \dfrac{1}{|a_n|\, r^n}\, |a_n|\, n\, r^{n-1}}{1/r}$$

$$= \lim_{r\to\infty} \frac{n}{\log(|a_n|\, r^n)} = 0$$

Hence the order of a polynomial is zero.

(ii) Here, $M(r) = e^{|a|\, r}$

$$\therefore \qquad \lim_{r\to\infty} \frac{\log\log M(r)}{\log r} = \lim_{r\to\infty} \frac{\log(|a|r)}{\log r} \qquad \left| \frac{\infty}{\infty} \text{ form} \right.$$

$$= \lim_{r\to\infty} \frac{\frac{1}{|a|r}|a|}{1/r} = 1$$

Hence order of e^{az} is 1

(iii) Since

$$\cos z = 1 - \frac{z^2}{\lfloor 2} + \frac{z^4}{\lfloor 4} - \frac{z^6}{\lfloor 6} + ... \;,$$

we find that

$$|\cos z| \le 1 + \frac{|z|^2}{\lfloor 2} + \frac{|z|^4}{\lfloor 4}$$

$$\le 1 + \frac{r^2}{\lfloor 2} + \frac{r^4}{\lfloor 4} + ...$$

$$= \frac{e^r + e^{-r}}{2}, \text{ in the disc } |z| \le r$$

Thus $|\cos z| \le \dfrac{e^r + e^{-r}}{2}$ if $|z| \le r$

Hence $M(r) = \dfrac{e^r + e^{-r}}{2} = e^r\left(\dfrac{1+e^{-2r}}{2}\right)$

$$\Rightarrow \qquad \log M(r) = r + \log\left(\frac{1+e^{-2r}}{2}\right)$$

$$= r\left[1 + \frac{1}{r}\log\left(\frac{1+e^{-2r}}{2}\right)\right]$$

$$\therefore \qquad \lim_{r\to\infty} \frac{\log\log M(r)}{\log r} = \lim_{r\to\infty} \frac{\log r + \log\left[1 + \frac{1}{r}\left(\frac{1+e^{-2r}}{2}\right)\right]}{\log r}$$

$$= \lim_{r\to\infty}\left[1 + \frac{\log\left[1 + \frac{1}{r}\left(\frac{1+e^{-2r}}{2}\right)\right]}{\log r}\right]$$

$$= 1$$

Thus it follows that order of cos z is 1

(iv) Proceeding as above, we find that order of sin z is also 1

(v) Here, we observe that

$$M(r) = \frac{e^{\sqrt{r}} + e^{-\sqrt{r}}}{2}$$

and thus as in (iii), the order of cos $\sqrt{z}$ comes out to be $\frac{1}{2}$.

(vi) Here, $M(r) = e^{r^\lambda}$ and so by definition, order of e^{z^λ} is λ

or
$$\lim_{r\to\infty} \frac{\log\log M(r)}{\log r} = \lim_{r\to\infty} \frac{\lambda \log r}{\log r} = \lambda$$

(vii) In this case,
$$M(r) = e^{e^r}$$

so that
$$\lim_{r\to\infty} \frac{\log\log M(r)}{\log r} = \lim_{r\to\infty} \frac{r}{\log r} \qquad \Big| \frac{\infty}{\infty} \text{form}$$
$$= \lim_{r\to\infty} \frac{1}{1/r} = \lim_{r\to\infty} r = \infty$$

Hence e^{e^z} is of infinite order.
For further discussion, we shall need the following theorem.

1.3. Theorem. If the real part of an entire function g(z) satisfies the inequality Re $g(z) < r^{\rho+\in}$ for every $\in > 0$ and all sufficiently large r, then g(z) is a polynomial of degree not exceeding ρ .

Proof. Since g(z) is entire function, so by Taylor's expansion, we have
$$g(z) = a_0 + a_1 z + a_2 z^2 + \ldots\ldots = \sum_{n=0}^{\infty} a_n z^n$$

where
$$a_n = \frac{1}{2\pi i}\int_C \frac{g(z)}{z^{n+1}}\,dz \qquad (1)$$

C being the circle |z| = r.
Now when n > 0,
$$\frac{1}{2\pi i}\int_C \frac{\overline{g(z)}}{z^{n+1}}\,dz = \frac{1}{2\pi i}\int_C \left(\sum_{m=0}^{\infty} \bar{a}_m \bar{z}^m\right)\frac{dz}{z^{n+1}}$$
$$= \frac{1}{2\pi}\sum_{m=0}^{\infty}\int_0^{2\pi} \bar{a}_m r^m e^{-mi\theta}\,\frac{re^{i\theta}}{r^{n+1}e^{(n+1)i\theta}}\,d\theta \qquad |\ z = re^{i\theta}$$

where term by term integration being justified in view of the uniform convergence of the series.

So ,
$$\frac{1}{2\pi i}\int_C \frac{\overline{g(z)}}{z^{n+1}}\,dz = \frac{1}{2\pi}\sum_{m=0}^{\infty}\int_0^{2\pi} \bar{a}_m r^{m-n} e^{-(m+n)i\theta}\,d\theta$$
$$\mathbf{= 0} \qquad \mathbf{(2)}$$

Thus, from (1) and (2), we get
$$a_n = \frac{1}{2\pi i}\int_C \frac{g(z)}{z^{n+1}}\,dz + \frac{1}{2\pi i}\int_C \frac{\overline{g(z)}}{z^{n+1}}\,dz$$
$$= \frac{1}{2\pi i}\int_C \frac{g(z)+\overline{g(z)}}{z^{n+1}}\,dz$$
$$= \frac{1}{2\pi i}\int_C \frac{2\,\mathrm{Re}\,g(z)}{z^{n+1}}\,dz = \frac{1}{\pi}\int_0^{2\pi} \mathrm{Re}\,g(re^{i\theta})\frac{d\theta}{r^n e^{in\theta}}$$

Thus, it follows that

$$|a_n| \leq \frac{1}{\pi r^n}\int_0^{2\pi} |\text{Re } g(re^{i\theta})|\, d\theta \qquad \Big|\, |e^{in\theta}| = 1$$

or $$|a_n|\, r^n \leq \frac{1}{\pi}\int_0^{2\pi} |\text{Re } g(re^{i\theta})|\, d\theta \tag{3}$$

On the other hand

$$a_0 = \frac{1}{2\pi i}\int_C \frac{g(z)}{z}dz = \frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})\, d\theta$$

so $\text{Re}(a_0) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \text{Re } g(re^{i\theta})\, d\theta$ 4)

Hence from (3) and (4), we conclude that

$$2\text{ Re }(a_0) + |a_n|\, r^n \leq \frac{1}{\pi}\int_0^{2\pi} \{|\text{Re } g(z)| + \text{Re } g(z)\}\, d\theta$$

But the integrand is equal to 2 Re g(z) or 0 according as Re g(z) > 0 or ≤ 0. Since, by hypothesis Re $g(z) < r^{\rho+\in}$, it follows that

$$\begin{aligned} 2\text{ Re } a_0 + |a_n|\, r^n &\leq \frac{1}{\pi}\int_0^{2\pi} 2\text{ Re } g(z)\, d\theta \\ &< \frac{1}{\pi}\int_0^{2\pi} 2\, r^{\rho+\in}\, d\theta \\ &= \frac{2}{\pi} r^{\rho+\in}\, 2\pi = 4r^{\rho+\in} \end{aligned}$$

which holds for ∈>0 and all sufficiently large r.

If we write this inequality in the form

$$|a_n| < 4\, r^{\rho+\in-n} + (2\text{ Re } a_0)r^{-n}$$

and then make $r\to\infty$, we see that $a_n = 0$ when $n > \rho$ and so g(z) is a polynomial of degree not exceeding ρ and hence the proof of the theorem.

1.4. An Estimation of Number of Zeros. We shall denote by N(r) the number of zeros of an entire function *f*(z) in the closed disc |z| ≤ r.

1.5. Theorem. If *f*(z) is an entire function of order ρ, then for every ∈ > 0, the inequality $N(r) \leq r^{\rho+\in}$ holds for all sufficiently large r.

Proof. Without loss of generality, we may suppose that *f*(0) = 1. For if *f*(z) has a zero of order m at the origin, we may consider $g(z) = \dfrac{cf(z)}{z^m}$, where c is chosen so that g(0) = 1 and since the functions *f*(z) and g(z) have the same order, for our consideration it will be unimportant that the number of zeros of *f*(z) and g(z) differ by m.

We also assume at first that *f*(z) has no zero on |z| = 2r and we suppose that the zeros z_i of *f*(z) are arranged in non-decreasing order of their moduli so that $|z_i| \leq |z_{i+1}|$.

We apply Jensen's formula (Unit-IV) with R replaced by 2r and n = N(2r). We thus have

$$\frac{1}{2\pi}\int_0^{2\pi} \log |f(2re^{i\phi})|\, d\phi = \log |f(0)| + \sum_{i=1}^{N(2r)} \log\left(\frac{2r}{|z_i|}\right)$$

or $$\sum_{i=1}^{N(2r)} \log\left(\frac{2r}{|z_i|}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(2re^{i\phi})|\, d\phi \tag{1}$$

Now, $$\sum_{i=1}^{N(2r)} \log\left(\frac{2r}{|z_i|}\right) \geq \sum_{i=1}^{N(r)} \log\left(\frac{2r}{|z_i|}\right)$$
$$\geq \mathbf{N(r)\log 2} \qquad \textbf{(2)}$$

since for large r, $\log\left(\frac{2r}{|z_i|}\right) \geq \log 2$

And $$\frac{1}{2\pi}\int_0^{2\pi} \log|f(2re^{i\phi})|\, d\phi \leq \frac{1}{2\pi}\int_0^{2\pi} \log M(2r)\, d\phi = \log M(2r) \qquad (3)$$

Also, since ρ is the order (assumed finite) of the function $f(z)$, then by definition, for every $\in > 0$, we have

$$\log M(2r) \leq (2r)^{\rho+\in/2} \qquad (4)$$

Thus, we conclude from (1), (2), (3) and (4) that

$$N(r)\log 2 \leq (2r)^{\rho+\in/2}$$

or $$N(r) \leq \frac{2^{\rho+\in/2}\, r^{\rho+\in}}{(\log 2)\, r^{\in/2}} \leq r^{\rho+\in}, \qquad (5)$$

since r is large and $\in > 0$ implies that $\frac{2^{\rho+\in/2}}{(\log 2) r^{\in/2}} \leq 1$. The inequality (5) continuous to hold if we remove the restriction that there may be no zeros on $|z| = 2r$ for we may apply the inequality on slightly larger circles (since zeros are isolated i.e. cannot cluster) and use the fact that N(r) is right continuous. We note that if N(r) = m on $|z| = r$, then N(r) = m on [r, s] for some s > r, otherwise $f(z)$ would have a limit point of zeros. But zeros cannot cluster and hence N(r) is right continuous.

1.6. Exponent of Convergence. Let $\{z_1, z_2, \ldots\}$ be a sequence of non-zero complex numbers with $|z_n| \to \infty$. The exponent of convergence σ of the sequence is defined by

$$\sigma = \inf\{t > 0 : \sum_{i=1}^{\infty} |z_i|^{-t} < \infty\}$$

If the sequence is finite, we define $\sigma = 0$. It should be noted that $0 \leq \sigma \leq \infty$ and $\sigma = \infty$ iff $\sum_{i=1}^{\infty}|z_i|^{-t} = \infty$ for all t > 0. Also $\sigma = 0$ iff $\sum_{i=1}^{\infty}|z_i|^{-t} < \infty$ for all t > 0.

e.g. if $z_i = 2^i$, then $\sigma = 0$, since $\sum_{i=1}^{\infty}(2^i)^{-t}$ converges for all t > 0. We shall mainly use this concept in the case in which z_i are the zeros (counted according to multiplicity) of an entire function $f(z)$ and we shall always assume that $|z_i| \leq |z_{i+1}|$. We sometimes call σ as the convergence exponent of $f(z)$. or exponent of convergence of zeros of $f(z)$.

Thus, we observe that

$\sigma = 0 \Rightarrow$ existence of finite number of zeros of $f(z)$

$\sigma > 0 \Rightarrow$ existence of infinite number of zeros of $f(z)$

1.7. Remark. N(r) and σ are both measures of the growth of the numbers $|z_i|$. If the zeros are densely distributed, N(r) increases rapidly with r and since $|z_i| \to \infty$ slowly, σ is large. We may compare the definition of σ with that of the genus p already defined. It is evident that p, which is an integer, exists if $\sigma < \infty$ and in this case $p \leq \sigma \leq p+1$. The definition of p shows that $\sigma = p+1$ implies $\sum_{i=1}^{\infty}|z_i|^{-\sigma} < \infty$.

1.8. Theorem. If $f(z)$ is an entire function order ρ and convergence exponent σ, then $\sigma \le \rho$.

Proof. If ρ is infinite, the inequality, $\sigma \le \rho$ is trivial. Again, if the number of zeros is finite, then $\sigma = 0$ and $\sigma \le \rho$ holds. We may therefore suppose that ρ is finite and that there are infinitely many zeros which we arrange as a sequence $\{z_n\}$ such that

$$|z_n| \le |z_{n+1}| \text{ and } |z_n| \to \infty \text{ as } n \to \infty.$$

By definition of N(r), it is observed that

$$N(|z_n|) \ge n \qquad (1)$$

The strict inequality $N(|z_n|) > n$ will hold if

$$|z_n| = |z_{n+1}|$$

Also, we have proved that $N(r) \le r^{\rho+\epsilon}$, for large r.

Thus $$N(|z_n|) \le |z_n|^{\rho+\epsilon} \qquad (2)$$

if $\epsilon > 0$ and n is sufficiently large.

Thus from (1) and (2), we get

$$|z_n|^{\rho+\epsilon} \ge n \qquad (3)$$

for sufficiently large n.

Since $|z_n| \to \infty$, we may assume that $|z_n| \ge 1$. Given $t > \rho$, we may take $\epsilon < t-\rho$ so that $t/(\rho+\epsilon)>1$

Then for sufficiently large n, we have from (3),

$$|z_n|^t \ge n^{t/(\rho+\epsilon)}$$

Consequently, $$\sum_{n=1}^{\infty} |z_n|^{-t} \le \sum_{n=1}^{\infty} n^{-t/(\rho+\epsilon)} \qquad (4)$$

Since $t/(\rho+\epsilon) > 1$, the series on R. H. S. of (4) converges

and so $$\sum_{n=1}^{\infty} |z_n|^{-t} < \infty \qquad (5)$$

for every $t > \rho$.

Since by definition,

$$\sigma = \inf.\left\{ t > 0 : \sum_{n=1}^{\infty} |z_n|^{-t} < \infty \right\}$$

and since (5) holds for every $t > \rho$, we must have $\sigma \le \rho$

1.9. Borel's Theorem. The order of a canonical product is equal to the exponent of convergence of its zeros.

Proof. If ρ and σ be respectively the order and convergence exponent of the conical product, then we have proved in the previous theorem that $\sigma \le \rho$. So we only need to prove here that $\rho < \sigma$.

Let us recall the notation

$$E_K(w) = (1-w)\exp\left(w + \frac{1}{2}w^2 + \ldots\ldots + \frac{1}{K}w^K\right), K \ge 0$$

$$= (1-w)\exp\left(\sum_{n=1}^{K} \frac{w^n}{n}\right)$$

$$\therefore \quad |E_K(w)| \le (1+|w|)\exp\left|\sum_{n=1}^{K}\left(\frac{|w|^n}{n}\right)\right| \qquad (1)$$

Since $\exp |w| \ge 1+|w|$ and for large $|w|$,

$$\frac{|w|^K}{K} \ge \frac{|w|^{K-1}}{K-1} \ge \ldots \ge |w|$$

$$\Rightarrow \quad K\left(\frac{|w|^K}{K}\right) \geq \sum_{n=1}^{K} \frac{|w|^n}{n}$$

$$\Rightarrow \quad |w|^K \geq \sum_{n=1}^{K} \frac{|w|^n}{n}$$

Therefore (1) becomes

$$\begin{aligned} |E_K(w)| &\leq (\exp|w|)(\exp|w|^K) \\ &= \exp(|w| + |w|^K) \\ &\leq \exp(2|w|^K) \\ &\leq \exp(c|w|^\lambda) \qquad (2) \end{aligned}$$

where $c \geq 2$, $\lambda \geq K$ and $|w| \geq 1$.

On the other hand if $|w| \leq \frac{1}{2}$ and $K \geq 0$, then we have (Unit – III)

$$|E_K(w)| \leq \exp(2|w|^{K+1})$$

and so

$$|E_K(w)| \leq \exp[(2|w|^\lambda) \text{ for } \lambda \leq K + 1 \text{ and } |w| \leq \frac{1}{2} \qquad (3)$$

Again, if $1/2 \leq |w| \leq 1$ and $K > 0$, it can easily be shown that for some constant c,

$$|E_K(w)| \leq \exp(c|w|^{K+1})$$

Since $|w| \leq 1$, it follows that

$$|E_K(w)| \leq \exp(c|w|^\lambda) \qquad (4)$$

for $\lambda \leq K + 1$ and $\frac{1}{2} \leq |w| \leq 1$

From (2), (3) and (4), we conclude that for $K \leq \lambda \leq K + 1$, there exists c such that for all w

$$|E_K(w)| \leq \exp(c|w|^\lambda) \qquad (5)$$

Now, let $P(z) = \prod_{n=1}^{\infty} E_p\left(\frac{z}{z_n}\right)$ be the canonical product with zeros at z_n, n = 1, 2,… Then by definition, p is the genus of the canonical product and we know that p satisfies the inequality $p \leq \sigma \leq p + 1$. If $\sigma = p + 1$, let $\lambda = p+1$ while if $\sigma < p+1$, let λ satisfy $\sigma < \lambda < p + 1$.

Since $\qquad \sigma = \inf\{t > 0 : \sum_{i=1}^{\infty} |z_i|^{-t} < \infty\}$

we conclude that

$$\sum_{i=1}^{\infty} |z_i|^{-\lambda} < \infty. \text{ Let } \sum_{i=1}^{\infty} |z_i|^{-\lambda} = a \text{ (say)}$$

Then, we get

$$\begin{aligned} |P(z)| &\leq \sum_{n=1}^{\infty} \exp\left(c\left|\frac{z}{z^n}\right|^\lambda\right) \\ &= \exp\left(c|z|^\lambda \sum_{n=1}^{\infty} |z_n|^{-\lambda}\right) \\ &= \exp(ac|z|^\lambda) \qquad (6) \end{aligned}$$

which holds for every $\lambda > \sigma$ and all z.

Since (6) is of the form

$$M(r) \leq \exp(r^{\lambda+\epsilon})$$

and by definition.

$$\rho = \text{inf. } \{\lambda \geq 0 : M(r) \leq \exp(r^{\lambda})\}$$

therefore, we conclude that

$$\rho \leq \sigma$$

which completes the proof.

1.10. Theorem. Let P(z) be a canonical product of finite order ρ and $q > 0$ and $\in > 0$. Then for all sufficiently large $|z|$

$$|z - z_i| > |z_i|^{-q} \text{ implies } \log |P(z)| > -|z|^{\rho+\in}.$$

Proof. Let $|z| = r$, $|z_i| = r_i$. We have

$$E_K(w) = (1-w) \exp \left(w + \frac{1}{2}w^2 + ... + \frac{1}{K}w^K \right) \; K \geq 0$$

which gives

$$E_K(w) = \exp\left(- \sum_{n=K+1}^{\infty} \frac{w^n}{n} \right) \tag{1}$$

Also, for $|w| \leq \frac{1}{2}$, we have

$$|E_K(w)| \leq \exp(2|w|^{K+1}) \tag{2}$$

Now, from (1), we get

$$E_K(w) = 1\Big/ \exp\left(\sum_{n=K+1}^{\infty} \frac{w^n}{n} \right)$$

$$\Rightarrow \qquad |E_K(w)| \geq 1\Big/ \exp\left(\sum_{n=K+1}^{\infty} \frac{|w|^n}{n} \right)$$

$$\geq 1/\exp(2|w|^{K+1}) \qquad | \text{ using } (2)$$

$$= \exp(-2|w|^{K+1})$$

$$\geq \exp(-2|w|^{\lambda}) \text{ if } \lambda \leq K + 1$$

Thus $\qquad \log |E_K(w)| \geq -2|w|^{\lambda}$ (3)

where $|w| \leq \frac{1}{2}$ and $\lambda \leq K + 1$

Also, for sufficiently large $A > 0$ and for all w satisfying $|w| \geq \frac{1}{2}$, we have

$$\sum_{n=1}^{K} \frac{|w|^n}{n} \leq A|w|^K$$

Further, if $K \leq \lambda \leq K + 1$, then $|2w|^K \leq |2w|^{\lambda}$ and consequently

$$|w|^K \leq 2^{\lambda-K} |w|^{\lambda} \leq 2|w|^{\lambda}$$

Hence

$$\log\left| \exp\left(\sum_{n=1}^{K} \frac{w^n}{n} \right) \right| \leq \sum_{n=1}^{K} \frac{|w|^n}{n} \leq A|w|^K \leq 2A|w|^{\lambda} \tag{4}$$

provided $K < \lambda \leq K + 1$ and $|w| \geq \frac{1}{2}$

Now, we consider a fixed z with $|z| = r > 1$. We shall estimate separately the factors of

$$P(z) = \prod_{i=1}^{\infty} E_p\left(\frac{z}{z_i} \right)$$

for which $|z/z_i| > \frac{1}{2}$ i.e. $r_i < 2r$ and those for which $|z/z_i| \le \frac{1}{2}$ i.e. $r_i \ge 2r$.

We note that p is the genus of P(z). We thus write

$$|P(z)| = \prod_{r_i<2r}\left|E_p\left(\frac{z}{z_i}\right)\right|\prod_{r_i<2r}\left|E_p\left(\frac{z}{z_i}\right)\right| \tag{5}$$

Now, we estimate log |P(z)| in (5) by using (4) to write the logarithm of each factor corresponding to $r_i < 2r$ as

$$\log\left|E_p\left(\frac{z}{z_i}\right)\right| = \log\left|\left(1-\frac{z}{z_i}\right)\exp\left(\sum_{n=1}^{p}\frac{1}{n}\left(\frac{z}{z_i}\right)^n\right)\right|$$

$$= \log\left|1-\frac{z}{z_i}\right| + \log\left|\exp\sum_{n=1}^{p}\frac{1}{n}\left(\frac{z}{z_i}\right)^n\right|$$

$$\ge \log\left|1-\frac{z}{z_i}\right| - 2A\left|\frac{z}{z_i}\right|^{\lambda} \tag{6}$$

provided $p \le \lambda \le p + 1$.

Again by applying (3) to the factors in (5) for which $r_i \ge 2r$, we conclude for each of them that

$$\log\left|E_p\left(\frac{z}{z_i}\right)\right| \ge -2\left|\frac{z}{z_i}\right|^{\lambda} \tag{7}$$

provided $\lambda \le p + 1$.

Now, the order ρ of P(z) is also the exponent of convergence of the sequence $\{z_i\}$. If $\rho < p+1$, let λ satisfy $\rho < \lambda < p + 1$ and if $\rho = p + 1$, let $\lambda = p + 1$. Thus in either case, the definition of convergence exponent gives

$$\sum_{i=1}^{\infty}|z_i|^{-\lambda} < \infty \text{ with } p \le \lambda \le p + 1.$$

Let $\sum_{i=1}^{\infty}|z_i|^{-\lambda} = B$, where B is a finite constant.

Using the estimates (6) and (7), we obtain from (5) that

$$\log|P(z)| \ge \sum_{r_i<2r}\log\left|1-\frac{z}{z_i}\right| - 2A|z|^{\lambda}\sum_{r_i<2r}\left|\frac{1}{z_i}\right|^{\lambda} - 2|z|^{\lambda}\sum_{r_i\ge 2r}\left|\frac{1}{z_i}\right|^{\lambda}$$

$$\ge \sum_{r_i<2r}\log\left|1-\frac{z}{z_i}\right| - 2(A+1)|z|^{\lambda}\sum_{i=1}^{\infty}\left|\frac{1}{z_i}\right|^{\lambda}$$

$$= \sum_{r_i<2r}\log\left|1-\frac{z}{z_i}\right| - 2(A+1)B|z|^{\lambda}$$

Thus

$$\log|P(z)| \ge \sum_{r_i<2r}\log\left|1-\frac{z}{z_i}\right| - D|z|^{\lambda} \tag{8}$$

where D is a suitable constant.

Since by our assumption $|z| \geq 1$, we see that the inequality (8) must hold for every $\lambda > \rho$. By hypothesis

$$|z - z_i| > |z_i|^{-q} \text{ i.e. } \left|\frac{z}{z_i} - 1\right| > |z_i|^{-q-1}$$

so we get

$$\sum_{r_i < 2r} \log\left|1 - \frac{z}{z_i}\right| > \sum_{r_i < 2r} \log |z_i|^{-q-1}$$

$$\geq -(q+1)\, N(2r) \log (2r) \qquad \textbf{(9)}$$

Also, we know the result

$$N(2r) \leq (2r)^{\rho + \in/2} \qquad (10)$$

which holds for all sufficiently large r and arbitrary $\in > 0$. Using (9) and (10), we may write (8) as

$$\log |P(z)| \geq -(q+1)\,(2r)^{\rho+\in/2} \log (2r) - D\, r^{\lambda} \qquad (11)$$

We choose $\lambda < \rho + \in$ which is always possible for any prescribed $\in$. Then since r is large, we may have

$$-(q+1)\,(2r)^{\rho+\in/2} \log (2r) \geq -\frac{1}{2} r^{\rho+\in} \text{ and } -D\, r^{\lambda} \geq -\frac{1}{2} r^{\rho+\in}$$

Substituting these estimates in (11), we obtain

$$\log |P(z)| \geq -r^{\rho+\in}$$

which proves the required result.

1.11. Hadmard's Factorization Theorem. If $f(z)$ is an entire function of finite order ρ, then

$$f(z) = z^m e^{g(z)} P(z)$$

where m is the order of zeros of z at z = 0, g(z) is a polynomial of degree not exceeding ρ and P(z) is the canonical product associated with the sequence of non-zero zeros of $f(z)$

Proof. We have already shown in Weierstrass's factorization theorem that an entire function $f(z)$ can be expressed as

$$f(z) = z^m e^{g(z)} P(z) \qquad (1)$$

where g(z) is itself an entire function. Here, we shall use the addition hypothesis, that $f(z)$ is of finite order ρ, to show that g(z) is a polynomial of degree not exceeding ρ. It is clear hat the division of $f(z)$ by cz^m does not affect either the hypothesis or the conclusion of the theorem and so it is sufficient to consider the representation $f(z) = e^{g(z)} P(z)$, so that

$$e^{g(z)} = \frac{f(z)}{P(z)} \Rightarrow |e^{g(z)}| = \left|\frac{f(z)}{P(z)}\right|$$

i.e.

$$e^{\mathrm{Re}\, g(z)} = \left|\frac{f(z)}{P(z)}\right|$$

Taking logarithm, we get

$$\mathrm{Re}\, g(z) = \log |f(z)| - \log |P(z)| \qquad (2)$$

By the definition of order, it follows that

$$|f(z)| \leq \exp (r^{\rho+\in})$$

for sufficiently large $|z| = r$ and all $\in > 0$.

Thus $\log |f(z)| \leq r^{\rho+\in}$ (3)

If σ is the convergence exponent of the non-zero zeros of $f(z)$, then we have proved the result that $\sigma \le \rho$. Also, by Borel's theorem, σ is the order of the canonical product P(z) and so it follows from theorem 1.10 that

$$\log |P(z)| > -r^{\sigma+\epsilon} \text{ for large } |z| = r.$$

Thus

$$-\log |P(z)| < r^{\sigma+\epsilon} \le r^{\rho+\epsilon} \qquad (4)$$

where $\sigma \le \rho$ and r is large.

From (3) and (4), we get

$\log | f(z)| - \log |P(z)| \le 2r^{\rho+\epsilon}$

and thus (2) gives

$$\text{Re } g(z) \le 2r^{\rho+\epsilon}$$

Since r is large, we conclude from theorem 1.3 that g(z) is a polynomial of degree not exceeding ρ.

1.12. Example. Using Hadmard's factorization theorem, prove that

$$\sin \pi z = \pi z \sum_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

Solution. The zeros of sin πz are at z = 0, $\pm$ 1 $\pm$ 2…, i.e. non-zero zeros of sin πz are $\pm$ 1, $\pm$ 2… . Since the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so that p = 1 is the least integer such that $\Sigma \left|\frac{1}{\pm n}\right|^{p+1}$ ($n \ne 0$) converges. Thus the genus of the canonical product is 1 and thus the canonical product associated with non-zero zeros of sin πz is of the form

$$P(z) = \prod_{\substack{n=-\infty \\ n \ne 0}}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n}$$

$$= \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$$

$$= \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

Now, order of sin πz is 1. Since z = 0 is a simple zero of sin πz, Hadmard's factorization of sin πz may be written as

$$\sin \pi z = z e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

where g(z) is a polynomial of degree not exceeding 1 i.e. order of sin πz. let $g(z) = a_0 + a_1 z$.

$$\therefore \qquad \sin \pi z = z e^{a_0 a_1 z} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

We are to find a_0 and a_1. For this we write

$$\frac{\sin \pi z}{z} = e^{a_0 + a_1 z} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \qquad (1)$$

Since $\frac{\sin \pi z}{z} \to \pi$ as $z \to 0$, so making $z \to 0$ in (1) we get

$$\pi = e^{a_0}$$

$$\therefore \quad \frac{\sin \pi z}{z} = \pi e^{a_1 z} \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) \qquad (2)$$

Again, replacing z by –z in (2), we get

$$\frac{\sin \pi z}{z} = \pi e^{-a_1 z} \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) \qquad (3)$$

Equations (2) and (3) give $e^{a_1 z} = e^{-a_1 z} \Rightarrow a_1 = 0$

Hence $\sin \pi z = \pi z \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$.

2. The Range of an Analytic Function

Here, the range of an analytic function is investigated. A generic problem of this type is :

Let Φ be a family of analytic functions on a region G which satisfy some property P. What can be said about $f(G)$ for each f in Φ ? Are the sets $f(G)$ uniformly big in some sense ? Does there exists a ball $B(a; r)$ such that $f(G) \supset B(a ; r)$ for each f in Φ ? The answers to these questions depend on the property P that is used to define Φ.

We start with the following lemma.

2.1. Lemma. Let f be analytic in $D = \{z : |z| < 1\}$ such that $f(0) = 0$, $f'(0) = 1$ and $|f(z)| \le M$ for all z in D. Then $M \ge 1$ and

$$f(D) \supset B\left(0; \frac{1}{6M}\right)$$

Proof. Let $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$

Since $f(z)$ is analytic in $D = B(0 ; 1)$ so by Cauchy's Estimate,

$$|a_n| \le M \text{ for } n \ge 1$$

$$\Rightarrow \quad |a_1| \le M$$

$$\Rightarrow \quad M \ge 1 \qquad [\because a_1 = 1]$$

Let $z \in D$ s.t. $|z| = \frac{1}{4M}$. Then

$$|f(z)| \ge |z| - \sum_{n=2}^{\infty} |a_n z^n|$$

$$\ge \frac{1}{4M} - \sum_{n=2}^{\infty} M.\left(\frac{1}{4M}\right)^n$$

$$= \frac{1}{4M} - \frac{1}{16M}\left(1+\frac{1}{4M}+\frac{1}{16M^2}+\ldots..\right)$$

$$= \frac{1}{4M} - \frac{1}{16M}.\frac{1}{\left(1-\frac{1}{4M}\right)} = \frac{1}{4M} - \frac{1}{16M-4} = \frac{1}{4M}\left(\frac{12M-4}{16M-4}\right) \ge \frac{1}{6M}$$

$$\left[\because \text{Min value of } \frac{12M-4}{16M-4} \text{ is } \frac{2}{3} \text{ when } M = 1\right]$$

Suppose $w \in B\left(0; \frac{1}{6M}\right)$ Then $|w| < \frac{1}{6M}$

Consider the function $g(z) = f(z) - w$.

For $|z| = \frac{1}{4M}$, $|f(z) - g(z)| = |w| < \frac{1}{6M} \le |f(z)|$

So by Rouche's theorem, f and g have the same number of zeros in $B\left(0; \frac{1}{4M}\right)$. Since $f(0) = 0$

so $g(z_0) = 0$ for some $z_0 \in B\left(0; \frac{1}{4M}\right)$

$\therefore \quad f(z_0) - w = 0$ for some $z_0 \in D \qquad \left[\because \; B\left(0; \frac{1}{4M}\right) \subset D\right]$

i.e. $\quad w = f(z_0)$ for some $z_0 \in D$

i.e. $\quad w \in f(D)$

Hence $\quad B\left(0; \frac{1}{6M}\right) \subset f(D)$

2.2. Lemma. Suppose g(z) is analytic on B(0; R), g(0) = 0, $|g'(0)| = \mu > 0$ and $|g(z)| \le M$ for all z, then

$$g(B(0; R)) \supset B\left(0; \frac{R^2\mu^2}{6M}\right)$$

Proof. Let $f(z) = \frac{g(Rz)}{R\,g'(0)}$ for $z \in D$ where $D = \{z : |z| < 1\}$.

Then f is analytic on D, $f(0) = 0$, $f'(0) = 1$ and

$$|f(z)| = \left|\frac{g(Rz)}{R\,g'(0)}\right| = \frac{|g(Rz)|}{R\mu} = \frac{M}{R\mu} \text{ for all z in D.}$$

So by lemma 2.1,

$$B\left(0; \frac{\mu R}{6M}\right) \subset f(D)$$

To show $B\left(0; \frac{R^2\mu^2}{6M}\right) \subset g(B(0; R))$, let $w \in B\left(0; \frac{R^2\mu^2}{6M}\right)$

Then $\quad |w| < B\frac{R^2\mu^2}{6M}$

$$\Rightarrow \quad \left|\frac{w}{R\mu}\right| < \frac{R\mu}{6M}$$

$$\Rightarrow \quad \frac{w}{R\mu} \in B\left(0; \frac{\mu R}{6M}\right) \subset f(D)$$

$$\Rightarrow \quad \frac{w}{R\mu} = f(z) \text{ for some } z \in D.$$

$$\Rightarrow \quad \frac{w}{R\mu} = \frac{g(Rz)}{Rg'(0)} \text{ where } |z| < 1$$

$\Rightarrow \quad w = g(Rz)$ where $|Rz| < R$

$\Rightarrow \quad w \in g(B(0\,;R))$ $\qquad [\because Rz \in B(0\,;R)]$

Hence the result.

2.3. Lemma. Let f be an analytic function on the disk $B(0\,;r)$ such that

$|f'(z) - f'(a)| < |f'(a)|$ for all z in $B(a\,;r)$, $z \neq a$; then f is one one.

Proof. Suppose z_1 and z_2 are points in $B(a\,;r)$ s. t. $z_1 \neq z_2$. Let γ be the line segment $[z_1, z_2]$ then

$$|f(z_1) - f(z_2)| = \left|\int_\gamma f'(z)dz\right|$$

$$\geq \left|\int_\gamma f'(a)dz\right| - \left|\int_\gamma [f'(z) - f'(a)]dz\right|$$

$$\left[\because \left|\int_\gamma f'(a)dz\right| \leq \left|\int_\gamma [f'(a) - f'(z)dz\right| + \left|\int_\gamma f'(z)dz\right|\right]$$

$$\geq |f'(a)|\,|z_1 - z_2| - \int_\gamma |f'(z) - f'(a)|\,|dz|$$

$$> |f'(a)|\,|z_1 - z_2| - |f'(a)|\,|z_1 - z_2| = 0$$

$\Rightarrow \quad f(z_1) \neq f(z_2)$

Hence f is one one.

2.4. Bloch's Theorem. Let f be an analytic function on a region containing the closure of the disk $D = \{z : |z| < 1\}$ and satisfying $f(0) = 0$, $f'(0) = 1$. Then there is a disk $S \subset D$ on which f is one-one and such that $f(S)$ contains a disk of radius $\frac{1}{72}$.

Proof. Let $K(r) = \max\{|f'(z)| : |z| = r\}$ and let $h(r) = (1-r)\,K(r)$.
Then $h : [0, 1] \to \mathbb{R}$ is continuous such that $h(0) = K(0) = 1$ and $h(1) = 0$.
Let $r_0 = \sup\{r : h(r) = 1\}$.
Then $h(r_0) = 1$, $r_0 < 1$ as $h(1) = 0$ and $h(r) < 1$ if $r > r_0$. Choose a s.t. $|a| = r_0$ and $|f'(a)| = K(r_0)$.

Then $\qquad |f'(a)| = \dfrac{1}{1-r_0}$ $\qquad [\because (1-r_0)\,K(r_0) = h(r_0) = 1]$

Let $z \in B(a\,;\rho_0)$ where $\rho_0 = \frac{1}{2}(1-r_0)$.

Then $|z-a| < \rho_0$.

So $|z| = |z-a+a| < |z-a| + |a| < \frac{1}{2}(1-r_0) + r_0 = \frac{1}{2}(1+r_0)$

$$\therefore \quad |f'(z)| \leq K\left(\frac{1}{2}(1+r_0)\right) = \frac{h\left(\frac{1}{2}(1+r_0)\right)}{1-\frac{1}{2}(1+r_0)} < \frac{1}{\frac{1}{2}(1-r_0)} = \frac{1}{\rho_0} \qquad \left[\because h\left(\frac{1}{2}(1+r_0)\right) < 1\right]$$

$\Rightarrow \quad |f'(z)| < \dfrac{1}{\rho_0}$ for $|z-a| < \rho_0$.

So $|f'(z) - f'(0)| \leq |f'(z)| + |f'(a)|$

$$< \frac{1}{\rho_0} + \frac{1}{2\rho_0} = \frac{3}{2\rho_0} \qquad \left[\because |f'(a)| = \frac{1}{1-r_0} = \frac{1}{2\rho_0}\right]$$

Using Schwarz's Lemma, this gives

$$|f'(z) - f'(a)| < \frac{3}{2\rho_0^2}|z-a| \text{ for } z \in B(a\,;\rho_0)$$

Let $S = B\left(a\,;\frac{1}{3}\rho_0\right)$ Then for $z \in S$,

$$|f'(z) - f'(a)| < \frac{1}{2\rho_0} \qquad \left[\because |z-a| < \frac{\rho_0}{3}\right]$$

i.e. $\quad |f'(z) - f'(a)| < |f'(a)|\}$ $\qquad \left[\because |f'(a)| = \frac{1}{1-r_0} = \frac{1}{2\rho_0}\right]$

So by Lemma 2.3, f is one-one on S.

Now we will show that $f(S)$ contains a disk of radius $\frac{1}{72}$.

Define a function $g : B\left(0;\frac{1}{3}\rho_0\right) \to \mathbb{C}$

as $g(z) = f(z+a) - f(a)$

Then $\qquad g(0) = f(a) - f(a) = 0$

and $\qquad |g'(0)| = |f'(0)| = \frac{1}{2\rho_0}$

Let $z \in B\left(0;\frac{1}{3}\rho_0\right)$. Then the line segment $\gamma = [a, z+a]$ lies in $S \subset B(a\,;\rho_0)$

So

$$|g(z)| = \left|\int_\gamma f'(w)dw\right|$$
$$\leq \int_\gamma |f'(w)||dw|$$
$$< \frac{1}{\rho_0}|z| < \frac{1}{3}$$

By Lemma 2.2, we get $g\left(B\left(0;\frac{1}{3}\rho_0\right)\right) \supset B(0\,;\sigma)$

where

$$\sigma = \frac{\left(\frac{1}{3}\rho_0\right)^2\left(\frac{1}{2\rho_0}\right)^2}{6\left(\frac{1}{3}\right)} = \frac{1}{72}$$

$$\therefore \qquad B\left(0;\frac{1}{72}\right) \subset g\left(B\left(0;\frac{1}{3}\rho_0\right)\right)$$

Now we show $\qquad B(f(a)\,;\frac{1}{72}) \subset f(S)$.

Let $w \in B(f(a)\,;\frac{1}{72})$. Then $|w - f(a)| < \frac{1}{72}$

$$\Rightarrow \qquad w - f(a) \in B\left(0;\frac{1}{72}\right)$$

$\Rightarrow$ $w - f(a) = g(z)$ for some $z \in B\left(0; \frac{1}{3}\rho_0\right)$

$\therefore$ $w - f(a) = f(z+a) - f(a)$ where $|z| < \frac{1}{3}\rho_0$

$\Rightarrow$ $w = f(z+a)$ where $|z+a-a| < \frac{1}{3}\rho_0$

$\Rightarrow$ $w \in f(S)$ as $z + a \in S$.

Hence $B\left(f(a); \frac{1}{72}\right) \subset f(S)$.

2.5. Corollary. Let f be analytic function on a region containing the closure of B(0 ; R); then $f(B(0; R))$ contains a disk of radius $\frac{1}{72} R|f'(0)|$

Proof. If $f'(0) = 0$ then the result is trivial.

So assume that $f'(0) \neq 0$.

Consider the function $g(z) = \dfrac{f(Rz) - f(0)}{Rf'(0)}$

Then g is analytic on a region containing the closure of B(0; 1). Also g(0) = 0, g′(0) = 1.

So by Bloch's theorem,

$$B\left(g(a); \frac{1}{72}\right) \subset g(B(0\,;1)) \text{ for some } a\,.$$

We claim that $B\left(f(aR)\,; \frac{1}{72} R|f'(0)|\right) \subset f(B(0; R))$

Let $w \in B\left(f(a\,R)\,; \frac{1}{72} R|f'(0)|\right)$

Then $|w - f(a\,R)| < \frac{1}{72} R\,|f'(0)|$

$\Rightarrow$ $|w - (g(a)\,Rf'(0) + f(0)| < \frac{1}{72} R\,|f'(0)|$

$\Rightarrow$ $\left|\dfrac{w}{Rf'(0)} - \dfrac{f(0)}{Rf'(0)} - g(a)\right| < \dfrac{1}{72}$

$\Rightarrow$ $\dfrac{w}{Rf'(0)} - \dfrac{f(0)}{Rf'(0)} = g(z)$ where $z \in B(0\,;1)$

$\Rightarrow$ $w - f(0) = R\,f'(0)\,g(z)$

$\Rightarrow$ $w - f(0) = f(Rz) - f(0)$ where $z \in B(0; 1)$

$\Rightarrow$ $w = f(Rz)$ where $|Rz| < R$

$\Rightarrow$ $w \in f(B(0; R))$ as $Rz \in B(0; R)$

Hence $B\left(f(aR)\,; \frac{1}{72} R|f'(0)|\right) \subset f(B(0; R))$

2.6. Definition. Let Φ be the set of all functions f analytic on a region containing the closure of the disk $D = \{z : |z| < 1\}$ and satisfying $f(0) = 0$, $f'(0) = 1$. For each f in Φ, let $\beta(f)$ be the supremum of all numbers r such that there is a disk S in D an which f is one-one and such that f(S) contains a disk of radius r. Then $\beta(f) \geq \frac{1}{72}$

Bloch's constant B is defined as

$$B = \inf \{\beta(f) : f \in \Phi\}$$

By Bloch's theorem, $B \geq \frac{1}{72}$

If we take $f(z) = z$ then $B \leq 1$. So $\frac{1}{72} \leq B \leq 1$.

However, better estimates than these are known. In fact, it is known that 0.43 ≤ B ≤ 0.47. Although the exact value of B remains unknown, but it has been conjectured (guessed) that

$$B = \frac{\Gamma(1/3\Gamma(11/12)}{\sqrt{1+\sqrt{3}}\Gamma(1/4}$$

2.7. Definition. Let Φ be the set of all function f analytic on a region containing the closure of the disk $D = \{z : |z| < 1\}$ and satisfying $f(0) = 0$, $f'(0) = 1$.

For each f in Φ, define

$$\lambda(f) = \sup \{r : f(D) \text{ contains a disk of radius } r\}$$

Landau's constant L is defined by

$$L = \inf \{\lambda(f) : f \in \Phi\}$$

Clearly $L \geq B$ and $L \leq 1$. Although exact value of L is unknown but it can be proved that

$$0.50 \leq L \leq 0.56$$

In particular, L > B.

2.8. Proposition. If f is analytic on a region containing the closure of the disk $D = \{z : |z| < 1\}$ and $f(0) = 0$ $f'(0) = 1$; then $f(D)$ contains a disk of radius L, where L is Landau's constant.

Proof. We shall show that $f(D)$ contains a disk of radius λ where

$$\lambda = \lambda(f)$$

Since $\lambda = \sup \{r : f(D)$ contains a disk of radius $r\}$ for each $n \in N$, there is a point α_n in $f(D)$ such that

$$B\left(\alpha_n ; \lambda - \frac{1}{n}\right) \subset f(D)$$

Now $\alpha_n \in f(D) \subset f(\bar{D})$ and $f(\bar{D})$ is compact. Since every compact metric space is sequentially compact, $f(\bar{D})$ is sequentially compact. So every sequence of points in $f(\bar{D})$ contains a convergent subsequence. In particular, $<\alpha_n>$ contains a subsequence $<\alpha_{n_k}>$ such that $\alpha_{n_k} \to \alpha \in f(\bar{D})$.

We may assume that $\alpha = \lim \alpha_n$. We show that $B(\alpha ; \lambda) \subset f(D)$. Let $w \in B(\alpha ; \lambda)$. Then $|w-\alpha| < \lambda$.

Choose n_0 s.t. $|w-\alpha| < \lambda - \frac{1}{n_0}$

Since $\alpha_n \to \alpha$ so there exists an integer $n_1 > n_0$ s.t.

$$|\alpha_n - \alpha| < \lambda - \frac{1}{n_0} - |w-\alpha| \text{ for } n \geq n_1.$$

$$\therefore \quad |w-\alpha_n| = |w-\alpha + \alpha-\alpha_n|$$
$$\leq |w-\alpha| + |\alpha - \alpha_n|$$

$$< \lambda - \frac{1}{n_0}$$

$$< \lambda - \frac{1}{n} \text{ if } n \geq n_1$$

$$\Rightarrow \quad w \in B\left(\alpha_n ; \lambda - \frac{1}{n}\right) \subset f(D)$$

Hence $B(\alpha; \lambda) \subset f(D)$.

2.9. Corollary. Let f be analytic on a region that contains the closure of $B(0 ; R)$; then $f(B(0 ;R))$ contains a disk of radius $R|f'(0)|$ L.

Proof. If $f'(0) = 0$ then the result is trivial. So assume that $f'(0) \neq 0$. Applying the above theorem to the function $g(z) = \frac{f(Rz) - f(0)}{Rf'(0)}$, we get the required result.

2.10. Definition. If G is an open connected set in $\forall$ and $f : G \to \forall$ is a continuous function such that $z = \exp f(z)$ for all z in G then f is a branch of the logarithm.

2.11. Lemma. Let G be a simply connected region and suppose that f is an analytic function on G that does not assume the values 0 or 1. Then there is an analytic function g on G such that

$$f(z) = -\exp (i\pi \cosh [2g(z)] \text{ for z in G.}$$

Proof. Since f never vanishes, there is a branch l of $\log f(z)$ defined on G; that is $e^l = f$.

Let $$F(z) = \frac{1}{2\pi i} l(z)$$

We claim that F does not assume any integer value. If F(a) = n for some integer n, then

$f(a) = \exp(2\pi \text{ i } F(a)) = \exp(2\pi \text{ in}) = 1$ which is not possible.

Since F cannot assume the values 0 and 1, it is possible to define

$$H(z) = \sqrt{F(z)} - \sqrt{F(z) - 1}$$

Now $H(z) \neq 0$ for any z so that it is possible to define a branch g of log H on G, that is, $e^g = H$.

$$\therefore \quad \cosh(2g) + 1 = \frac{e^{2g} + e^{-2g}}{2} + 1$$

$$= \frac{(e^g + e^{-g^2})}{2}$$

$$= \frac{1}{2}\left(H + \frac{1}{H}\right)^2$$

$$= 2F \qquad \left[\begin{array}{l} \because H(z) = \sqrt{F(z)} - \sqrt{F(z) - 1} \\ \text{and } \frac{1}{H(z)} = \sqrt{F(z)} + \sqrt{F(z) - 1} \end{array}\right]$$

$$= \frac{1}{\pi i} l$$

This gives

$$f = \exp(l) = \exp[\pi \text{ i} + \pi \text{ i} \cosh(2g)]$$

$$\mathbf{= \exp(\pi\, i).\ \exp(\pi\, i \cosh(2g))}$$

$$= -\exp(\pi \text{ i} \cosh(2g))$$

Hence the result.

2.12. Lemma. Let G be a simply connected region and suppose that *f* is an analytic function on G that does not assume the values 0 or 1. Let g be an analytic function on G such that

$$f(z) = -\exp(i\,\pi \cosh[2g(z)]) \text{ for } z \text{ in } G.$$

Then g(G) contains no disk of radius 1.

Proof. Let n be a positive integer and m any integer.
We claim that g cannot assume any of the values

$$\{\pm \log(\sqrt{n}+\sqrt{n-1})+\frac{1}{2}im\pi : n\geq 1,\ m = 0, \pm 1, \pm 2, \ldots\}$$

If there is a point a in G such that

$$g(a) = \pm \log\ \sqrt{n}+\sqrt{n-1})+\frac{1}{2}im\pi$$

then
$$\begin{aligned}
2\cosh[2g(a)] &= e^{2g(a)} + e^{-2g(a)}\\
&= e^{im\pi}(\sqrt{n}+\sqrt{n-1})^{\pm 2} + e^{-im\pi}(\sqrt{n}+\sqrt{n-1})^{\pm 2}\\
&= (-1)^m[(\sqrt{n}+\sqrt{n-1})^2 + (\sqrt{n}-\sqrt{n-1})^2]\quad [\because e^{i\pi} = -1]\\
&= (-1)^m[2(2n-1)]
\end{aligned}$$

$$\therefore \quad \cosh(2g(a)) = (-1)^m(2n-1)$$

So
$$\begin{aligned}
f(a) &= -\exp[(-1)^m(2n-1)\pi i]\\
&= 1 \qquad [\because (2n-1) \text{ is odd}]
\end{aligned}$$

which is not true. Hence g cannot assume any of the values

$$\{\pm \log\ (\sqrt{n}+\sqrt{n-1})+\frac{1}{2}im\pi : n\geq 1,\ m = 0, \pm 1, \pm 2, \ldots\}$$

These points form the vertices of a grid of rectangles in the plane. The height of an arbitrary rectangle is

$$\left|\frac{1}{2}im\pi - \frac{1}{2}i(m+1)\pi\right| = \frac{\pi}{2} < \sqrt{3}$$

The width is $\log(\sqrt{n+1}+\sqrt{n}) - \log(\sqrt{n}+\sqrt{n-1}) > 0$.

Now, $\phi(x) = \log\ (\sqrt{x+1}+\sqrt{x}) - \log(\sqrt{x}+\sqrt{x-1})$ is a decreasing function so that the width of any rectangle $\leq \phi(1) = \log\ (1+\sqrt{2}) < \log e = 1$. So the diagonal of the rectangle < 2. Hence g(G) contains no disk of radius 1.

2.13. Little Picard Theorem. If *f* is an entire function that omits two values then *f* is a constant.

Proof. Let a and be two values omitted by *f*. so that

$$f(z) \neq a \text{ and } f(z) \neq b \text{ for all } z.$$

Then the function $\dfrac{f-a}{b-a}$ omits the values 0 and 1

So assume that $f(z) \neq 0$ and $f(z) \neq 1$ for all z. By lemma 2.12, there is an entire function g such that g(∀) contains no disk of radius 1.

We want to prove *f* is a constant function. Let, if possible, suppose that *f* is not a constant function. Then g is also not a constant so there is a point z_0 with $g'(z_0) \neq 0$ by considering $g(z + z_0)$ if necessary, we may suppose that

$$g'(0) \neq 0$$

$\therefore$ g(B(0 ; R)) contains a disk of radius LR $|g'(0)|$. Choosing R sufficiently large we get that g($\forall$) contains a disk of radius 1. A contradiction. Hence f must be constant.

2.14. Schottky's Theorem. For each α and β, $0 < \alpha < \infty$ and $0 \le \beta \le 1$, there is a constant $C(\alpha, \beta)$ such that if f is an analytic function on some simply connected region containing the closure of B(0; 1) that omits the values 0 and 1 and such that $|f(0)| \le \alpha$; then $|f(z)| \le C(\alpha, \beta)$ for $|z| \le \beta$.

Proof. We consider two cases

Case I. Suppose $\frac{1}{2} \le |f(0)| \le \alpha$.

Since f never vanishes there is a branch l of log $f(z)$ s.t. $0 \le \text{Im } l(0) < 2\pi$.

Let $$F(z) = \frac{1}{2\pi i} l(z)$$

Then
$$|F(0)| = \frac{1}{2\pi} |\log |f(0)| + i \text{ Im } l(0)|$$
$$< \frac{1}{2\pi}(\log \alpha + 2\pi) = \frac{1}{2\pi}\log \alpha + 1 = C_0(\alpha) \text{ (say)}$$

$$\therefore \quad |F(0)| < C_0(\alpha) \tag{1}$$

Also
$$|\sqrt{F(0)} \pm \sqrt{F(0)-1}| \le |\sqrt{F(0)}| + |\sqrt{F(0)-1}|$$
$$= |F(0)|^{1/2} + |F(0)-1|^{1/2}$$
$$\le [C_0(\alpha)]^{1/2} + [C_0(\alpha)+1]^{1/2} = C_1(\alpha) \text{ (say)}$$

Now F cannot assume the values 0 and 1, it is possible to define
$$H(z) = \sqrt{F(z)} - \sqrt{F(z)-1}$$

Also $H(z) \neq 0$ for any z so define a branch g of log H s.t.
$$0 \le \text{Im } g(0) \le 2\pi.$$

If $H(0) \ge 1$, then
$$|g(0)| = |\log |H(0)| + i \text{ Im } g(0)|$$
$$\le \log |H(0)| + 2\pi$$
$$\le \log C_1(\alpha) + 2\pi$$

If $|H(0)| < 1$ then
$$|g(0)| \le -\log |H(0)| + 2\pi$$
$$= \log \left(\frac{1}{|H(0)|}\right) + 2\pi$$
$$= \log |\sqrt{F(0)} + \sqrt{F(0)-1}| + 2\pi$$
$$\le \log C_1(\alpha) + 2\pi$$

$$\therefore \quad |g(0)| \le C_2(\alpha) \text{ where } C_2(\alpha) = \text{Log } C_1(\alpha) + 2\pi.$$

If $|a| < 1$ then g(B(a ; 1–|a|)) contains a disk of radius $L(1-|a|)\,|g'(a)|$ [By cor. 2.9]
On the other hand g(B(0 ; 1)) contains no disk of radius 1. So we must have
$$L(1-|a|)\,|g'(a)| < 1 \text{ for } |a| < 1$$

i.e.
$$|g'(a)| < \frac{1}{(1-|a|)L} \text{ for } |a| < 1 \tag{2}$$

If $|a| < 1$, let γ be the line segment [0, a]; then
$$|g(a)| = |g(0) + g(a) - g(0)|$$

$$\leq |g(0)| + |g(a) - g(0)|$$
$$\leq C_2(\alpha) + \left|\int_\gamma g'(z)dz\right|$$
$$\leq C_2(\alpha) + |a| \max\{|g'(z)| : z \in [0, a]\}$$
$$\leq C_2(\alpha) + \frac{|a|}{L(1-|a|)} \qquad \text{(using (2))} \qquad (3)$$

Let $C_3(\alpha, \beta) = C_2(\alpha) + \dfrac{\beta}{L(1-\beta)}$

Then (3) gives

$$|g(z)| \leq C_3(\alpha, \beta) \text{ if } |z| \leq \beta.$$

Consequently, if $|z| \leq \beta$,

$$|f(z)| = |\exp[\pi i \cosh 2g(z)]|$$
$$\leq \exp[\pi |\cosh 2g(z)|]$$
$$\leq \exp[\pi e^{2|g(z)|}]$$
$$\leq \exp[\pi e^{2C_3(\alpha,\beta)}] = C_4(\alpha, \beta) \quad \text{(say)}$$

Case II. Suppose $0 < |f(0)| < \dfrac{1}{2}$

In this case $(1-f)$ satisfies the condition of case (I) so that

$$|1-f(z)| \leq C_4(2, \beta) \text{ if } |z| \leq \beta.$$

Hence $|f(z)| = |-f(z)| = |1-f(z)+1| \leq |1-f(z)| + 1 \leq 1 + C_4(2, \beta)$ if $|z| \leq \beta$

If we define $C(\alpha, \beta) = \max.\{C_4(\alpha, \beta), 1 + C_4(2, \beta)\}$, we have

$$|f(z)| \leq C(\alpha, \beta) \text{ if } |z| \leq \beta$$

2.15. Corollary. Let f be analytic on a simply connected region containing $\bar{B}(0\,;R)$ and suppose that f omits the values 0 and 1. If $C(\alpha, \beta)$ is the constant obtained in Schottky's Theorem and $|f(0)| \leq \alpha$ then

$$|f(z)| \leq C(\alpha, \beta) \text{ for } |z| \leq \beta^R$$

Proof. It follows by considering the function $f(Rz)$ for $|z| \leq 1$.

2.16. Montel-Coratheodory Theorem. If Φ is the family of all analytic functions on a region G that do not assume the values 0 and 1, then Φ is normal in $C(G, \forall_\infty)$.

Proof. Fix a point z_0 in G and define the families $\boldsymbol{G}$ and $\boldsymbol{H}$ by

$$\boldsymbol{G} = \{f \in \Phi : |f(z_0)| \leq 1\},$$
$$\boldsymbol{H} = \{f \in \Phi : |f(z_0)| \geq 1\}$$

So that $\Phi = \boldsymbol{G} \cup \boldsymbol{H}$

We shall show that $\boldsymbol{G}$ is normal in H(G) and $\boldsymbol{H}$ is normal in $C(G, \forall_\infty)$. To show $\boldsymbol{G}$ is normal in H(G), it is sufficient to show that $\boldsymbol{G}$ is locally bounded in view of Montel's theorem.

Let a be any point in G and γ be a curve in G from z_0 to a. Let $D_0, D_1, D_2, \ldots, D_n$ be disks in G with centres $z_0, z_1, z_2, \ldots, z_n =$ a on $\{\gamma\}$ and such that z_{k-1} and z_k are in $D_{k-1} \cap D_k$ for $1 \leq k \leq n$. Also assume that $\bar{D}_k \subset G$ for $0 \leq k \leq n$. By Schottky's theorem for D_0, there is a constant C_0 such that

$$|f(z)| \leq C_0 \text{ for z in } D_0$$

In particular

$$|f(z_1)| \leq C_0$$

So by Schottky's theorem $\boldsymbol{G}$ is uniformly bounded by C_1 on D_1 . Continuing, we have $\boldsymbol{G}$ is uniformly bounded on D_n. Since a was arbitrary, this gives that $\boldsymbol{G}$ is locally bounded. Hence by Montel's theorem, $\boldsymbol{G}$ is normal in H(G).

Now consider $\boldsymbol{H} = \{f \in \Phi : |f(z_0| \geq 1 \}$

If f ε $\boldsymbol{H}$ then 1/f is analytic on G because f never vanishes. Also 1/f omits the values 0 and 1 and $|(\frac{1}{f})(z_0)| \leq 1$.

Hence $\tilde{\boldsymbol{H}} = \{\frac{\mathbf{1}}{\mathbf{f}} : f \in \boldsymbol{H}\} \subset \boldsymbol{G}$

and $\tilde{\boldsymbol{H}}$ is normal in H(G).

So if $\{f_n\}$ is a sequence in $\boldsymbol{H}$ there is a subsequence $\{f_{n_k}\}$ and an analytic function h on G such that $\left\{\frac{1}{f_{n_k}}\right\}$ converges in H(G) to h. So either h ≡ 0 or h never vanishes (By Cor. to Hurwitz's theorem, Unit III). If h ≡ 0 then $f_{nk}(z) \to \infty$ uniformly on compact subsets of G. If h never vanishes then $\frac{1}{h}$ is analytic and it follows that $f_{n_k}(z) \to \frac{1}{h(z)}$ uniformly on compact subsets of G.

2.17. Great Picard Theorem. Suppose an analytic function f has an essential singularity at z = a. Then in each neighbourhood of a, f assumes each complex number, with one possible exception, an infinite number of times.

Proof. We prove the theorem by taking a = 0 .

Suppose that there is an R such that there are two numbers not in $\{f(z) : 0 < |z| < R\}$. Also suppose that $f(z) \neq 0$ and $f(z) \neq 1$ for $0 < |z| < R$. Let $G = B(0; R) - \{0\}$

Define $f_n : G \to \forall$ by

$$f_n(z) = f(\frac{z}{n})$$

Then each f_n is analytic and no f_n assumes the value 0 or 1. So by Montel-Caratheodory Theorem, $\{f_n\}$ is a normal family in $C(G, \forall_\infty)$.

Let $\{f_{n_k}\}$ be a subsequence of $\{f_n\}$ such that $f_{n_k} \to \phi$ uniformly on $\{z : |z| = \frac{1}{2}R\}$ where φ is either analytic on G or $\phi \equiv \infty$.

If φ is analytic, let $M = \max\{ |\phi(z)| : |z| = \frac{1}{2}R\}$

Then $|f(\frac{z}{n_k})| = |f_{n_k}(z)| \leq | f_{n_k}(z) - \phi(z)| + |\phi(z)| \leq 2M$ for n_k sufficiently large and $|z| = \frac{1}{2}R$.

Thus $|f(z)| \leq 2M$ for $|z| = \frac{R}{2n_k}$ and for sufficiently large n_k.

By Maximum Modulus Principle, f is uniformly bounded on concentric annuli about zero. This gives that f is bounded by 2M on a deleted neighbourhood of zero and so z = 0 must be a removable singularity. Therefore φ cannot be analytic. So $\phi \equiv \infty$. In this case f has a pole at zero. A contradiction. So at most one complex number is never assumed.

If there is a complex number w which is assumed only a finite number of times then by taking a sufficiently small disk, we again arrive at a punctured disk in which f fails to assume two values.

2.18 Remark. An alternate framing of the above theorem is as follows :

If f has an isolated singularity at z = a and if there are two complex numbers that are not assumed infinitely often by f then z = a is either a pole or a removable singularity.

2.19 Corollary. If f is an entire function that is not a polynomial then f assumes every complex number, with one exception, an infinite number of times.

Proof. Consider the function $g(z) = f(\frac{1}{z})$.

Since f is not a polynomial, g has an essential singularity at z = 0. So result follows from Great Picard Theorem. We observe that Cor. 2.19 is an improvement of the Little Picard Theorem.

3. Univalent Function

The theory of conformal mappings on simply connected regions is of special significance from the point of view of geometric function theory. Essentially there are three types of simply connected spaces in the extended complex plane

(i) The open unit disc
(ii) The entire finite complex plane
(iii) The extended complex plane or the Riemann sphere.

The cases (i), (ii) and (iii) are referred to as hyperbolic, parabolic and elliptic cases respectively.

A simple geometric restriction namely the injectivity imposed on functions defined on the unit disc throws a wealth of information on the geometric and analytic properties of such functions.

In general, we say that a function $f(z)$ is univalent (simple, schlicht) in a region D if it is analytic, one-valued, and does not take any value more than once in D. The function $w = f(z)$ then represents the region D of the z-plane on a region D′ of the w-plane, in such a way that there is a one-one correspondence between the points of the two regions. In other words, a univalent function assumes each value in its range precisely once.

3.1. Theorem. If $f(z)$ is univalent in D, then $f'(z) \neq 0$ in D.

Proof. On the contrary, suppose that $f'(z_0) = 0$. Then $f(z) - f(z_0)$ has a zero of order n ($n \geq 2$) at z_0. Since $f(z)$ is not constant, we can find a circle $|z-z_0| = \delta$ on which $f(z) - f(z_0)$ does not vanish, and inside which $f'(z)$ has no zeros except z_0. Let m be the lower bound of $|f(z)- f(z_0)|$ on this circle. Then by Rouche's theorem, if $0 < |a| < m$, $f(z) - f(z_0) - a$ has n zeros in the circle, it cannot have a double zero, since $f'(z)$ has no other zeros in the circle. This is contrary to the hypothesis that $f(z)$ does not take any value more than once.

3.2. Remarks. (i) If $f(z)$ is univalent in D, then the mapping $w = f(z)$ is conformal at every point of D.

(ii) A univalent function of a univalent function is univalent

Proof. If $f(z)$ is univalent in D and F(w) in D′, then $F\{f(z)\}$ is univalent in D, since $F\{f(z_1)\} = F\{f(z_2)\}$ implies $f(z_1) = f(z_2)$ as F is univalent and this further implies $z_1 = z_2$, since f is univalent.

(iii) In the above relationship, to every point of D′ corresponds just one point of D. We can therefore consider z as a function of w, say $z = \phi(w)$. This is called the inverse function of $w = f(z)$.

(iv) The inverse function is univalent in D′, since it is one-valued and analytic.

3.3. Theorem. A univalent function $w = f(z)$ which represents a unit circle on itself, so that the centre and a given direction through it remain unaltered, is the identical transformation $w = z$.

Proof. We have $|f(z)| = 1$ for $|z| = 1$ and $f(0) = 0$.
Hence by Schwarz's lemma (unit –II)

$$|w| = |f(z)| \le |z|$$

But, applying Schwarz's lemma to the inverse function, we have $|z| \le |w|$

Hence, we get

$$|w| = |z|, \text{ i.e. } |f(z)/z| = 1, |z| \le 1.$$

Since a function of constant modulus is constant, it follows that

$$f(z)/z = a, \text{ i.e., } f(z) = az$$

where $|a| = 1$. The remaining conditions then show that a = 1 and thus we obtain w = z.

3.4. Remark. The class of functions $f(z)$ which are univalent for the open disc $|z| < 1$ and such that $f(0) = 0$, $f'(0) = 1$, has been studied in great detail. The function w = z belongs to this class and represents the unit circle on itself. We denote this class by and the unit open disc by U. As an immediate consequence of Taylor's series development, to every $f(z)$ in F we have a power series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which is valid throughout U. We further observe that the class F is not closed under either addition or multiplication.

as $$(f+g)'(0) = 2 \text{ and } (fg)'(0) = 0 \text{ for } f, g \in F.$$

3.5. Bieberbach's Conjecture. The study of geometric function theory has been given a solid foundation by Riemann with his fundamental mapping theorem. Later on, Koebe and others formulated the theory of univalent functions which are normalized by $f(0) = 0$ $f'(0) = 1$ (the class F) Normal families of analytic functions was introduced by Montel and Coratheodary established his kernel theorem on sequences of univalent functions.

Ludwig Bieberbach proposed his famous conjecture in 1916 which says : If

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

is univalent in the open unit disc $|z| < 1$ then $|a_n| \le n$ (n = 2, 3,….) with equality if and only if $f(z)$ is a rotation of the Koebe function

$$K(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \dots$$

Bieberbach proved his conjecture for n = 2 using the area principle which was established by T. H. Gronwall. Several sub-classes of univalent functions for which the conjecture can be easily verified by other geometric means have been introduced and studied.

3.6. Theorem. $\left(\textbf{The "}\frac{1}{4}\textbf{ Theorem"}\right)$: For any function of the class F, no boundary point of the map of the unit circle is nearer to the origin than the point $\frac{1}{4}$.

The 1/4 theorem may be stated in another form as follows.

If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in F$ then $|a_2| \le 2$ with equality if and only if

$f(z) = \dfrac{z}{(1-\alpha z)^2}$ with $|\alpha| = 1$. Further, $f(U)$ contains all w with $|w| < \frac{1}{4}$ and the constant $\frac{1}{4}$ cannot be improved for all $f \in F$ where U denotes the units open disc.