MECHANICS OF SOLIDS

M.A./M.Sc. Mathematics (Final)

MM-504 & MM 505
(Option-A_1)

Directorate of Distance Education
Maharshi Dayanand University
ROHTAK – 124 001

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Contents

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Max. Marks : 100

Time : 3 Hours

Note: Question paper will consist of three sections. Section I consisting of one question with ten parts covering whole of the syllabus of 2 marks each shall be compulsory. From Section II, 10 questions to be set selecting two questions from each unit. The candidate will be required to attempt any seven questions each of five marks. Section III, five questions to be set, one from each unit. The candidate will be required to attempt any three questions each of fifteen marks.

Unit I

Analysis of Strain: Affine transformation. Infinite simal affine deformation. Geometrical interpretation of the components of strain. Strain quadric of Cauchy. Principal strains and invariants. General infinitesimal deformation. Saint-Venant's equations of Compatibility. Finite deformations.

Unit II

Equations of Elasticity: Generalized Hooke's law. Homogeneous isotropic media. Elasticity moduli for isotropic media. Equilibrium and dynamic equations for an isotropic elastic solid. Strain energy function and its connection with Hooke's law. Uniquness of solution. Beltrami-Michell compatibility equations. Saint-Venant's principle.

Unit III

Two – dimensional Problems: Plane stress. Generalized plane stress. Airy stress function. General solution of Biharmonic equation. Stresses and displacements in terms of complex potentials. Simple problems. Stress function appropriate to problems of plane stress. Problems of semi-infinite solids with displacements or stresses prescribed on the plane boundary.

Unit IV

Torsional Problem: Torsion of cylindrical bars. Tortional rigidity. Torsion and stress functions. Lines of shearing stress. Simple problems related to circle, elipse and equilateral triangle.

Variational Methods: Theorems of minimum potential energy. Theorems of minimum complementary energy. Reciprocal theorem of Betti and Rayleigh. Deflection of elastic string, central line of a beam and elastic membrane. Torsion of cylinders. Variational problem related to biharmonic equation. Solution of Euler's equation by Ritz, Galerkin and Kantorovich methods.

Unit V

Elastic Waves: Propagation of waves in an isotropic elastic solid medium. Waves of dilatation and distortion Plane waves. Elastic surface waves such as Rayleigh and Love waves.

Chapter-1
Cartesian Tensors

1.1 INTRODUCTION

There are physical quantities which are independent or invariant of any particular coordinate system that may be used to describe them. Mathematically, such quantities are represented by tensors. That is, a tensor is a quantity which describes a physical state or a physical phenomenon.

As a mathematical entity, a tensor has an existence independent of any coordinate system. Yet it may be specified in a particular coordinate system by a certain set of quantities, **known as its components**. Specifying the components of a tensor in one coordinate system determines the components in any other system according to some definite law of transformation.

In dealing with general coordinate transformations between arbitrary curvilinear coordinate systems, the tensors defined are known as **general tensors**. When one is dealing with cartesian rectangular frames of reference only, the tensor involved are referred to as **cartesian tensors**. **From now onwards, the word "tensor" means "cartesian tensors"** unless specifically stated otherwise.

1.2 COORDINATE TRANSFORMATIONS

Let us consider a right handed system of rectangular cartesian axes x_i with a fixed origin O. Let P be a general point whose coordinates with respect to this system O $x_1x_2x_3$ are (x_1, x_2, x_3).

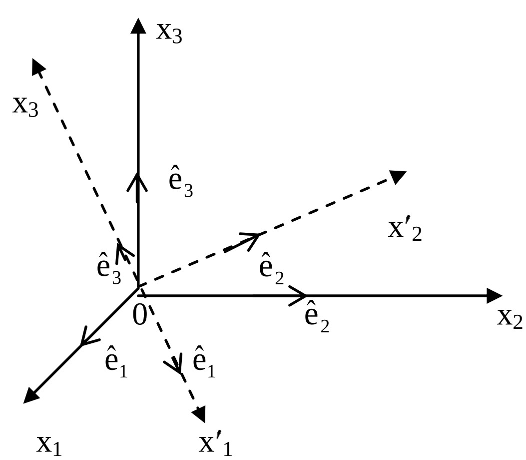


Fig (1.1)

Let $\bar{r}$ be the position vector of P w.r.t. O.

Then

$$\bar{r} = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3 \qquad (1)$$

and (x_1, x_2, x_3) are the components of the vector $\overline{OP}$. Here, $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ are unit vectors along axes.

Let a new system $Ox_1'x_2'x_3'$ of axes be obtained by rotating the "**old system**" of axes about some line in space through O. The position vector $\bar{\mathbf{r}}$ of P has the following representation in the new system

$$\bar{r} = x_1'\hat{e}_1' + x_2'\hat{e}_2' + x_3'\hat{e}_3' \qquad (2)$$

where $\hat{e}_i$ is the unit vector directed along the positive x_i'-axis, and

$$\hat{e}_i'.\hat{e}_j' = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

and

$$\hat{e}_1' \times \hat{e}_2' = \hat{e}_3', \text{ etc.}$$

and (x_1', x_2', x_3') are the new components of $\overline{OP} = \bar{r}$ relative to the new axes $Ox_1'x_2'x_3'$. Let a_{pi} be the direction cosines of new x_p'-axis w.r.t. the old x_i-axis.

That is,

$a_{pi} = \cos(x_p', x_i)$ = cosine of the angle between the positive x_p'-axis (new axis) and the positive x_i-axis (old axis)

$$= \bar{e}_p'.\ \bar{e}_i \qquad (3)$$

Form (2), we write

$$\bar{r}.\hat{e}'_p = x'_p$$

$$\Rightarrow x'_p = \bar{r}.\ \hat{e}'_p = (x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3).\hat{e}'_p\ ,$$

$$\Rightarrow x'_p = a_{p1}x_1 + a_{p2}x_2 + a_{p3}x_3 = a_{pi}x_i\ . \qquad (4)$$

Here p is the free suffix and i is dummy.

In the above, the following **Einstein summation convection** is used.

"Unless otherwise stated specifically, whenever a suffix is repeated, it is to be given all possible values (1, 2,3) and that the terms are to be added for all".

Similarly

$$x_i = \bar{r}\ .\ \hat{e}_i$$

$$= (x_1'\ \hat{e}'_1 + x_2\hat{e}'_2 + x_3\hat{e}'_3).\hat{e}_i\ ,$$

$$= a_{1i}\,x'_1 + a_{2i}x'_2 + a_{3i}x'_3$$

$$= a_{pi}\, x'_p \,. \qquad (5)$$

Here i is a free suffix and p is dummy. When the orientations of the new axes w.r.t. the old axes are known, the coefficients a_{pi} are known. Relation (4) represent the law that transforms the old triplet x_i to the new triplet x'_p and (5) represent the inverse law, giving old system in terms of new system.

Remark 1. The transformation rules (4) and (5) may be displaced in the following table

	x_1	x_2	x_3
x'_1	a_{11}	a_{12}	a_{13}
x'_2	a_{21}	a_{22}	a_{23}
x'_3	a_{31}	a_{32}	a_{33}

(6)

Remark 2. The transformation (4) is a linear transformation given by

$$\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (7)$$

The matrix

$$[L] = (a_{ij})_{3\times 3} \qquad (8)$$

may be thought as an operator operating on the vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and giving the vector $\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix}$.

Remark 3. Since this transformation is rotational only, so the matrix L of the transformation is non-symmetric.

Remark 4. Relations (4) and (5) yield

$$\frac{\partial x'_p}{\partial x_i} = a_{pi} \qquad (9)$$

and

$$\frac{\partial x_i}{\partial x'_p} = a_{pi} \,. \qquad (10)$$

1.3. THE SYMBOL δ_{ij}

It is defined as

$$\delta_{ij} = \begin{matrix} 1 & \text{if} & i = j \\ 0 & \text{if} & i \neq j \end{matrix} \,. \qquad (1)$$

That is,

$$\delta_{11} = \delta_{22} = \delta_{33} = 1,$$

$$\delta_{12} = \delta_{21} = \delta_{31} = \delta_{23} = \delta_{32} = 0 \ .$$

The symbol δ_{ij} is known as the Kronecker δ symbol, named after the **German mathematician** Leopold Kronecker (1827-1891). The following property is inherent in the definition of δ_{ij}

$$\delta_{ij} = \delta_{ji} \ .$$

By summation convention

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3 \ . \qquad (2)$$

The unit matrix of order 3 is

$$I_3 = (\delta_{ij}) \text{ and } \det(\delta_{ij}) = 1.$$

The orthonormality of the base unit vectors $\hat{e}_i$ can be written as

$$\hat{e}_i . \hat{e}_j = \delta_{ij} \ . \qquad (3)$$

We know that

$$\frac{\partial x_i}{\partial x_j} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} .$$

Therefore,

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} \ . \qquad (4)$$

Theorem 1.1. Prove the following (**known as substitution properties of δ_{ij}**)

(i) $u_j = \delta_{ij} u_i$

(ii) $\delta_{ij} u_{jk} = u_{ik}, \ \delta_{ij} u_{ik} = u_{jk}$

(iii) $\delta_{ij} u_{ij} = u_{11} + u_{22} + u_{33} = u_{ii}$

Proof. (i) Now $\delta_{ij} u_i = \delta_{1j} u_1 + \delta_{2j} u_2 + \delta_{3j} . u_3$

$$= u_j + \sum_{\substack{i=1 \\ i \neq j}}^{3} \delta_{ij} u_i$$

$$= u_j$$

(ii) $$\delta_{ij} u_{jk} = \sum_{j=1}^{3} \delta_{ij} u_{jk}$$

$= \delta_{ii} u_{ik}$ (for $j \neq i$, $\delta_{ij} = 0$), here summation over i is not taken

$$= u_{ik}$$

(iii) $$\delta_{ij}u_{ij} = \sum_i \left[\sum_j \delta_{ij} u_{ij} \right]$$

$$= \sum_i (1.u_{ii}) \ , \quad \text{in } u_{ii} \text{ summation is not being taken}$$

$$= \sum_i u_{ii}$$

$$= u_{ii} = u_{11} + u_{22} + u_{33} \, .$$

Question. Given that

$$a_{ij} = \alpha\delta_{ij}b_{kk} + \beta b_{ij}$$

where $\beta \neq 0$, $3\alpha+\beta \neq 0$, find b_{ij} in terms of a_{ij} .

Solution. Setting j = i and summing accordingly, we obtain

$$a_{ii} = \alpha.3.b_{kk} + \beta b_{ii} = (3\alpha + \beta)b_{kk}$$

$$\Rightarrow \qquad b_{kk} = \frac{1}{3\alpha+\beta} a_{kk} \, .$$

Hence

$$b_{ij} = \frac{1}{\beta} a_{ij} - \alpha\delta_{ij}b_{kk}$$

$$\Rightarrow \qquad b_{ij} = \frac{1}{\beta}\left[a_{ij} - \frac{\alpha}{3\alpha+\beta}\delta_{ij}a_{kk} \right] .$$

Theorem 1.2. Prove that

(i) $a_{pi}a_{qi} = \delta_{pq}$

(ii) $a_{pi}a_{pj} = \delta_{ij}$

(iii) $|a_{ij}| = 1, \quad (a_{ij})^{-1} = (a_{ij})'$.

Proof. From the transformation rules of coordinate axes, we have

$$x'_p = a_{pi}\, x_i \tag{1}$$

$$x_i = a_{pi}\, x'_p \tag{2}$$

where

$$a_{pi} = \cos(x'_p, x_i) \tag{3}$$

(i) Now

$$x'_p = a_{pi}\, x_i$$

$$= a_{pi}(a_{qi}\, x'_q)$$

$$= a_{pi}\, a_{qi}\, x'_q \tag{4}$$

Also

$$x'_p = \delta_{pq} x'_q \tag{5}$$

Therefore,

$$(a_{pi}a_{qi} - \delta_{pq})x'_q = 0$$

or

$$a_{pi}a_{qi}\ \ \delta_{pq} = 0$$

or

$$a_{pi}\ a_{qi} = \delta_{pq} \tag{6}$$

This proves that (i).

(ii) Similarly, $x_i = a_{pi}\, x'_p$

$$= a_{pi}a_{pj}x_j \tag{7}$$

Also

$$x_i = \delta_{ij}x_j \tag{8}$$

Hence,

$$a_{pi}a_{pj} = \delta_{ij}\ . \tag{9}$$

(iii) Relation (6) gives, in the expanded form,

$$a_{11}^2 + a_{12}^2 + a_{13}^2 = 1,$$

$$a_{21}^2 + a_{22}^2 + a_{23}^2 = 1,$$

$$a_{31}^2 + a_{32}^2 + a_{33}^2 = 1$$

$$a_{11}a_{21}+a_{12}a_{22}+a_{13}a_{23} = 0\ ,$$

$$a_{21}a_{31}+a_{22}a_{32} + a_{23}a_{33} = 0\ ,$$

$$a_{31}a_{11}+a_{32}a_{12} + a_{33}a_{13} = 0\ \ . \tag{10}$$

The relations (6) and (9) are referred to as the orthonormal relations for a_{ij} . In matrix notation, relations (6) and (9) may be represented respectively, as follows

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12}$$

or

$$LL' = L'L = 1\ . \tag{13}$$

These expressions show that the matrix $L = (a_{ij})$ is non-singular and that

$$(a_{ij})^{-1} = (a_{ij})' \text{ and } |a_{ij}| = 1 . \qquad (14)$$

The transformation matrix $L = (a_{ij})$ is called the **proper orthogonal matrix**. For this reason, the transformation laws (3) and (4), determined by the matrix $L = (a_{ij})$, are called **orthogonal transformations**.

Example. The x'_i-system is obtained by rotating the x_i-system about the x_3-axis through an angle θ in the sense of right handed screw. Find the transformation matrix. If a point P has coordinates (1,1, 1) in the x_i-system, find its coordinate in the x'_i-system. If a point Q has coordinate (1, 1,1) in the x'_i-system, find its coordinates in the x_i-system.

Solution. The figure (1.2) shows how the x'_i-system is related to the x_i-system. The table of direction cosines for the given transform is

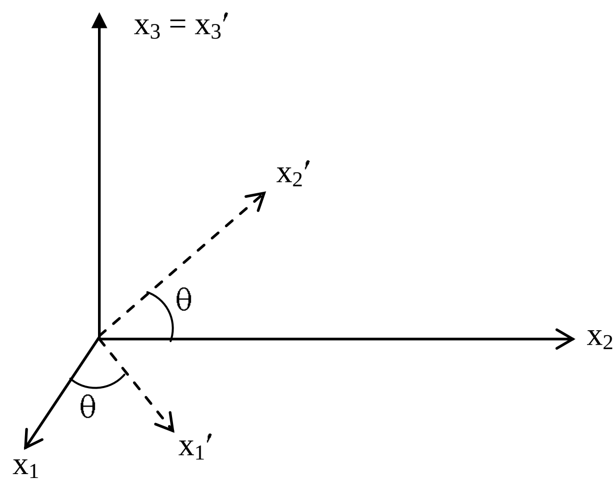


Fig. (1.2)

	$\hat{e}_1$	$\hat{e}_2$	$\hat{e}_3$
$\hat{e}'_1$	$\cos\theta$	$\sin\theta$	0
$\hat{e}'_2$	$-\sin\theta$	$\cos\theta$	0
$\hat{e}'_3$	0	0	1

Hence, the matrix of this transformation is

$$(a_{ij}) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (1)$$

The transformation rules for coordinates are

$$x'_p = a_{pi}x_i, \qquad (2)$$

$$x_i = a_{pi}\, x'_p . \qquad (3)$$

The coordinates $P(x'_1, x'_2, x'_3)$ of the point P(1, 1, 1) in the new system are given by

$$x'_1 = a_{1i}x_i = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = \cos\theta + \sin\theta$$

$$x'_2 = a_{2i}x_i = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = \cos\theta - \sin\theta$$

$$x'_3 = a_{3i}x_i = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 1. \qquad (4)$$

Therefore, coordinates of P in the x'_i-system are $(\cos\theta + \sin\theta, \cos\theta - \sin\theta, 1)$.

The coordinates (x_1, x_2, x_3) of a w.r.t. old system are given as

$$x_1 = a_{p1}x'_p = a_{11}x'_1 + a_{21}x'_2 + a_{31}x'_3 = \cos\theta - \sin\theta$$

$$x_2 = a_{p2}x'_p = a_{12}x'_1 + a_{22}x'_2 + a_{32}x'_3 = \cos\theta + \sin\theta$$

$$x_3 = a_{p3}x_p = a_{13}x'_1 + a_{23}x'_2 + a_{33}x'_3 = 1\ . \qquad (5)$$

Hence, the coordinates of the point Q in the old x_i-system are $(\cos\theta\text{-}\sin\theta, \cos\theta + \sin\theta, 1)$.

1.4 SCALARS AND VECTORS

Under a transformation of certesian coordinate axes, a scalar quantity, such as the **density or the temperature**, remains unchanged. This means that a scalar is an invariant under a coordinate transformation. Scalaras are called **tensors of zero rank.**

We know that a scalar is represented by a single quantity in any coordinate system. Accordingly, a tensor of zero rank (or order) is specified in any coordinate system in three-dimensional space by one component or a single number.

All physical quantities having magnitude only are tensors of zero order.

Transformation of a Vector

Let $\bar{u}$ be any vector having components (u_1, u_2, u_3) along the x_i-axes and components (u'_1, u'_2, u'_3) along the x'_i-axes so that vector $\bar{u}$ is represented by three components/quantities. Then we have

$$\bar{u} = u_i\ \hat{e}_i \qquad (1)$$

and $$u = u'_i\ \hat{e}'_i \qquad (2)$$

where $\hat{e}_i$ is the unit vector along x_i-direction and $\hat{e}'_i$ is the unit vector along x'_i-direction.

Now

$$u'_p = \bar{u}\,.\,\hat{e}'_p$$

$$= (u_i\ \hat{e}_i) \cdot e'_p$$

$$= (\hat{e}'_p \cdot \hat{e}_i) u_i$$

$$\Rightarrow \quad u'_p = a_{pi}\ u_i \,. \tag{3}$$

where

$$a_{pi} = \hat{e}'_p \cdot \hat{e}_i = \cos(x'_p, x_i) \,. \tag{4}$$

Also

$$u_i = \bar{u} \cdot \hat{e}_i$$

$$= (u'_p\ \hat{e}'_p) \cdot \hat{e}_1$$

$$= (\hat{e}'_p \cdot \hat{e}'_i)\ u'_p$$

$$= a_{pi}\ u'_p \,, \tag{5}$$

where (a_{pi}) is the proper orthogonal transformation matrix.
Relations (3) and (5) are the rules that determine u'_p in terms of u_i and vice-versa. Evidently, these relations are analogous to rules of transformation of coordinates.

Definition (Tensor of rank one)

A cartesian tensor of rank one is an entity that may be represented by a set of three quantities in every cartesian coordinate system with the property that its components u_i relative to the system $ox_1x_2x_3$ are related/connected with its components u'_p relative to the system $ox'_1x'_2x'_3$ by the relation

$$u'_p = a_{pi}u_i$$

where the law of transformation of coordinates of points is

$$x'_p = a_{pi}x_i \quad \text{and } a_{pi} = \cos(x'_p, x_i) = \hat{e}'_p . \hat{e}_i \ .$$

Note: We note that **every vector in space is a tensor of rank one.** Thus, physical quantities possessing both magnitude and direction such as **force**, **displacement, velocity, etc.** are all tensors of rank one. In three-dimensional space, 3 real numbers are needed to represent a tensor of order 1.

Definition (Tensor of order 2)

Any entity representable by a set of 9 (real) quantities relative to a system of rectangular axes is called a tensor of rank two if its components w_{ij} relative to system $ox_1x_2x_3$ are connected with its components w'_{pq} relative to the system $ox'_1x'_2x'_3$ by the transformation rule

$$w'_{pq} = a_{pi}\ q_{qj}\ w_{ij}$$

when the law of transformation of coordinates is

$$x'_p = a_{pi}x_i \,,$$

$$a_{pi} = \cos(x'_p, x_i) = \hat{e}'_p . \hat{e}_i \, .$$

Note : Tensors of order 2 are also called **dyadics**. For example, strain and stress tensors of elasticity are, each of rank 2. In the theory of elasticity, we shall use tensors of rank 4 also.

Example. In the x_i-system, a vector $\bar{u}$ has components (–1, 0, 1) and a second order tensor has the representation

$$(w_{ij}) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix} .$$

The x'_i-system is obtained by rotating the x_i-system about the x_3-axis through an angle of 45^o in the sense of the right handed screw. Find the components of the vector $\bar{u}$ and the second ordered tensor in the x'_i-system.

Solution. The table of transformation of coordinates is

	x_1	x_2	x_3
x'_1	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0
x'_2	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0
x'_3	0	0	1

If u'_p are the components of vector in the new system, then

$$u'_p = a_{pi}\, u_i \, , \ \ a_{pi} = \hat{e}'_p . \hat{e}_i \ . \tag{1}$$

This gives $u'_1 = -\dfrac{1}{\sqrt{2}}$

$$u'_2 = \frac{1}{\sqrt{2}} \, ,$$

$$u'_3 = 1 \, .$$

Let w'_{pq} be the components of the given second order tensor in the x'_i-system. Then the transformation law for second order tensor yields

$$w'_{pq} = a_{pi} a_{qj} w_{ij} \tag{2}$$

We find (left an exercise to readers)

$$w'_{pq} = \begin{pmatrix} 0 & 1 & \sqrt{2} \\ -1 & 0 & \sqrt{2} \\ -\sqrt{2} & -\sqrt{2} & 0 \end{pmatrix} \tag{3}$$

Definition 1. A second order tensor u_{ij} is said to be symmetric if

$$u_{ij} = u_{ji} \quad \text{for all } i \text{ and } j .$$

Definition 2. A second order tensor u_{ij} is said to be skew-symmetric if

$$u_{ij} = -u_{ji} \text{ for all } i \text{ and } j .$$

Tensors of order n

A tensor of order n has 3^n components. If $u_{ijk\ldots}$…..are components of a tensor of order n , then, the transformation law is

$$u'_{pqr\ldots\ldots} = a_{pi}\, a_{qj}\, a_{rk} \ldots .\, u_{ijk\ldots\ldots}$$

where the law of transformation of coordinates is

$$x'_p = a_{pi}\, x_i,$$

and

$$a_{pi} = \cos\,(x'_p, x_i) .$$

Importance of the Concept of Tensors

(a) Tensors are quantities describing the same phenomenon regardless of the coordinate system used. Therefore, tensors provide an important guide in the formulation of the correct forms of physical laws.

(b) The tensor concept gives us a convenient means of transforming an equation from one system of coordinates to another.

(c) An advantage of the use of cartesian tensors is that once the properties of a tensor of order n have been established, they hold for all such tensors regardless of the physical phenomena they present.

Note : For example, in the study of strain, stress, inertia properties of rigid bodies, the common bond is that they are all symmetric tensors of rank two.

(d) With the use of tensors, equations are condensed, such as

$$\tau_{ij\,,\,j} + f_i = 0,$$

is the equation of equilibrium in tensor form. It consists of 3 equations and each equation has 4 terms.

(e) Equations describing physical laws must be tensorially homogeneous, which means that every term of the equation must be a tensor of the same rank.

1.5 PROPERTIES OF TENSORS

Property 1 : If all components of a tensor are 0 in one coordinate system, then they are 0 in all coordinate systems.

Proof : Let $u_{ijkl\ldots}$ and $u'_{pqrs\ldots}$ be the components of a nth order tensor in two systems $0x_1x_2x_3$ and $0x_1'x_2'x_3'$, respectively.

Suppose that

$$u_{ijkl\ldots} = 0 .$$

Then, the transformation rule yields

$$u'_{pqrs\ldots} = a_{pi}\, a_{qj}\, a_{rk}\, a_{sm} \ldots u_{ijkl\ldots}$$

giving

$$u'_{pqrs\ldots} = 0 .$$

This proves the result.

Zero Tensor

A tensor whose all components in one Cartesian coordinates system are 0 is called a zero tensor.

A zero tensor may have any order n.

Property 2 : If the corresponding components of two tensors of the same order are equal in one coordinate system, then they are equal in all coordinate systems.

Corollary : A tensor equation which holds in one Cartesian coordinate system also holds in every other Cartesian coordinate system.

Equality of Tensors

Two tensors of the same order whose corresponding components are equal in a coordinate system (and hence in all coordinates) are called equal tensors.

Note : Thus, in order to show that two tensors are equal, it is sufficient to show that their corresponding components are equal in any one of the coordinate systems.

Property 3 (Scalar multiplication of a tensor)

If the components of a tensor of order n are multiplied by a scalar α, then the resulting components form a tensor of the same order n.

Proof : Let $u_{ijk\ldots}$ be a tensor of order n. Let $u'_{pqr\ldots}$ be the corresponding components in the dashed system $0\ x_1'\ x_2'\ x_3'$. The transformation rule for a tensor of order n yields

$$u'_{pqrs\ldots} = a_{pi}\, a_{qj} \ldots u_{ijkl\ldots} \tag{1}$$

where
$$a_{pi} = \cos (x_p', x_i) . \tag{2}$$

Now
$$(\alpha\, u'_{pqr\ldots}) = (a_{pi}\, a_{qj}\ldots)\, (\alpha\, u_{ijk\ldots}) \tag{3}$$

This shows that components $\alpha\, u_{ijk\ldots}$ form a tensor of rank n.

Tensor Equations

An equation of the form

$$\alpha_{ijk} - \beta_{ij}\, u_k = 0$$

is called a **tensor equation.**

For checking the correctness of a tensor equation, there are following two rules :

Rule (i) In a correctly written tensor equation, no suffix shall appear more than twice in a term, otherwise, the operation will not be defined. For example, an equation

$$u_j' = \alpha_{ij}\, u_j\, v_j$$

is not a tensor equation.

Rule (ii) If a suffix appears only once in a term, then it must appear only once in the remaining terms also. For example, an equation

$$u_j' - l_{ij}\, u_j = 0$$

is not a tensor equation.

Here j appears once in the first term while it appears twice in the second term.

Property 4 (Sum and Difference of tensors)

If $u_{ijk\ldots}$ and $v_{ijk\ldots}$ are two tensors of the same rank n then the sums

$$(u_{ijk\ldots} + v_{ijk\ldots})$$

of their components are components of a tensor of the same order n.

Proof : Let

$$w_{ijk}\ldots = u_{ijk}\ldots + v_{ijk}\ldots.. \qquad (1)$$

Let $u'_{pqr}\ldots..$ and $v'_{pqr}\ldots$ be the components of the given tensors of order n relative to the new dashed system $0\, x_1'\, x_2'\, x_3'$. Then, transformation rules for these tensors are

$$u'_{pqr}\ldots.. = a_{pi}\, a_{qj}\, \ldots u_{ijk}\ldots \qquad (2)$$

and

$$v'_{pqrs}\ldots.. = a_{pi}\, a_{qj}\, \ldots v_{ijkl}\ldots \qquad (3)$$

where

$$a_{pi} = \cos\,(x_p'\,,\, x_i). \qquad (4)$$

Let

$$w'_{pqr}\ldots\ldots = u'_{pqr}\ldots\ldots v'_{pqr}\ldots.. \qquad (5)$$

Then equations (2) – (5) give

$$w'_{pqr}\ldots.. = a_{pi}\, a_{ij}\, \ldots\, w_{ijk}\ldots\ldots. \qquad (6)$$

Thus quantities $w_{ijk}\ldots..$ obey the transformation rule of a tensor of order n. Therefore, they are components of a tensor of rank n.

Corollary : Similarly, $u_{ijk}\ldots - v_{ikl}\ldots$ are components of a tensor of rank n.

Property 5 (Tensor Multiplication)

The product of two tensors is also a tensor whose order is the sum of orders of the given tensors.

Proof : Let $u_{ijk}\ldots..$ and $v_{pqr}\ldots..$ be two tensors of order m and n respectively. We shall show that the product

$$w_{ijk\ldots pqr}\ldots = u_{ijk}\ldots\, v_{pqr}\ldots \qquad (1)$$

is a tensor of order m + n.

Let $u'_{i_1 j_1}\ldots$ and $v'_{p_1 q_1}\ldots..$ be the components of the given tensors of orders m and n relative to the new system $0\, x_1'\, x_2'\, x_3'$. Then

$$u'_{i_1 j_1}\ldots = a_{i_1 i}\, a_{j_1 j}\ldots\ldots u_{ijk}\ldots\ldots\ldots \qquad (2)$$

$$v'_{p_1 q_1 \ldots\ldots} = a_{p_1 p}\, a_{q_1 q} \ldots\ldots v_{pqr \ldots\ldots} \qquad (3)$$

where

$$a_{pi} = \cos(x_p', x_i)\,. \qquad (4)$$

Let

$$w'_{i_1 j_1 \ldots\ldots p_1 q_1 \ldots\ldots} = u'_{i_1 j_1 \ldots\ldots} v'_{p_1 q_1 \ldots\ldots} \qquad (5)$$

Multiplying (2) an (3), we get

$$w'_{i_1 j_1 \ldots\ldots p_1 q_1 \ldots\ldots} = (a_{i_1 i}\, a_{j_1 j} \ldots)\, a_{p_1 p} a_{q_1 q} \ldots\ldots) w_{ijk \ldots pqr \ldots} \qquad (6)$$

This shows that components $w_{ijk\ldots pqr\ldots}$ obey the transformation rule of a tensor of order (m + n). Hence $u_{ijk\ldots}v_{pqr\ldots}$ are components of a (m + n) order tensor.

Exercise 1 : If u_i and v_i are components of vectors, then show that $u_i\, v_j$ are components of a second – order tensor.

Exercise 2 : If a_{ij} are components of a second – order tensor and b_i are components of a vector, show that $a_{ij}\, b_k$ are components of a third order tensor.

Exercise 3 : If a_{ij} and b_{ij} are components of two second – order tensors show that $a_{ij}\, b_{km}$ are components of fourth – order tensor.

Exercise 4 : Let u_i and v_i be two vectors. Let $w_{ij} = u_i\, v_j + u_j\, v_i$ and $\alpha_{ij} = u_i\, v_j - u_j\, v_i$. Show that each of w_{ij} and α_{ij} is a second order tensor.

Property 6 (Contraction of a tensor)

The operation or process of setting two suffices equal in a tensor and then summing over the dummy suffix is called a contraction operation or simply a contraction.

The tensor resulting from a contraction operation is called a contraction of the original tensor.

Contraction operations are applicable to tensor of all orders (higher than 1) and each **such operation reduces the order of a tensor by 2.**

Theorem : Prove that the result of applying a contraction to a tensor of order n is a tensor of order n – 2.

Proof : Let $u_{ijk\ldots\ldots}$ and $u'_{pqr\ldots\ldots}$ be the components of the given tensor of order n relative to two cartesian coordinate systems $0\,x_1\,x_2\,x_3$ and $0\,x_1'\,x_2'\,x_3'$. The rule of transformation of tensors is

$$u'_{pqr} \ldots\ldots = a_{pi}\, a_{qj}\, a_{rk} \ldots\ldots u_{ijk} \ldots\ldots \qquad (1)$$

where

$$a_{pi} = \cos(x_p', x_i)\,. \qquad (2)$$

Without loss of generality, we contract the given tensor by setting j = i and using summation convention . Let

$$v_{kl} \ldots = u_{iikl} \ldots\ldots \qquad (3)$$

Now
$$u'_{pqr} \ldots = (a_{pi}\, a_{qj})\, a_{rk} \ldots\ldots u_{iikl} \ldots\ldots$$

$$= \delta_{pq}\, a_{rk} \ldots\, u_{kl} \ldots$$

This gives

$$u'_{pprs} \ldots = a_{rk}\, a_{sl} \ldots\, v_{kl} \ldots\ldots$$

or

$$v'_{rs} \ldots\ldots = a_{rk} \ldots v_{kl} \ldots \qquad (4)$$

Property 7 (Quotient laws)

Quotient law is the partial converse of the contraction law.

Theorem : If there is an entity representable by the set of 9 quantities u_{ij} relative to any given system of cartesian axes and if $u_{ij}\, v_j$ is a vector for an arbitrary vector v_i, then show that u_{ij} is second order tensor.

Proof : Let

$$w_i = u_{ij}\, v_j \qquad (1)$$

Suppose that u'_{pq}, u'_p, w'_p be the corresponding components in the dashed system 0 $x_1'\, x_2'\, x_3'$. Then

$$v'_q = a_{qj}\, v_j, \qquad (2)$$

$$w'_p = a_{pi}\, w_i, \qquad (3)$$

where

$$a_{pi} = \cos(x_p', x_i)\,. \qquad (4)$$

Equation (1) in the dashed system is

$$w'_p = u'_{pq}\, v'_q\,. \qquad (5)$$

Inverse laws of (2) and (3) are

$$v_j = a_{qj}\, v'_q, \qquad (6)$$

$$w_i = a_{pi}\, w'_p. \qquad (7)$$

Now $\quad u'_{pq}\, v_q' = w_p'$

$$= a_{pi}\, w_i$$

$$= a_{pi}\, (u_{ij}\, v_j)$$

$$= a_{pi}\, (a_{qj}\, v_q')\, u_{ij}$$

$$= a_{pi}\, a_{qj}\, u_{ij}\, v_q'.$$

This gives

$$(u'_{pq} - a_{pi}\, a_{qj}\, u_{ij})\, v_q' = 0, \qquad (8)$$

for an arbitrary vector v_q'. Therefore, we must have

$$u'_{pq} = a_{pi}\, a_{qj}\, u_{ij}. \qquad (9)$$

This rule shows that components u_{ij} obey the tensor law of transformation of a second order.

Hence, u_{ij} is a tensor of order two.

Question : Show that δ_{ij} and a_{ij} are tensors, each of order two.

Solution : Let u_i be any tensor of order one.

(a) By the substitution property of the Kronecker delta tensor δ_{ij}, we have

$$u_i = \delta_{ij}\, u_j. \qquad (1)$$

Now u_i and v_j are, each of tensor order 1. Therefore, by quotient law, we conclude that δ_{ij} is a tensor of rank two.

(b) The transformation law for the first order tensor u_i is

$$u_p' = a_{pi}\, u_i, \qquad (2)$$

where

$$a_{pi} = \cos\,(x_p', x_i). \qquad (3)$$

Now u_i is a vector and $a_{pi}\, u_i$ is a vector by contraction property. Therefore, by quotient law, the quantities a_{pi} are components of a second order tensor.

Hence the result.

Note (1) The tensor δ_{ij} is called **a unit tensor** or an **identity tensor** of order two.

Note (2) We may call the tensor a_{ij} as the **transformation tensor of rank two.**

Exercise 1 : Let a_i be an ordered triplet and b_i be a vector, referred to the x_i – axis. If $a_i\, b_i$ is a scalar, show that a_i are components of a vector.

Exercise 2 : If there is an entity representable by a set of 27 quantities u_{ijk} relative to o $x_1\, x_2\, x_3$ system and if $u_{ijk}\, v_{jk}$ is a tensor of order one for an arbitrary tensor v_{jk} of order 2, show that u_{ijk} is a tensor of order 3.

Exercise 3 : If $u_{ijk}\, v_k$ is a tensor of order 2 for an arbitrary tensor v_k of order one, show that u_{ijk} is tensor of order 3.

1.6 THE SYMBOL $\in_{ijk}$

The symbol $\in_{ijk}$ is known as the Levi – civita $\in$ - symbol , named after the Italian mathematician Tullio Levi – civita (1873 – 1941).

The $\in$ - symbol is also referred to as the **permutation symbol / alternating symbol** or **alternator.**

In terms of mutually orthogonal unit vectors $\bar{e}_1, \bar{e}_2, \bar{e}_3$ along the cartesian axes , it is defined as

$$\bar{e}_i \cdot (\bar{e}_j \times \bar{e}_k) = \in_{ijk},$$

for i , j , k = 1 , 2 , 3. Thus, the symbol $\in_{ijk}$ gives

$$\in_{ijk} = \begin{cases} 1 & \text{if } i, j, k \text{ take values in the cyclic order} \\ -1 & \text{if } i, j, k \text{ takes value in the acyclic order} \\ 0 & \text{if two or all of } i, j, k \text{ take the same value} \end{cases}$$

These relations are 27 in number.

The $\in$ - symbol is useful in expressing the vector product of two vectors and scalar triple product.

(i) We have

$$\bar{e}_i \times \bar{e}_j = \in_{ijk}\, \bar{e}_k.$$

(ii) For two vectors a_i and b_i , we write

$$\bar{a} \times \bar{b} = (a_i\, e_i) \times (b_j\, e_k) = a_i\, b_j\, (e_i \times e_j) = \in_{ijk}\, a_i\, b_j\, e_k.$$

(iii) For vectors

$$a = a_i\, e_i\,,\; b = b_j\, e_j\,,\; c = c_k\, e_k,$$

we have

$$[\bar{a}, \bar{b}, \bar{c}] = (\bar{a} \times \bar{b}) \cdot \bar{c}$$

$$= (\in_{ijk} a_i b_j e_k) \cdot (c_k e_k)$$

$$= \in_{ijk} a_i b_j c_k$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Question: Show that $\in_{ijk}$ is a tensor of order 3.

Solution: Let $\bar{a} = a_i$ and $\bar{b} = b_j$ be any two vectors. Let

$$\bar{c} = c_i = \bar{a} \times \bar{b} .$$

Then

$$c_i = \in_{ijk} a_j b_k. \qquad \text{(i)}$$

Now $a_j b_k$ is a tensor of order 2 and $\in_{ijk} a_j b_k$ is a tensor of order one. Therefore , by quotient law, $\in_{ijk}$ is a tensor of order 3.

Note (1) Due to tensorial character of the $\in$ - symbol , it is called an **alternating tensor** or **permutation tensor.**

Note (2) The symbols δ_{ij} and $\in_{ijk}$ were introduced earlier to simplifying the writing of some equations.

Vector of a Second Order Tensor

Let u_{ij} be a second order tensor. The vector

$$\in_{ijk} u_{jk}$$

is called the vector of the tensor u_{jk}.

Example 1: Show that $w_{ij} = \in_{ijk} u_k$ is a skew – symmetric tensor, where u_k is a vector and $\in_{ijk}$ is an alternating tensor.

Solution: Since $\in_{ijk}$ is a tensor of order 3 and u_k is a tensor of order one , so by contraction , the product $\in_{ijk} u_k$ is a tensor of order 2 . Further

$$w_{ji} = \in_{jik} u_k$$

$$= - \in_{ijk} u_k$$

$$= - w_{ji}.$$

This shows that w_{ij} is a tensor which is skew – symmetric.

Example 2: Show that u_{ij} is symmetric iff $\in_{ijk} u_{ij} = 0$.

Solution: We find

$$\in_{ij1} u_{ij} = \in_{231} u_{23} + \in_{321} u_{32} = u_{23} - u_{32}$$

$$\in_{ij2} u_{ij} = u_{31} - u_{13} \, , \in_{ij3} u_{ij} = u_{12} - u_{21}$$

Thus, u_{ij} is symmetric iff

$$u_{ij} = u_{ji}$$

or

$$u_{23} = u_{32} \, , u_{13} = u_{31} \, , u_{12} = u_{21}.$$

1.7 ISOTROPIC TENSORS

Definition: A tensor is said to be an isotropic tensor if its components **remain unchanged / invariant** however the axes are rotated.

Note (1) An isotropic tensor possesses no directional properties. Therefore a non – zero vector (or **a non – zero tensor of rank 1) can never be an isotropic tensor.**

Tensors of higher orders , other than one , can be isotropic tensors.

Note (2) Zero tensors of **all orders** are isotropic tensors.

Note (3) By definition , a scalar (or a tensor of rank zero) is an isotropic tensor.

Note (4) A scalar multiple of an isotropic tensor is an isotropic tensor.

Note (5) The sum and the differences of two isotropic tensors is an isotropic tensor.

Theorem: Prove that substitution tensor δ_{ij} and alternating tensor $\in_{ijk}$ are isotropic tensors.

Proof: Let the components δ_{ij} relative to x_i system are transformed to quantities δ'_{pq} relative to x_i' - system. Then , the tensorial transformation rule is

$$\delta'_{pq} = a_{pi} \, a_{qj} \, \delta_{ij} \qquad (1)$$

where

$$a_{pi} = \cos(x_p' \, , x_i) \, . \qquad (2)$$

Now

$$\text{RHS of (1)} = a_{pi}\,[a_{qj}\,\delta_{ij}]$$

$$= a_{pi}\,a_{qi}$$

$$= \delta_{pq}$$

$$= \begin{cases} 0 & if\;\; p \neq q \\ 1 & if\;\; p = q \end{cases}. \tag{3}$$

Relations (1) and (3) show that the components δ_{ij} are transformed into itself under all co-ordinate transformations. Hence , by definition , δ_{ij} is an isotropic tensor.

We know that $\in_{ijk}$ is a system of 27 numbers. Let

$$\in_{ijk} = [\,\bar{e}_i\,,\ \bar{e}_j\,,\ \bar{e}_k] = \bar{e}_i\,.\,(\bar{e}_j \times \bar{e}_k), \tag{4}$$

be related to the x_i axes. Suppose that these components are transformed to $\in'_{pqr}$ relative to x_i' - axis. Then , the third order tensorial law of transformation gives

$$\in'_{pqr} = a_{pi}\,a_{qj}\,a_{rk}\,\in_{ijk} \tag{5}$$

where l_{pi} is defined in (2)

we have already checked that (exercise)

$$\in_{ijk}\,a_{pi}\,a_{qj}\,a_{rk} = \begin{vmatrix} a_{p1} & a_{p2} & a_{p3} \\ a_{q1} & a_{q2} & a_{q3} \\ a_{r1} & a_{r2} & a_{r3} \end{vmatrix} \tag{6}$$

and

$$[\,\bar{e}_p{}'\,,\ \bar{e}_q{}'\,,\ \bar{e}_r{}'] = \begin{vmatrix} a_{p1} & a_{p2} & a_{p3} \\ a_{q1} & a_{q2} & a_{q3} \\ a_{r1} & a_{r2} & a_{r3} \end{vmatrix} \tag{7}$$

From (5) – (7) , we get

$$\in'_{pqr} = [\,\bar{e}_p{}'\,,\ \bar{e}_q{}'\,,\ \bar{e}_r{}']$$

$$= \bar{e}_p{}'\,.\,(\bar{e}_q{}' \times \bar{e}_r{}')$$

$$= \begin{cases} 1 & if\ p,q,r\ are\ in\ cyclic\ order \\ -1 & if\ p,q,r\ are\ in\ cyclic\ order \\ 0 & if\ any\ two\ or\ all\ suffices\ are\ equal \end{cases} \qquad (8)$$

This shows that components $\in_{ijk}$ are transformed into itself under all coordinate transformations. Thus, the third order tensor $\in_{ijk}$ is an isotropic tensor.

Theorem: If u_{ij} is an isotropic tensor of second order , then show that

$$u_{ij} = \alpha\, \delta_{ij}$$

for some scalar α.

Proof: As the given tensor is isotropic , we have

$$u_{ij}' = u_{ij}\ , \qquad (1)$$

for all choices of the x_i' - system. In particular , we choose

$$x_1' = x_2\ ,\ x_2' = x_3\ ,\ x_3' = x_1 \qquad (2)$$

Then

$$a_{ij} = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \qquad (3)$$

and

$$u'_{pq} = a_{pi}\, a_{qj}\, u_{ij}\ . \qquad (4)$$

Now

$$u_{11} = u_{11}' = a_{1i}\, a_{1j}\, u_{ij}$$
$$= a_{12}\, a_{12}\, u_{23} = u_{22}\ ,$$
$$u_{22} = u_{22}' = a_{2i}\, a_{2j}\, u_{ij}$$
$$= a_{23}\, a_{23}\, u_{33} = u_{33}\ ,$$
$$u_{12} = u_{12}' = a_{1i}\, a_{2j}\, u_{ij}$$
$$= a_{12}\, a_{23}\, u_{23} = u_{23}\ ,$$
$$u_{23} = u_{23}' = a_{2i}\, a_{3j}\, u_{ij}$$
$$= a_{23}\, a_{31}\, u_{31} = u_{31}\ ,$$
$$u_{13} = u_{13}' = a_{1i}\, a_{3j}\, u_{ij}$$

$$= a_{12}\, a_{31}\, u_{21} = u_{21}\ ,$$

$$u_{21} = u_{21}' = a_{2i}\, a_{1j}\, u_{ij}$$

$$= a_{23}\, a_{12}\, u_{32} = u_{32}\ .$$

Thus

$$u_{11} = u_{22} = u_{33}\ ,$$

$$u_{12} = u_{23} = u_{31}\ ,$$

$$u_{21} = u_{32} = u_{13}\ . \tag{5}$$

Now , we consider the transformation

$$x_1' = x_2\ ,\ x_2' = -x_1\ ,\ x_3' = x_3\ . \tag{6}$$

Then

$$(a_{ij}) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{7}$$

$$u'_{pq} = a_{pi}\, a_{qj}\, u_{ij} \tag{8}$$

This gives

$$u_{13} = u'_{13} = a_{1i}\, a_{3j}\, u_{ij}$$

$$= a_{12}\, a_{33}\, u_{23} = u_{23}\ ,$$

$$u_{23} = u'_{23} = a_{2i}\, a_{3j}\, u_{ij}$$

$$= a_{21}\, a_{33}\, u_{13} = -\, u_{13}\ .$$

Thus

$$u_{13} = u_{23} = 0. \tag{9}$$

From (5) and (9), we obtain

$$a_{ij} = \alpha\, \delta_{ij}\ , \tag{10}$$

where

$$\alpha = a_{11} = a_{22} = a_{33}. \tag{11}$$

Note 1: If a_{ijk} are components of an isotropic tensor of third order, then

$$a_{ijk} = \alpha \,.\, \in_{ijk}$$

for some scalar α.

Note 2: If a_{ijkm} are components of a fourth – order isotropic tensor , then

$$a_{ijkm} = \alpha\, \delta_{ij}\, \delta_{km} + \beta\, \delta_{ik}\, \delta_{jm} + \gamma\, \delta_{im}\, \delta_{jk}$$

for some scalars α , β , γ .

Definition: (Gradient)

If $u_{pqr\ldots}(x_1 , x_2 , x_3)$ is a tensor of order n , then

$$v_{spqr\ldots\ldots} = \frac{\partial}{\partial x_s} u_{pqr\ldots\ldots}$$

$$= u_{pqr\ldots, s}$$

is defined as the gradient of the tensor field $u_{pqr\ldots}(x_1 , x_2 , x_3)$.

Theorem: Show that the gradient of a scalar point function is a tensor of order 1.

Proof: Suppose that $U = U(x_1 , x_2 , x_3)$ be a scalar point function and

$$v_i = \frac{\partial U}{\partial x_i} = U_{,i} = \text{gradient of U.} \qquad (1)$$

Let the components of the gradient of U in the dashed system o $x_1' \, x_2' \, x_3'$ be v_p', so that,

$$v_p' = \frac{\partial U}{\partial x_{p'}}, \qquad (2)$$

where the transformation rule of coordinates is

$$x_p' = a_{pi}\; x_i , \qquad (3)$$

$$x_i = a_{pi}\; x_p' \qquad (4)$$

$$a_{pi} = \cos\,(x_p' , x_i) . \qquad (5)$$

By chain rule

$$v_p' = \frac{\partial U}{\partial x_{p'}}$$

$$= \frac{\partial U}{\partial x_i} \quad \frac{\partial x_i}{\partial x_{p'}}$$

$$= a_{pi} \frac{\partial U}{\partial x_i}$$

$$= a_{pi}\, v_i \ ,$$

which is a transformation rule for tensors of order 1.

Hence gradient of the scalar point function U is a tensor of order one.

Theorem: Show that the **gradient of a vector** u_i is a tensor of order 2. Deduce that δ_{ij} is a tensor of order 2.

Proof: The **gradient of the tensor** u_i is defined as

$$w_{ij} = \frac{\partial u_i}{\partial x_j} = u_{i\,,\,j}. \tag{1}$$

Let the vector u_i be transformed to the vector u_p' relative to the new system o $x_1'\ x_2'\ x_3'$. Then the transformation law for tensors of order 1 yields

$$u_p' = a_{pi}\, u_i, \tag{2}$$

where the law of transformation of coordinates is

$$x_q' = a_{qj}\, x_j, \tag{3}$$

$$x_j = a_{qj}\, x_q', \tag{4}$$

$$a_{pi} = \cos\,(x'_p, x_i). \tag{5}$$

Suppose that the 9 quantities w_{ij} relative to new system are transformed to w'_{pq}. Then

$$w'_{pq} = \frac{\partial u'_p}{\partial x'_q}$$

$$= \frac{\partial}{\partial x_q'}\,(a_{pi}\, u_i)$$

$$= a_{pi} \frac{\partial u_i}{\partial x_q'}$$

$$= a_{pi} \frac{\partial u_i}{\partial x_j} \frac{\partial x_j}{\partial x_q'}$$

$$= a_{pi}\, a_{qj} \frac{\partial u_i}{\partial x_j}$$

$$= a_{pi}\, a_{qj}\, w_{ij}, \qquad (6)$$

which is the transformation rule for tensors of order 2.

Hence , w_{ij} is a tensor of order 2. Consequently , the gradient of the vector u_i is a tensor of order 2.

Deduction : We know that

$$\delta_{ij} = \frac{\partial x_i}{\partial x_j},$$

and that x_i is a vector. So , δ_{ij} is a gradient of the vector x_i. It follows that 9 quantities δ_{ij} are components of a tensor of order 2.

1.8 EIGENVALUES AND EIGEN VECTORS OF A SECOND ORDER SYMMETRIC TENSOR.

Definition: Let u_{ij} be a **second order** symmetric tensor. A scalar λ is called an **eigenvalue of** the tensor u_{ij} if there exists a **non – zero vector** $\mathbf{v_i}$ such that

$$u_{ij}\, v_j = \lambda\, v_i, \qquad \text{for } i = 1 , 2 , 3.$$

The **non – zero vector** $\mathbf{v_i}$ is then called an eigenvector of tensor u_{ij} corresponding to the eigen vector λ.

We observe that every (**non - zero**) scalar multiple of an eigenvector is also an eigen vector.

Article: Show that it is always possible to find three mutually orthogonal eigenvectors of a second order symmetric tensor.

Proof: Let u_{ij} be a second order symmetric tensor and λ be an eigen value of u_{ij} . Let v_i be an eigenvector corresponding to λ.

Then $\qquad u_{ij}\, v_j = \lambda\, v_i$

or

$$(u_{ij} - \lambda\, \delta_{ij})\, v_j = 0 . \qquad (1)$$

This is a set of three homogeneous simultaneous linear equations in three unknown v_1 , v_2 , v_3. These three equations are

$$\left.\begin{aligned}(u_{11}-\lambda)v_1+u_{12}v_2+u_{13}v_3=0\\ u_{21}v_1+(u_{22}-\lambda)v_2+u_{23}v_3=0\\ u_{31}v_1+u_{32}v_2+(u_{33}-\lambda)v_3=0\end{aligned}\right\} . \qquad (2)$$

This set of equations possesses a non – zero solution when

$$\begin{vmatrix} u_{11}-\lambda & u_{12} & u_{13}\\ u_{21} & u_{22}-\lambda & u_{23}\\ u_{31} & u_{32} & u_{33}-\lambda\end{vmatrix} = 0,$$

or

$$| u_{ij} - \lambda\, \delta_{ij} | = 0. \qquad (3)$$

Expanding the determinant in (3) , we find

$$(u_{11} - \lambda)\, [(u_{22} - \lambda)\, (u_{33} - \lambda) - u_{32}\, u_{23}]$$

$$- \quad u_{12}\, [u_{21}\, (u_{33} - \lambda) - u_{31}\, u_{23}] + u_{13}\, [u_{21}\, u_{32} - u_{31}\, (u_{22} - \lambda)] = 0$$

or

$$- \lambda^3 + (u_{11} + u_{22} + u_{33})\, \lambda^2 - (u_{11}\, u_{22} + u_{22}\, u_{33} + u_{33}\, u_{11} - u_{23}\, u_{32} - u_{31}\, u_{13} - u_{12}\, u_{21})\, \lambda + [u_{11}\, (u_{22}\, u_{33} - u_{23}\, u_{32}) - u_{12}\, (u_{21}\, u_{33} - u_{31}\, u_{23})$$

$$+ u_{13}(u_{21}\, u_{32} - u_{31}\, u_{22})] = 0\ . \qquad (4)$$

We write (4) as

$$- \lambda^3 + \lambda^2\, I_1 - \lambda\, I_2 + I_3 = 0\ , \qquad (5)$$

where

$$I_1 = u_{11} + u_{22} + u_{33}$$

$$= u_{ii}, \qquad (6)$$

$$I_2 = u_{11}\, u_{22} + u_{22}\, u_{33} + u_{33}\, u_{11} - u_{12}\, u_{21} - u_{23}\, u_{32} - u_{31}\, u_{13}$$

$$= \frac{1}{2}[u_{ii}\,u_{jj} - u_{ij}\,u_{ji}], \qquad (7)$$

$$I_3 = |\,u_{ij}\,|$$

$$= \in_{ijk} u_{i1}\,u_{j2}\,u_{k3}\,. \qquad (8)$$

Equation (5) is a cubic equation in λ. Therefore , it has three roots , say , λ_1 , λ_2 , λ_3 which may not be distinct (real or imaginary). These roots (which are scalar) are the three eigenvalues of the symmetric tensor u_{ij} .

Further

$$\lambda_1 + \lambda_2 + \lambda_3 = I_1 \qquad (9)$$

$$\lambda_1\,\lambda_2 + \lambda_2\,\lambda_3 + \lambda_3\,\lambda_1 = I_2 \qquad (10)$$

$$\lambda_1\,\lambda_2\,\lambda_3 = I_3 \qquad (11)$$

Each root λ_i , when substituted in equation (2) , gives a set of three linear equations (homogeneous) which are not all independent. By discarding one of equations and using the condition

$$v_1^2 + v_2^2 + v_3^2 = 1$$

for unit vectors , the eigenvector $\bar{v}_i$ is determined.

Before proceeding further, we state and prove two important lemmas.

Lemma 1: Eigenvalues of a real symmetric tensor u_{ij} are **real**.

Proof: Let λ be an eigenvalue with corresponding eigenvector u_i.

Then $$u_{ij}\,v_j = \lambda\,v_i. \qquad (I)$$

Taking the complex conjugate on both sides of (I) , we find

$$\overline{u_{ij}v_j} = \overline{\lambda v_i} \qquad \overline{u_{ij}}\,\overline{v_j} = \bar{\lambda}\,\overline{v_i}$$

$$u_{ij}\,\bar{v}_j = \bar{\lambda}\,\bar{v}_i \qquad (II)$$

since u_{ij} is a real tensor. Now

$$u_{ij}\,\bar{v}_j\,v_i = (u_{ij}\,\bar{v}_j)\,v_i$$

$$= (\bar{\lambda}\,\bar{v}_i)\,v_i$$

$$= \bar{\lambda}\,\bar{v}_i\,v_i. \qquad (III)$$

Also $$\overline{u_{ij}\,\overline{v_j}\,v_i} = \overline{u_{ij}}\,v_j\,\overline{v_i}$$

$$= u_{ij}\,v_j\,\bar{v}_i$$

$$= u_{ji}\,v_i\,\bar{v}_j$$

$$= u_{ij}\,\bar{v}_j\,v_i.$$

This shows that quantity $u_{ij}\ \bar{v}_j\ v_i$ is real. Hence $\bar{\lambda}\ \ \bar{v}_i\ v_i$ is real.

Since $\bar{v}_i\ v_i$ is always real, it follows that $\bar{\lambda}$ is real.

Therefore λ is real.

Lemma 2: Eigen vector corresponding to two distinct eigen values of the symmetric tensor u_{ij} are orthogonal.

Proof: Let $\lambda_1 \neq \lambda_2$ be two distinct eigenvalues of u_{ij}. Let A_i and B_i be the corresponding non – zero eigenvectors. Then

$$u_{ij}\ A_j = \lambda_1\ A_i,$$

$$u_{ij}\ B_j = \lambda_2\ B_i. \qquad \text{(I)}$$

We obtain

$$u_{ij}\ A_j\ B_i = \lambda_1\ A_i\ B_i,$$

$$u_{ij}\ B_j\ A_i = \lambda_2\ A_i\ B_i\ . \qquad \text{(II)}$$

Now

$$u_{ij}\ A_j\ B_i = u_{ji}\ A_i\ B_j$$

$$= u_{ij}\ B_j\ A_i\ . \qquad \text{(III)}$$

From (II) & (III) , we get

$$\lambda_1\ A_i\ B_i = \lambda_2\ A_i\ B_i$$

$$(\lambda_1 - \lambda_2)\ A_i\ B_i = 0$$

$$A_i\ B_i = 0. \qquad (\because\ \lambda_1 \neq \lambda_2)$$

Hence , eigenvectors A_i and B_i are mutually orthogonal.

This completes the proof of lemma 2.

Now we consider various possibilities about eigenvalues λ_1 , λ_2 , λ_3 of the main theorem.

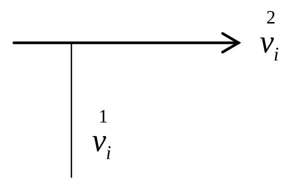
Case 1: If $\lambda_1 \neq \lambda_2 \neq \lambda_3$, i . e. , when all eigenvalues are different and real.

Then , by lemma 2 , three eigenvectors corresponding to λ_i are mutually orthogonal. Hence the result holds.

Case 2: If $\lambda_1 \neq \lambda_2 = \lambda_3$. Let $\overset{1}{v}_i$ be the eigenvector of the tensor u_{ij} corresponding to the

eigenvalue λ_1 and $\overset{2}{v}_i$ be the eigenvector corresponding to λ_2. Then

$$\overset{1}{v}_i \ \overset{2}{v}_i = 0.$$



Let p_i be **a** vector orthogonal to both $\overset{1}{v}_i$ and $\overset{2}{v}_i$. Then

$$p_i \overset{1}{v}_i = p_i \overset{2}{v}_i = 0 , \tag{12}$$

and

$$u_{ij} \overset{1}{v}_j = \lambda_1 \overset{1}{v}_i \quad ,$$

$$u_{ij} \overset{2}{v}_j = \lambda_2 \overset{2}{v}_i . \tag{13}$$

Let

$$u_{ij}\, p_j = q_i = \text{a tensor of order 1} \tag{14}$$

we shall show that q_i and p_i are parallel.

Now

$$\begin{aligned} q_i \overset{1}{v}_i &= u_{ij}\, p_j \overset{1}{v}_i \\ &= u_{ji} \overset{1}{v}_j\, p_i \\ &= u_{ij} \overset{1}{v}_j\, p_i \\ &= \lambda_1 \overset{1}{v}_i\, p_i \\ &= 0. \end{aligned} \tag{15}$$

Similarly

$$q_i \overset{2}{v_i} = 0. \tag{16}$$

Thus q_i is orthogonal to both orthogonal eigenvectors $\overset{1}{v_i}$ and $\overset{2}{v_i}$.

Thus q_i must be parallel to p_i. So , we may write

$$u_{ij}\, p_j = q_i = \alpha\, p_i \ , \tag{17}$$

for some scalar α .

Equation (10) shows tha α must be an eigenvalue and p_i must be the corresponding eigenvector of u_{ij}.

Let

$$\overset{3}{v_i} = \frac{p_i}{|\, p_i \,|} \ . \tag{18}$$

Since u_{ij} has only three eigenvalues λ_1 , $\lambda_2 = \lambda_3$, so α must be equal to $\lambda_2 = \lambda_3$.

Thus $\overset{3}{v_i}$ is an eigenvector which is orthogonal to both $\overset{1}{v_i}$ and $\overset{2}{v_i}$ where $\overset{1}{v_i} \perp \overset{2}{v_i}$.

Thus, there exists three mutually orthogonal eigenvectors.

Further , let w_i be any vector which lies in the plane containing the two eigenvectors $\overset{2}{v_i}$ and $\overset{3}{v_i}$ corresponding to the repeated eigenvalues. Then

$$w_i = k_1 \overset{2}{v_i} + k_2 \overset{3}{v_i}$$

for some scalars k_1 and k_2 and

$$w_i \overset{1}{v_i} = 0 \ ,$$

and

$$u_{ij}\, w_j = u_{ij}(k_1 \overset{2}{v_j} + k_2 \overset{3}{v_j})$$

$$= k_1 u_{ij} \overset{2}{v_j} + k_2 u_{ij} \overset{3}{v_j}$$

$$= k_1 \lambda_2 \overset{2}{v_i} + k_2 \lambda_3 \overset{3}{v_i}$$

$$= \lambda_2 (k_1 \overset{2}{v_i} + k_2 \overset{3}{v_i}) \qquad (\because \lambda_2 = \lambda_3)$$

$$= \lambda_2 w_i . \tag{19}$$

Thus w_i is orthogonal to $\overset{1}{v_i}$ and w_i is an eigenvector corresponding to λ_2.

Hence , any two orthogonal vectors that lie on the plane normal to $\overset{1}{v_i}$ can be chosen as the other two eigenvectors of u_{ij}.

Case 3: If $\lambda_1 = \lambda_2 = \lambda_3$.

In this case , the cubic equation in λ becomes

$$(\lambda - \lambda_1)^3 = 0,$$

or

$$\begin{vmatrix} \lambda_1 - \lambda & 0 & 0 \\ 0 & \lambda_1 - \lambda & 0 \\ 0 & 0 & \lambda_1 - \lambda \end{vmatrix} = 0 . \tag{20}$$

Comparing it with equation (3) , we find

$$u_{ij} = 0 \text{ for } i \neq j$$

and

$$u_{11} = u_{22} = u_{33} = \lambda_1.$$

Thus

$$\mathbf{u_{ij} = \lambda_1 \delta_{ij}} \tag{21}$$

Let v_i be any **non – zero vector.** Then

$$u_{ij} v_j = \lambda_1 \delta_{ij} v_j \quad ,$$

$$= \lambda_1 v_i. \tag{22}$$

This shows that v_i is an eigenvector corresponding to λ_1. Thus , **every non – zero vector in space is an eigenvector which corresponds to the same eigenvalue λ_1.** Of these vectors , we can certainly chose (at least) there vectors $\overset{1}{v_i}$, $\overset{2}{v_i}$, $\overset{3}{v_i}$ that are mutually orthogonal.

Thus , in every case , there exists (at least) three mutually orthogonal eigenvectors of u_{ij}.

Example: Consider a second order tensor u_{ij} whose matrix representation is

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}.$$

It is clear , the tensor u_{ij} is not symmetric. We shall find eigenvalues and eigenvectors of u_{ij}.

The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

or

$$(1 - \lambda) [(2 - \lambda) (3 - \lambda) - 2] - 1 [2 - 2(2 - \lambda)] = 0$$

or

$$(1 - \lambda) (2 - \lambda) (3 - \lambda) = 0 .$$

Hence , eigenvalues are

$$\lambda_1 = 1 , \lambda_2 = 2 , \lambda_3 = 3$$

which are all different.

We find that an unit eigenvector corresponding to $\lambda = 1$ is

$$\overset{1}{v_i} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) ,$$

an unit eigenvector corresponding to $\lambda = 2$ is

$$\overset{2}{v_i} = \left(\frac{2}{3} , -\frac{1}{3} , -\frac{2}{3} \right) ,$$

and an unit eigenvector corresponding to $\lambda = 3$ is

$$\overset{3}{v}_i = \left(\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}} \right) \ .$$

We note that

$$\overset{1}{v}_i\ \overset{2}{v}_i \neq 0,\ \overset{2}{v}_i\ \overset{3}{v}_i \neq 0,\ \overset{1}{v}_i\ \overset{3}{v}_i = 0.$$

This happens due to non – symmetry of the tensor u_{ij}.

Example 2: Let the matrix of the components of the second order tensor u_{ij} be

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find eigenvalues and eigenvectors of it.

We note that the tensor is symmetric. The characteristic equation is

$$\begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

or

$$\lambda(\lambda - 1)\ (\ \lambda - 4) = 0\ .$$

Thus eigenvalues are

$$\lambda_1 = 0\ ,\ \lambda_2 = 1\ ,\ \lambda_3 = 4\ ,$$

which are all different.

Let $\overset{1}{v}_i$ be the unit eigenvector corresponding to eigenvalue $\lambda_1 = 0$. Then , the system of homogegeous equations is

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \overset{1}{v}_1 \\ \overset{1}{v}_2 \\ \overset{1}{v}_3 \end{bmatrix} = 0\ .$$

This gives $\overset{1}{v}_1 + \overset{1}{v}_2 = 0\ ,\ \overset{1}{v}_1 + \overset{2}{v}_2 = 0\ ,\ \overset{1}{v}_3 = 0$.

we find

$$\overset{1}{v}_i = \left(\frac{1}{\sqrt{2}}\ ,\ \frac{-1}{\sqrt{2}}\ ,\ 0 \right)\ .$$

Similarly

$$\overset{2}{v}_i = (0\ ,\ 0\ ,\ 1)\ \ ,$$

and

$$\overset{3}{v}_i = \left(\frac{1}{\sqrt{2}}\ ,\ \frac{1}{\sqrt{2}}\ ,\ 0\right),$$

are eigen vectors corresponding to $\lambda_2 = 1$ and $\lambda_3 = 4$, respectively.

Moreover, these vector are mutually orthogonal.

Books Recommended

1. Y.C. Fung — Foundations of Solid Mechanics, Prentice Hall, Inc., New Jersey, 1965

2. T.M. Atanackovic, A. Guran — Theory of Elasticity for Scientists and Engineers, Birkhauser , Boston, 2000

3. Saada, A.S. — Elasticity – Theory and Applications, Pergamon Press, Inc., NY, 1974.

4. Sokolnikoff, I.S. — Mathematical Theory of Elasticity, Tata McGraw Hill Publishing Company, Ltd., New Delhi, 1977.

5. Garg, N.R. and Sharma, R.K. — Generation of Displacements and Stresses in a Multilayered Half-Space due to a Strip-Loading, Journal ISET, Vol 28, 1991, pp 1-26.

Chapter-2
Analysis of Stress

2.1 INTRODUCTION

Deformation and motion of an elastic body are generally caused by external forces such as surface loads or internal forces such as earthquakes, nuclear explosions, etc. When an elastic body is subjected to such forces , its behaviour depends upon the **magnitude of the forces ,** upon **their direction** , and upon the inherent **strength of the material** of which the body is made. Such forces **give rise to interactions between neighbouring portions** in the interior parts of the elastic solid. Such interactions **are studied** through the **concept of stress.** The concepts of stress vector on a surface and state of stress at a point of the medium shall be discussed.

An approach to the solution of problems in elastic solid mechanics is to examine deformations initially , and then consider stresses and applied loads. Another approach is to establish relationships between applied loads and internal stresses first and then to consider deformations. Regardless of the approach selected , it is necessary to derive the component relations individually.

2.2. BODY FORCES AND SURFACE FORCES

Consider a continuous medium. We refer the points of this medium to a rectangular cartesian coordinate system. Let τ represents the region occupied by the body in the deformed state. A deformable body may be acted upon by two different types of external forces.

(i) Body forces : **These forces are those forces** which act on every volume element of the body and hence on the entire volume **of the body. For example ,** gravitational force **is a body force** (magnetic forces **are also body forces). Let ρ denotes the density of a volume element $\Delta\tau$ of the body τ. Let g be the gravitational force / acceleration. Then , the force acting on the mass** $\rho\Delta\tau$ **contained in volume $\Delta\tau$ is g.**$\rho\Delta\tau$**.**

(ii) Surface forces : **These forces are those which act upon every surface element of the body. Such forces are** also called contact forces. Loads applied **over the exterior surface or bounding surface are examples of surface forces.** A hydrostatic pressure **acting on the surface of a body** submerged in a liquid **/ water is a surface force.**

Internal forces : **In addition to the external forces , there are internal forces (such** as earthquakes , nuclear explosions**) which arise** from the mutual interaction between various parts **of the elastic body.**

Now , we consider an elastic body in its undeformed state with no forces acting on it. Let a system of forces be applied to it. Due to these forces , the body is deformed and a system of internal forces is set up to oppose this deformation. **These internal forces** give rise to stress within the body**. It is therefore necessary to consider** how external forces are transmitted through the medium.

2.3 STRESS VECTOR ON A PLANE AT A POINT

Let us consider an elastic body in equilibrium under the action of a system of external forces. Let us pass a fictitious plane π through a point P(x_1 , x_2 , x_3) in the interior of this body. The body can be considered as consisting of two parts , say , A
and B and these parts are in welded contact at the interface π. Part A of the body is in equilibrium under forces (external) and the effect of part B on the plane π. We assume that this effect is continuously distributed over the surface of intersection.

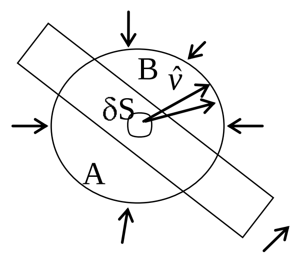


Fig. (2.1)
Around the point P , let us consider a small surface δS (on the place π) and let $\hat{v}$ be an outward unit normal vector (for the part A of the body). The effect of part B on this small surface element can be reduces to a force $\bar{Q}$ and a vector couple $\bar{C}$. Now, let δS shrink in size towards zero in a manner such that the point P always remain inside δS and $\hat{v}$ remains the normal vector.

We assume that $\frac{\bar{Q}}{\delta S}$ **tends to a definite limit $\bar{T}$ (x_1 , x_2 , x_3) and that** $\frac{\bar{C}}{\delta S}$ **tends to zero as δS tends to zero. Thus**

$$\lim_{\delta S \to 0} \frac{\bar{Q}}{\delta S} = \bar{T}(x_1, x_2, x_3),$$

$$\lim_{\delta S \to 0} \frac{\bar{C}}{\delta S} = 0 .$$

Now $\bar{T}$ is a surface force per unit area.

This force , $\bar{T}$, is called the stress vector **or** traction on the plane π at P.

Note 1: **Forces acting over the surface of a body are never idealized point forces , they are, in reality, forces per unit area applied over some finite area. These external forces per unit area are called also tractions.**

Note 2: Cauchy's stress postulate

If we consider another oriented plane π' containing the same point $P(x_i)$, then the stress vector is likely to have a different direction. For this purpose, Cauchy made the following postulated – known as Cauchy's stress postulate.

"The stress vector $\underset{\sim}{T}$ depends on the orientation of the plane upon which it acts".

Let $\hat{v}$ be the unit normal to plane π through the point P. This normal characterize the orientation of the plane upon which the stress vector acts. For this reason , we write the stress vector as $\underset{\sim}{\overset{\hat{v}}{T}}$**, indicating its dependence on the orientation $\hat{v}$.**

Cauchy's Reciprocal relation

When the plane π is in the interior of the elastic body , the normal $\hat{v}$ has two possible directions that are opposite to each other and we choose one of these directions.

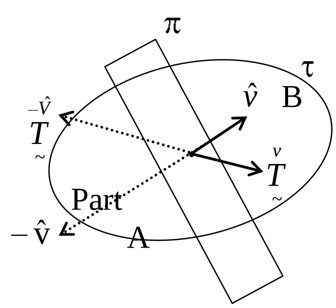


Fig. (2.2)

For a chosen $\hat{v}$, the stress vector $\underset{\sim}{\overset{\hat{v}}{T}}$ is interpreted as the internal surface force per unit area acting on plane π due to the action of part B of the material / body which $\hat{v}$ is directed upon the part A across the plane π.

Consequently , $\underset{\sim}{\overset{-\hat{v}}{T}}$ is the internal surface force per unit area acting on π due to the action of part A for which $\hat{v}$ is the outward drawn unit normal.

By Newton's third law of motion , vector $\underset{\sim}{T}^{-\hat{v}}$ and $\underset{\sim}{T}^{\hat{v}}$ balance each other as the body is in equilibrium. Thus

$$\underset{\sim}{T}^{-\hat{v}} = -\underset{\sim}{T}^{\hat{v}} ,$$

which is known as Cauchy's reciprocal relation.

Homogeneous State of Stress

If π and π' are any two parallel planes through any two points P and P′ of a continuous elastic body , and if the stress vector on π at P is equal to the stress on π' at P′ , then the state of stress in the body is said to be a homogeneous state of stress.

2.4 NORMAL AND TANGENTIAL STRESSES

In general , the stress vector $\underset{\sim}{T}^{v}$ is inclined to the plane on which it acts and need not be in the direction of unit normal $\hat{v}$. The projection of $\underset{\sim}{T}^{v}$ on the normal $\hat{v}$ is called the normal stress. **It is denoted by σ or σ_n. The projection of $\underset{\sim}{T}^{v}$ on the plane π , in the plane of $\underset{\sim}{T}^{\hat{v}}$ and $\hat{v}$, is called the** tangential **or** shearing stress. **It is denoted by τ or σ_t.**

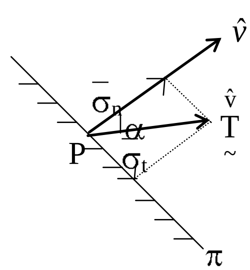


Fig. (2.3)

Thus,

$$\sigma = \sigma_n = \underset{\sim}{T}^{v} \cdot \hat{v} \qquad , \qquad \tau = \sigma_t = \underset{\sim}{T}^{\hat{v}} \cdot \hat{t} , \qquad \textbf{(1)}$$

$$|\underset{\sim}{T}^{\hat{v}}|^2 = \sigma_n^2 + \sigma_t^2 \qquad \textbf{(2)}$$

where $\hat{t}$ is a unit vector normal to $\hat{v}$ and lies in the place π.

A stress in the direction of the outward normal is considered positive (i.e. $\sigma > 0$) and is called a tensile stress**. A stress in the opposite direction is considered negative ($\sigma < 0$) and is called a** compressible stress.

If σ **= 0 ,** $\underset{\sim}{\overset{\hat{v}}{T}}$ **is perpendicular to** $\hat{v}$**. Then , the stress vector** $\underset{\sim}{\overset{\hat{v}}{T}}$ **is called a** pure shear stress **or a** pure tangential stress.

If τ **= 0 , then** $\underset{\sim}{\overset{\hat{v}}{T}}$ **is parallel to** $\hat{v}$**. The stress vector** $\underset{\sim}{\overset{\hat{v}}{T}}$ **is then called** pure normal stress.

When $\underset{\sim}{\overset{\hat{v}}{T}}$ **acts opposite to the normal** $\hat{v}$ **, then the pure normal stress is called** pressure **($\sigma < 0, \tau = 0$).**

From (1) , we can write

$$\underset{\sim}{\overset{\hat{v}}{T}} = \sigma\, \hat{v} + \tau\, \hat{t} \qquad \textbf{(3)}$$

and

$$\tau = \sqrt{|\underset{\sim}{\overset{\hat{v}}{T}}|^2 - \sigma^2} \qquad \textbf{(4)}$$

Note : $\sigma_t = \tau = |\underset{\sim}{\overset{v}{T}}|$ **sin** $\alpha \;\Rightarrow\; |\sigma| = |\underset{\sim}{\overset{\hat{v}}{T}} \times \hat{v}|$, **as** $|\hat{v}| = 1$.

This τ in magnitude is given by the magnitude of vector product of $\underset{\sim}{\overset{\hat{v}}{T}}$ **and** $\hat{v}$**.**

2.5 STRESS COMPONENTS

Let $P(x_i)$ be any point of the elastic medium whose coordinates are (x_1 , x_2 , x_3) relative to rectangular cartesian system o x_1 x_2 x_3.

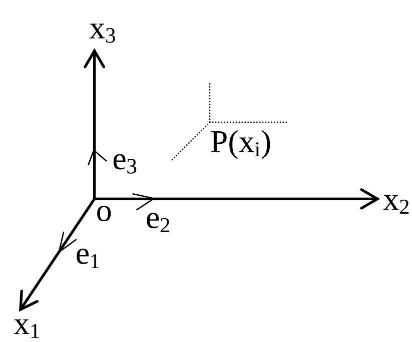


Fig. (2.4)

Let $\overset{1}{\underset{\sim}{T}}$ denote the stress vector on the plane , with normal along x_1 – axis , at the point P. Let the stress vector $\overset{1}{\underset{\sim}{T}}$ has components τ_{11} , τ_{12} , τ_{13} , i.e. ,

$$\overset{1}{\underset{\sim}{T}} = \tau_{11}\ \hat{e}_1 + \tau_{12}\ \hat{e}_2 + \tau_{13}\ \hat{e}_3$$

$$= \tau_{1j}\ \hat{e}_j\,. \tag{1}$$

Let $\overset{2}{\underset{\sim}{T}}$ be the stress vector acting on the plane || to $x_1\ x_3$ – plane at P. Let

$$\overset{2}{\underset{\sim}{T}} = \tau_{21}\ \hat{e}_1 + \tau_{22}\ \hat{e}_2 + \tau_{23}\ \hat{e}_3$$

$$= \tau_{2j}\ \hat{e}_j\,. \tag{2}$$

Similarly

$$\overset{3}{\underset{\sim}{T}} = \tau_{31}\ \hat{e}_1 + \tau_{32}\ \hat{e}_2 + \tau_{33}\ \hat{e}_3$$

$$= \tau_{3j}\ \hat{e}_j\,. \tag{3}$$

Equations (1) – (3) can be condensed in the following form

$$\overset{i}{\underset{\sim}{T}} = \tau_{ij}\ \hat{e}_j\,. \tag{4}$$

Then $$\overset{i}{\underset{\sim}{T}} \cdot \hat{e}_k = (\tau_{ij}\ \hat{e}_j) \cdot \hat{e}_k$$

$$= \tau_{ij}\ \delta_{jk}$$

$$= \tau_{ik}. \tag{5}$$

Thus , for given i & j , the quantity τ_{ij} represent the jth components of the stress vector $\overset{i}{\underset{\sim}{T}}$ acting on a plane having $\hat{e}_i$ **as the unit normal. Here , the first suffix i indicates the direction of the normal to the plane through P and the second suffix j indicates the direction of the stress component. In all , we have 9 components τ_{ij} at the point $P(x_i)$ in the o $x_1\ x_2\ x_3$ system. These quantities are called** stress – components. **The matrix**

$$(\tau_{ij}) = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix},$$

whose rows are the components of the three stress vectors, is called the matrix of the state of stress at P. The dimensions of stress components are $\frac{force}{(Length)^2}$ **= M L^{-1} T^{-2}.**

The stress components τ_{11} , τ_{22} , τ_{33} are called normal stresses **and other components τ_{12} , τ_{13} , τ_{21} , τ_{23} , τ_{31} , τ_{32} are called** shearing stresses **(**$\because \overset{1}{\underset{\sim}{T}} \cdot \hat{e}_1$ = **e_{11} ,** $\overset{1}{\underset{\sim}{T}} \cdot \hat{e}_2$ **= e_{12} , etc)**

In CGS system , the stress is measured in dyne per square centimeter**.**

In English system , it measured in pounds per square inch **or** tons per square inch.

Dyadic Representation of Stress

It may be helpful to consider the stress tensor as a vector – like quantity having a magnitude and associated direction (s) , specified by unit vector. The dyadic is such a representation. We write the stress tensor **or** stress dyadic **as**

$$\bar{\bar{\tau}} = \tau_{ij}\, \hat{e}_i\, \hat{e}_j$$

$$= \tau_{11}\, \hat{e}_1 \hat{e}_1 + \tau_{12}\, \hat{e}_1 \hat{e}_2 + \tau_{13}\, \hat{e}_1 \hat{e}_3 + \tau_{21}\, \hat{e}_2 \hat{e}_1 + \tau_{22}\, \hat{e}_2 \hat{e}_2$$

$$+ \tau_{23}\, \hat{e}_2 \hat{e}_3 + \tau_{31} \hat{e}_3 \hat{e}_1 + \tau_{32} \hat{e}_3 \hat{e}_2 + \tau_{33} \hat{e}_3 \hat{e}_3 \qquad \textbf{(1)}$$

where the juxtaposed double vectors are called dyads.

The stress vector $\overset{i}{\underset{\sim}{T}}$ **acting on a plane having normal along** $\hat{e}_i$ **is evaluated as follows :**

$$\overset{i}{\underset{\sim}{T}} = \bar{\bar{\sigma}} \cdot \hat{e}_i$$

$$= (\tau_{jk}\, \hat{e}_j\, \hat{e}_k) \cdot \hat{e}_i$$

$$= \tau_{jk}\ \hat{e}_j\ (\delta_{ki})$$

$$= \tau_{ji}\ \hat{e}_j$$

$$= \tau_{ij}\ \hat{e}_j\ .$$

2.6 STATE OF STRESS AT A POINT−THE STRESS TENSOR

We shall show that the state of stress at any point of an elastic medium on an oblique plane is completely characterized by the stress components at P.

Let $\underset{\sim}{\overset{v}{T}}$ be the stress vector acting on an oblique plane at the material point P , the unit normal to this plane being $\hat{v}$ = v_i .

Through the point P , we draw three planar elements parallel to the coordinate planes. A fourth plane ABC at a distance h from the point P and parallel to the given oblique plane at P is also drawn. Now, the tetrahedron PABC contains the elastic material.

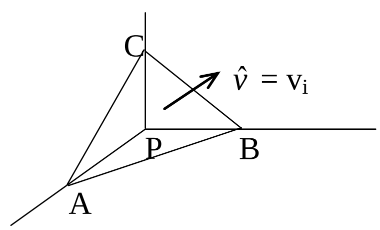


Fig. (2.5)

Let τ_{ij} be the components of stress at the point P. Regarding the signs (negative or positive) of scalar quantities τ_{ij} , we adopt the following convention.

If one draws an exterior normal (outside the medium) to a given face of the tetrahedron PABC ,then the positive values of components τ_{ij} are associated with forces acting in the positive directions of the coordinate axes. On the other hand , if the exterior normal to a given face is pointing in a direction opposite to that of the coordinate axes , then the positive values of τ_{ij} are associated with forces directed oppositely to the positive directions of the coordinate axes.

Let σ be the area of the face ABC of the tetrahedron in figure. Let σ_1 , σ_2 , σ_3 be the areas of the plane faces PBC , PCA and PAB (having normals along x_1 - , x_2 - & x_3 – axes) respectively.

Then

$$\sigma_i = \sigma \cos (x_i , \hat{\nu}) = \sigma \nu_i \qquad (1)$$

The volume of the tetrahedron is

$$v = \frac{1}{3} h \sigma. \qquad (2)$$

Assuming the continuity of the stress vector $\overset{\nu}{\underset{\sim}{T}} = \overset{\nu}{T_i}$, the x_i – component of the stress force acting on the face ABC of the tetrahedron PABC (made of elastic material) is

$$(\overset{\nu}{T_i} + \in_i) \sigma , \qquad (3)$$

provided $$\lim_{h \to 0} \in_i = 0 . \qquad (3a)$$

Here , $\in_i$'s are inserted because the stress force act at points of the oblique plane ABC and not on the given oblique plane through P. Under the assumption of continuing of stress field , quantities $\in_i$'s are infinitesimals.

We note that the plane element PBC is a part of the boundary surface of the material contained in the tetrahedron. As such , the unit outward normal to PBC is $-\hat{e}_1$. Therefore , the x_i – component of force due to stress acting on the face PBC of area σ_1 is

$$(-\tau_{1i} + \in_{1i}) \sigma_1 \qquad (4a)$$

where $$\lim_{h \to 0} \in_{1i} = 0.$$

Similarly forces on the face PCA and PAB are

$$\left.\begin{array}{l} (-\tau_{2i} + \in_{2i}) \sigma_2 , \\ (-\tau_{3i} + \in_{3i}) \sigma_3 \end{array}\right\}$$

with $\lim\limits_{h \to 0}$

$$\in_{2i} = \lim_{h \to 0} \in_{3i} = 0. \qquad (4b)$$

On combining (4a) and (4b) , we write

$$(-\tau_{ji} + \in_{ji}) \sigma_j, \qquad (5)$$

as the x_i – component of stress force acting on the face of area σ_j provided

$$\lim_{h \to 0} \in_{ji} = 0.$$

In equation (5) , the stress components τ_{ij} are taken with the negative sign as the exterior normal to a face of area σ_j is in the negative direction of the x_j – axis.

Let F_i be the body force per unit volume at the point P. Then the x_i – component of the body force acting on the volume of tetrahedron PABC is

$$\frac{1}{3} h\,\sigma(F_i + \in_i') \tag{6}$$

where $\in_i$'s are infinitesimal and

$$\lim_{h \to 0} \in_i' = 0.$$

Since the tetrahedral element PABC of the elastic body is in equilibrium, therefore , the resultant force acting on the material contained in PABC must be zero. Thus

$$(\overset{v}{T}_i + \in_i)\,\sigma + (-\,\tau_{ji} + \in_{ji})\sigma_j + \frac{1}{3}(F_i + \in_i')h\,\sigma = 0.$$

Using (1) , above equation (after cancellation of σ) becomes

$$(\overset{v}{T}_i + \in_i) + (-\tau_{ji} + \in_{ji})v_j + \frac{1}{3}(F_i + \in_i')\,h = 0. \tag{7}$$

As we take the limit $h \to 0$ in (7) , the oblique face ABC tends to the given oblique plane at P. Therefore , this limit gives

$$\overset{v}{T}_i - \tau_{ji}\,v_j = 0$$

or

$$\overset{v}{T}_i = \tau_{ji}\,v_j \tag{8}$$

This relation connecting the stress vector $\overset{v}{\underset{\sim}{T}}$ and the stress components τ_{ij} is known as Cauchy's law or formula**.**

It is convenient to express the equation (8) in the matrix notation. This has the form

$$\begin{bmatrix} \overset{v}{T}_1 \\ \overset{v}{T}_2 \\ \overset{v}{T}_3 \end{bmatrix} = \begin{bmatrix} \tau_{11} & \tau_{21} & \tau_{31} \\ \tau_{12} & \tau_{22} & \tau_{32} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{8a}$$

As $\overset{v}{T}_i$ and v_i are vectors. Equation (8) shows , by quotient law for tensors , that new components τ_{ij} form a second order tensor**.**

This stress tensor is called the CAUCHY'S STRESS TENSOR.

We note that , through a given point , there exists infinitely many surface plane elements. On every one of these elements we can define a stress vector. The totality of all these stress vectors is called the state of stress **at the point. The relation (8) enables us to find the stress vector on any surface element at a point by knowing the stress tensor at that point. As such , the state of stress at a point is completely determined by the stress tensor at the point.**

Note : **In the above , we have assumed , first , that stress can be defined everywhere in a body , and , second , that the stress field is continuous. These are the basic assumptions of continuum mechanics. Without these assumptions , we can do very little. However , in the further development of the theory , certain mathematical discontinuities will be permitted / allowed.**

2.7 BASIC BALANCE LAWS

(A) Balance of Linear Momentum :

So far , we have discussed the state of stress at a point. If it is desired to move from one point to another , **the stress components will change. Therefore, it is necessary to investigate the equations / conditions which control the way in which they change.**

While the strain tensor e_{ij} has to satisfy six compatibility conditions , the components of stress tensor must satisfy three linear partial differential equations of the first order**. The principle of balance of linear momentum gives us these differential equations. This law , consistent with the** Newton's second law of motion **, states that** the time rate of change of linear momentum **is equal to the** resultant force on the elastic body**.**

Consider a continuous medium in equilibrium with volume τ and bounded by a closed surface σ. Let F_i be the components of the body force per unit volume **and $\overset{v}{T_i}$ be the component of the surface force in the x_i – direction.**

For equilibrium of the medium , the resultant force acting on the matter within τ must vanish . That is

$$\int_\tau F_i \, d\tau + \int_\sigma \overset{v}{T_i} \, d\sigma = 0, \qquad \text{for } i = 1, 2, 3. \tag{1}$$

We know the following Cauchy's formula

$$\overset{v}{T_i} = \tau_{ji} \, v_j, \qquad (i = 1, 2, 3), \tag{2}$$

where τ_{ij} is the stress tensor and v_j is the unit normal to the surface. Using (2) into equation (1) , we obtain

$$\int_\tau F_i \, d\tau + \int_\sigma \tau_{ji} v_j \, d\sigma = 0, \quad (i = 1, 2, 3). \tag{3}$$

We assume that stresses τ_{ij} and their first order partial derivatives are also continuous and single valued in the region τ. Under these assumptions , Gauss – divergence theorem can be applied to the surface integral in (3) and we find

$$\int_\sigma \tau_{ji} v_j \, d\sigma = \int_\tau \tau_{ji,j} \, d\tau. \tag{4}$$

From equations (3) and (4) , we write

$$\int_\tau (\tau_{ji,j} + F_i) \, d\tau = 0, \tag{5}$$

for each i = 1, 2 , 3. Since the region τ of integration is arbitrary (every part of the medium is in equilibrium) and the integrand is continuous, so , we must have

$$\tau_{ji,j} + F_i = 0, \tag{6}$$

for each i = 1, 2 , 3 and at every interior point of the continuous elastic body. These equations are

$$\begin{aligned}
&\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + F_1 = 0 \ , \\
&\frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} + F_2 = 0 \ , \\
&\frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \tau_{33}}{\partial x_3} + F_3 = 0 \ .
\end{aligned} \tag{7}$$

These equations are referred to as Cauchy's equations of equilibrium. **These equations are also called** stress equilibrium equations. **These equations are associated with undeformed cartesian coordinates.**

These equations were obtained by Cauchy in 1827.

Note 1 : **In the case of motion of an elastic body , these equations (due to balance of linear momentum) take the form**

$$\tau_{ji,j} + F_i = \rho\ \ddot{u}_i\ , \qquad (8)$$

where $\ddot{u}_i$ **is the acceleration vector and** ρ **is the density (mass per unit volume) of the body.**

Note 2 : **When body force** F_i **is absent (or negligible) , equations of equilibrium reduce to**

$$\tau_{ji,j} = 0. \qquad (9)$$

Example: **Show that for zero body force, the state of stress for an elastic body given by**

$$\tau_{11} = x^2 + y + 3\,z^2\ ,\ \tau_{22} = 2\,x + y^2 + 2\,z\ ,\ \tau_{33} = -2x + y + z^2$$

$$\tau_{12} = \tau_{21} = -x\,y + z^3\ ,\ \tau_{13} = \tau_{31} = y^2 - x\,z\ ,\ \tau_{23} = \tau_{32} = x^2 - y\,z$$

is possible.

Example: **Determine the body forces for which the following stress field describes a state of equilibrium**

$$\tau_{11} = -2x^2 - 3y^2 - 5z\ ,\ \tau_{22} = -2y^2 + 7\ ,\ \tau_{33} = 4x + y + 3z - 5$$

$$\tau_{12} = \tau_{21} = z + 4\,x\,y - 6\ ,\ \tau_{13} = \tau_{31} = -3x + 2\,y + 1\ ,\ \tau_{23} = \tau_{32} = 0.$$

Example: **Determine whether the following stress field is admissible in an elastic body when body forces are negligible.**

$$[\tau_{ij}] = \begin{bmatrix} yz+4 & z^2+2x & 5y+z \\ . & xz+3y & 8x^3 \\ . & . & 2xyz \end{bmatrix}$$

(B) Balance of Angular momentum

The principle of balance of angular momentum for an elastic solid is –

"The time rate of change of angular momentum about the origin **is equal to** the resultant moment about of origin of body and surface forces."

This law assures the symmetry of the stress tensor τ_{ij}**.**

Let a continuous elastic body in equilibrium occupies the region τ **bounded by surface** σ**. Let** F_i **be the body force acting at a point** $P(x_i)$ **of the body. Let the position vector of the point P relative to the origin be** $\vec{r}$ =

$x_i\ \hat{e}_i$. Then , the moment of force $\vec{F}$ is $\vec{r} \times \vec{F} = \in_{ijk} x_j F_k$, where $\in_{ijk}$ is the alternating tensor.

As the elastic body is in equilibrium , the resultant moment due to body and surface forces must be zero. So ,

$$\int_\tau \in_{ijk} x_j F_k \, d\tau + \int_\sigma \in_{ijk} x_j \overset{v}{T_k} \, d\sigma = 0, \qquad (1)$$

for each i = 1, 2 , 3.

Since , the body is in equilibrium , so the Cauchy's equilibrium equations gives

$$F_k = -\tau_{lk,l}. \qquad (2a)$$

The stress vector $\overset{v}{T_k}$ in terms of stress components is given by

$$\overset{v}{T_k} = \tau_{lk} v_l \qquad (2b)$$

The Gauss – divergence theorem gives us

$$\int_\sigma \in_{ijk} x_j \tau_{lk} v_l \, d\sigma = \int_\tau [\in_{ijk} x_j \tau_{lk}]_{,l} \, d\tau$$

$$= \int_\tau \in_{ijk} [x_j \tau_{lk,l} + \delta_{jl} \tau_{lk}] \, d\tau$$

$$= \int_\tau \in_{ijk} [x_j \tau_{lk,l} + \tau_{jk}] \, d\tau. \qquad (3)$$

From equations (1) , (2a) and (3) ; we write

$$\int_\tau \in_{ijk} x_j (-\tau_{lk,l}) \, d\tau + \int_\tau \in_{ijk} [x_j \tau_{lk,l} + \tau_{jk}] \, d\tau = 0.$$

This gives

$$\int_\tau \in_{ijk} \tau_{jk} \, d\tau = 0, \qquad (4)$$

for i = 1 , 2 , 3. Since the integrand is continuous and the volume is arbitrary , so

$$\in_{ijk} \tau_{jk} = 0, \qquad (5)$$

for i = 1 , 2 , 3 and at each point of the elastic body. Expanding (5) , we write

$$\in_{123}\tau_{23} + \in_{132}\tau_{32} = 0$$

$$\Rightarrow \quad \tau_{23} - \tau_{32} = 0,$$

$$\in_{213}\tau_{13} + \in_{231}\tau_{31} = 0$$

$$\Rightarrow \quad \tau_{13} = \tau_{31}, \tag{6}$$

$$\in_{312}\tau_{12} + \in_{321}\tau_{21} = 0$$

$$\Rightarrow \quad \tau_{12} = \tau_{21}.$$

That is

$$\tau_{ij} = \tau_{ji} \qquad \text{for } i \neq j \tag{7}$$

at every point of the medium.

This proves the symmetry of stress tensor.

This law is also referred to as Cauchy's second law. **It is due to** Cauchy **in** 1827.

Note 1 : **On account of this symmetry , the state of stress at every point is specified by six instead of nine functions of position.**

Note 2 : **In summary , the six components of the state of the stress must satisfy three partial differential equations ($\tau_{ji,j} + F_i = 0$) within the body and the three relations ($\overset{v}{T_i} = \tau_{ji}\, v_j$) on the bounding surface. The equations**

$\overset{v}{T_i} = \tau_{ji}\, v_j$ **are called the boundary conditions.**

Note 3 : **Because of symmetry of the stress – tensor , the equilibrium equations may be written as**

$$\tau_{ij,j} + F_i = 0.$$

Note 4 : **Since** $\overset{i}{T_j} = \tau_{ij}$ **, equations of equilibrium (using symmetry of τ_{ij})**

may also be expressed as

$$\overset{i}{T}_{j,j} = -F_i$$

or

$$\text{div}\, \underset{\sim}{\overset{i}{T}} = -F_i.$$

Note 5 : **Because of the symmetry of τ_{ij} , the boundary conditions can be expressed as**

$$\overset{v}{T_i} = \tau_{ij}\, v_j$$

Remark : **It is obvious that the three equations of equilibrium do not suffice for the determination of the six functions that specify the stress field. This may be expressed by the statement that the stress field is statistically indeterminate. To determine the stress field , the equations of equilibrium must be supplemented by other relations that can't be obtained from static considerations.**

2.8 TRANSFORMATION OF COORDINATES

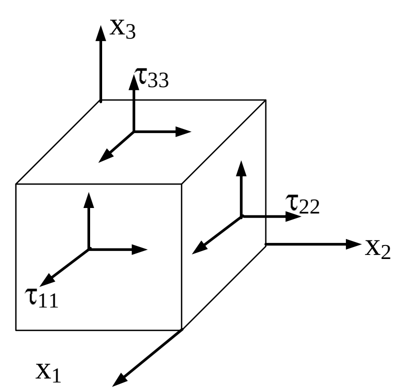
We have defined earlier the components of stress τ_{ij} with respect to cartesian system o $x_1\ x_2\ x_3$. Let $Ox_1'\ x_2'\ x_3'$ be any other cartesian system with the same origin but oriented differently. Let these coordinates be connected by the linear relations

$$x_p' = a_{pi}\, x_i \quad , \qquad (1)$$

where a_{pi} are the direction cosines of the x_p' - axis with respect to the x_i – axis. That is ,

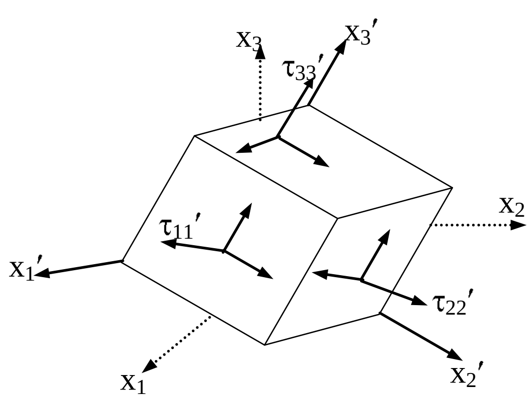
$$a_{pi} = \cos(x_p' , x_i) \ . \qquad (2)$$

Let τ'_{pq} be the components of stress in the new reference system (Fig.).



Given stresses

Fig. (2.6)



Desired stresses

Fig. (2.7) Transformation of stress components under rotation of co-ordinates system.

We shall now obtain a general formula ,in the form of the theorem given below , which enables one to compute the component in any direction $\hat{v}$ of the stress vector acting on any given element with $\hat{v}'$.

Theorem: **let the surface element $\Delta\sigma$ and $\Delta\sigma'$, with unit normals $\hat{v}$ and $\hat{v}'$, pass through the point P. Show that the component of the stress vector $\overset{\hat{v}}{\underset{\sim}{T}}$ acting on $\Delta\sigma$ in the direction of $\hat{v}'$ is equal to the component of the stress vector $\overset{\hat{v}'}{\underset{\sim}{T}}$ acting on $\Delta\sigma'$ in the direction of $\hat{v}$.**

Proof: **In this theorem , it is required to show that**

$$\overset{\hat{v}}{\underset{\sim}{T}} \cdot \hat{v}' = \overset{\hat{v}'}{\underset{\sim}{T}} \cdot \hat{v} \qquad \textbf{(3)}$$

The Cauchy's formula gives us

$$\overset{v}{T_i} = \tau_{ji}\, v_j \qquad \textbf{(4)}$$

and

$$\overset{v'}{T_i} = \tau_{ij}\, v_j', \qquad \textbf{(5)}$$

due to symmetry of stress tensors as with

$$\hat{v} = v_j \text{ and } \hat{v}' = v_j'.$$

Now $$\overset{v'}{\underset{\sim}{T}} \cdot \hat{v} = \overset{v'}{T_i}\, v_i$$

$$= (\tau_{ij}\, v_j')\, v_i$$

$$= \tau_{ji}\, v_i'\, v_j$$

$$= (\tau_{ij}\, v_j)\, v_i'$$

$$= \overset{v}{T_i}\ v_i'$$

$$= \underset{\sim}{\overset{v}{T}}\ .\ \hat{v}'. \qquad (6)$$

This completes the proof of the theorem.

Article : **Use the formula (3) to derive the formulas of transformation of the components of the stress tensor τ_{ij}.**

Solution : **Since the stress components τ'_{pq} is the projection on the x'_q – axis of the stress vector acting on a surface element normal to the x'_p – axis (by definition) , we can write**

$$\tau'_{pq} = \overset{p}{T_q} = \overset{\hat{v}'}{T}\ ,\ \hat{v} \qquad (7)$$

where

$$\left.\begin{array}{l} \hat{v}' \text{ is parallel to the } x'_p \text{ – axis ,} \\ \hat{v} \text{ is parallel to the } x'_q \text{ – axis.} \end{array}\right\} \qquad (8)$$

Equations (6) and (7) imply

$$\tau'_{pq} = \tau_{ij}\, v_i'\, v_j.. \qquad (9)$$

Since

$$\left.\begin{array}{l} v_i' = \cos(x'_p\ , x_i) = a_{pi} \\ v_j = \cos(x'_q\ , x_j) = a_{qj}. \end{array}\right\} \qquad (10)$$

Equation (9) becomes

$$\tau'_{pq} = a_{pi}\, a_{qj}\, \tau_{ij}. \qquad (11)$$

Equation (11) and definition of a tensor of order 2 show that the stress components τ_{ij} transform like a cartesian tensor of order 2. Thus , the physical concept of stress which is described by τ_{ij} agrees with the mathematical definition of a tensor of order 2 in a Euclidean space.

Theorem: **Show that the quantity**

$$\theta = \tau_{11} + \tau_{22} + \tau_{33}$$

is invariant relative to an orthogonal transformation of cartesian coordinates.

Proof: **Let τ_{ij} be the tensor relative to the cartesian system o x_1 x_2 x_3. Let these axes be transformed to o x_1' x_2' x_3' under the orthogonal transformation**

$$x'_p = a_{pi}\, x_i\,, \tag{1}$$

where

$$a_{pi} = \cos(x'_p\,,\, x_i). \tag{2}$$

Let τ'_{pq} be the stress components relative to new axes. Then these components are given by the rule for second order tensors ,

$$\tau'_{pq} = a_{pi}\, a_{qj}\, \tau_{ij}. \tag{3}$$

Putting q = p and taking summation over the common suffix , we write

$$\begin{aligned}\tau'_{pp} &= a_{pi}\, a_{pj}\, \tau_{ij}\\ &= \delta_{ij}\, \tau_{ij}\\ &= \tau_{ij}.\end{aligned}$$

This implies

$$\tau'_{11} + \tau'_{22} + \tau'_{33} = \tau_{11} + \tau_{22} + \tau_{33} = \theta \tag{4}$$

This proves the theorem.

Remark: **This theorem shows that whatever be the orientation of three mutually orthogonal planes passing through a given point , the sum of the normal stresses is independent of the orientation of these planes.**

Exercise 1 : **Prove that the tangential traction , parallel to a line l , across a plane at right angles to a line l′ , the two lines being at right angles to each other , is equal to the tangential traction , parallel to the line l′ , across a plane at right angles to l.**

Exercise 2 : **Show that the following two statements are equivalent.**

(a) The components of the stress are symmetric.

(b) Let the surface elements $\Delta\sigma$ and $\Delta\sigma'$ with respective normals $\hat{v}$ and $\hat{v}'$ pass through a point P. Then $\overset{v}{\underset{\sim}{T}} \cdot \hat{v}' = \overset{v'}{\underset{\sim}{T}} \cdot \hat{v}.$

Hint : **(b)** $\Rightarrow$ **(a). Let** $\hat{v} = \hat{i}$ **and** $\hat{v}' = \hat{j}$

Then $\underset{\sim}{\overset{v}{T}} \cdot \hat{v}' = \underset{\sim}{\overset{i}{T}} \cdot \hat{j} = \overset{i}{T_j} = \tau_{ij}$

and $\underset{\sim}{\overset{v'}{T}} \cdot \hat{v} = \underset{\sim}{\overset{j}{T}} \cdot \hat{i} = \overset{j}{T_i} = \tau_{ji}$

by assumption , $\underset{\sim}{\overset{v}{T}} \cdot \hat{v}' = \underset{\sim}{\overset{v'}{T}} \cdot \hat{v}$ **, therefore** $\tau_{ij} = \tau_{ji}$

This shows that τ_{ij} is symmetric**.**

Example 1: **The stress matrix at a point P in a material is given as**

$$[\tau_{ij}] = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 2 & -5 \\ 4 & -5 & 0 \end{bmatrix}$$

Find
(i) the stress vector on a plane element through P and parallel to the plane $2x_1 + x_2 - x_3 = 1$,
(ii) the magnitude of the stress vector , normal stress and the shear stress ,
(iii) the angle that the stress vector makes with normal to the plane.

Solution: **(i) The plane element on which the stress – vector is required is parallel to the plane $2x_1 + x_2 - x_3 = 1$. Therefore , direction ratios of the normal to the required plane at P are < 2 , 1 , -1>. So, the d.c.'s of the unit normal $\hat{v} = v_i$ to the required plane at P are**

$$v_1 = \frac{2}{\sqrt{6}}\ , v_2 = \frac{1}{\sqrt{6}}\ , v_3 = -\frac{1}{\sqrt{6}}\ .$$

Let $\underset{\sim}{\overset{v}{T}} = \overset{v}{T_i}$ **be the required stress vector. Then , Cauchy's formula gives**

$$\begin{bmatrix} \overset{v}{T_1} \\ \overset{v}{T_2} \\ \overset{v}{T_3} \end{bmatrix} \begin{bmatrix} 3 & 1 & 4 \\ 1 & 2 & -5 \\ 4 & -5 & 0 \end{bmatrix} \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix},$$

or

$$\overset{v}{T}_1 = \sqrt{3/2}\ ,\ \overset{v}{T}_2 = 3\sqrt{3/2}\ ,\ \overset{v}{T}_3 = \sqrt{3/2}\,.$$

So , the required stress vector at P is

$$\underset{\sim}{\overset{v}{T}} = \sqrt{\frac{3}{2}}(\hat{e}_1 + 3\hat{e}_2 + \hat{e}_3)$$

and

$$|\underset{\sim}{\overset{v}{T}}| = \sqrt{\frac{33}{2}}\,.$$

(ii) The normal stress σ is given by

$$\sigma = \underset{\sim}{\overset{v}{T}}\,.\,\hat{v} = \sqrt{\frac{3}{2}}\,.\,\frac{1}{\sqrt{6}}\,(2 + 3 - 1) = \frac{1}{2} \times 4 = 2,$$

and the shear stress τ is given by

$$\tau = \sqrt{|\underset{\sim}{\overset{v}{T}}|^2 - \sigma^2} = \sqrt{\frac{33}{2} - 4} = \frac{5}{\sqrt{2}}\,.$$

(As $\tau \neq 0$, so the stress vector $\underset{\sim}{\overset{v}{T}}$ need not be along the normal to the plane element)

(iii) Let θ be the angle between the stress vector $\underset{\sim}{\overset{v}{T}}$ and normal $\hat{v}$.

Then

$$\cos\theta = \frac{\underset{\sim}{\overset{v}{T}}.\hat{v}}{|\underset{\sim}{\overset{v}{T}}||\hat{v}|} = \frac{2}{\sqrt{\frac{33}{2}}} = \sqrt{\frac{8}{33}}\,.$$

This determines the required inclination.

Example 2: **The stress matrix at a point P(x_i) in a material is given by**

$$[\tau_{ij}] = \begin{bmatrix} x_3x_1 & x_3^{\ 2} & 0 \\ x_3^{\ 2} & 0 & -x_2 \\ 0 & -x_2 & 0 \end{bmatrix}.$$

Find the stress vector at the point Q (1 , 0 , -1) on the surface $x_2^2 + x_3^2 = x_1$.

Solution: **The stress vector $\overset{v}{\underset{\sim}{T}}$ is required on the surface element**

$$f(x_1 , x_2 , x_3) = x_1 - x_2^2 - x_3^2 = 0 ,$$

at the point Q(1 , 0 , -1).

We find $\nabla f = \hat{e}_1 + 2\hat{e}_3$ and $|\nabla f| = \sqrt{5}$ at the point Q.

Hence , the unit outward normal $\hat{v} = v_i$ to the surface f = 0 at the point Q(1 , 0 , –1) is

$$\hat{v} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{5}}(\hat{e}_1 + 2\hat{e}_3).$$

giving

$$v_1 = \frac{1}{\sqrt{5}} , v_2 = 0 , v_3 = \frac{2}{\sqrt{5}} .$$

The stress matrix at the point Q(1 , 0 , –1) is

$$[\tau_{ij}] = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let $\overset{v}{\underset{\sim}{T}} = \overset{v}{T_i}$ be the required stress vector at the point Q. Then , by Cauchy's law

$$\begin{bmatrix} \overset{v}{T_1} \\ \overset{v}{T_2} \\ \overset{v}{T_3} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{bmatrix}.$$

We find $\overset{v}{T_1} = \frac{-1}{\sqrt{5}} , \quad \overset{v}{T_2} = \frac{1}{\sqrt{5}} , \quad \overset{v}{T_3} = 0 .$

Hence , the required stress vector at Q is

$$\underset{\sim}{\overset{v}{T}} = \frac{1}{\sqrt{5}}(-\hat{e}_1 + \hat{e}_2).$$

Example 3: **The stress matrix at a certain point in a material is given by**

$$[\tau_{ij}] = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}.$$

Find the normal stress and the shear stress on the octahedral plane element through the point.

Solution: **An octahedral plane is a plane whose normal makes equal angles with positive directions of the coordinate axes. Hence , the components of the unit normal $\hat{v} = \hat{v}_i$ are**

$$v_1 = v_2 = v_3 = \frac{1}{\sqrt{3}}.$$

Let $\underset{\sim}{\overset{v}{T}} = \overset{v}{T_i}$ be the stress vector through the specified point. Then , Cauchy's formula gives

$$\begin{bmatrix} \overset{v}{T_1} \\ \overset{v}{T_2} \\ \overset{v}{T_3} \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix}$$

The magnitude of this stress vector is

$$|\underset{\sim}{\overset{v}{T}}| = \sqrt{\frac{43}{3}}.$$

Let σ be the normal stress and τ be the shear stress. Then

$$\sigma = \underset{\sim}{\overset{v}{T}} \cdot \hat{v} = \frac{1}{3}(5 + 3 + 3) = \frac{11}{3},$$

and

$$\tau = \sqrt{\frac{43}{3} - \frac{121}{9}} = \sqrt{\frac{129 = 121}{9}} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}.$$

Since $\sigma > 0$, the normal stress on the octahedral plane is tensile.

Example 4: **The state of stress at a point P in cartesian coordinates is given by**

$$\tau_{11} = 500 \, , \; \tau_{12} = \tau_{21} = 500 \, , \; \tau_{13} = \tau_{31} = 800 \, , \; \tau_{22} = 1000 \, ,$$

$$\tau_{33} = -300 \, , \; \tau_{23} = \tau_{32} = -750.$$

Compute the stress vector $\underset{\sim}{T}$ and the normal and tangential components on the plane passing through P whose outward normal unit vector is

$$\hat{v} = \frac{1}{2}\hat{e}_1 + \frac{1}{2}\hat{e}_2 + \frac{1}{\sqrt{2}}\hat{e}_3 .$$

Solution: **The stress vector**

$$\underset{\sim}{T} = T_i \, \hat{e}_i$$

is given by

$$T_i = \tau_{ji} \, v_j .$$

We find

$$T_1 = \tau_{11} \, v_1 + \tau_{21} \, v_2 + \tau_{31} \, v_3 = 250 + 250 + 400\sqrt{2}$$

$$= 500 + 400 \times (1.41)$$

$$= 500 + 564 = 1064, \quad \text{approx.}$$

$$T_2 = \tau_{12} \, v_1 + \tau_{22} \, v_2 + \tau_{32} \, v_3 = 250 + 250 + \frac{750}{\sqrt{2}}$$

$$= 221, \quad \text{App.}$$

$$T_3 = \tau_{13} \, v_1 + \tau_{23} \, v_2 + \tau_{33} \, v_3 = 400 - 375 - 150\sqrt{2} = 25 - 150(1.41)$$

$$= -187, \text{ app.}$$

2.9 STRESS QUADRIC

In a trirectangular cartesian coordinate system o x_1 x_2 x_3 , consider the equation

$$\tau_{ij} \, x_i \, x_j = \pm \, k^2 \qquad (1)$$

where (x_1 , x_2 , x_3) are the coordinates a point P relative to the point P^o whose coordinates relative to origin O are (x_1^o , x_2^o , x_3^o) , τ_{ij} is the stress tensor at the point $P^o(x_i^o)$, and k is a real constant **.**

The sign + or – is so chosen that the quadric surface (1) is real**.**
The quadric surface (1) is known as the stress quadric of Cauchy **with its centre at the point $P^o(x_i^o)$.**

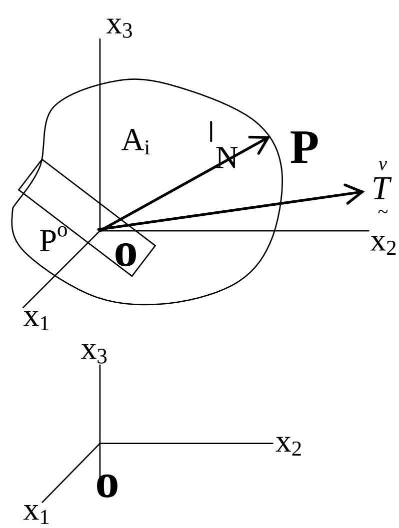


Fig. (2.8)

Let A_i be the radius vector , of magnitude A , on this stress quadric surface which is normal on the plane π through the point P^o having stress tensor τ_{ij}. Let $\hat{v}$ be the unit vector along the vector A_i. Then

$$v_i = A_i / A = x_i / A. \qquad (2)$$

Let $\underset{\sim}{\overset{v}{T}}$ denote the stress vector on the plane π at the point P^o. Then , the normal stress N on the plane π is given by

$$N = \underset{\sim}{\overset{v}{T}} . \hat{v} = \overset{v}{T_i}\ v_i = \tau_{ij}\, v_j\, v_i = \tau_{ij}\, v_i\, v_j. \qquad (3)$$

From equations (1) and (2) , we obtain

$$\tau_{ij}\,(A\, v_i)\,(A\, v_j) = \pm\, k^2$$

$$\tau_{ij}\, v_i\, v_j = \pm\, k^2 / A^2$$

$$N = \pm k^2/A^2. \tag{4}$$

This gives the normal stress acting on the plane π **with orientation** $\hat{v} = v_i$ in terms of the length of the radius vector **of the stress quadric from the point (centre) P^o along the vector v_i.**

The relation (4) shows that the normal stress N on the plane π through P^o with orientation along A_i is inversely proportional to the square of that radius vector $A_i = \overline{P^oP}$ of the stress quadric.

The positive sign in (1) or (4) is chosen whenever the normal stress N represents tension (i.e., N > 0) and negative sign when N represents compression (i.e. , N < 0).

The Cauchy's stress quadric (1) possesses another interesting property. This property is

"The normal to the quadric surface at the end of the radius vector A_i is parallel to the stress vector $\overset{v}{\underset{\sim}{T}}$ acting on the plane π at P^o."

To prove this property , let us write equation (1) in the form

$$G(x_1, x_2, x_3) \equiv \tau_{ij} x_i x_j \mp k^2 = 0. \tag{5}$$

Then the direction of the normal to the stress quadric surface is given by the gradient of the scalar point function G. The components of gradient are

$$\begin{aligned}\frac{\partial G}{\partial x_n} &= \tau_{ij}(\delta_{in}) x_j + \tau_{ij} x_i(\delta_{jn}) = 2\,\tau_{nj} x_j \\ &= 2A\,\tau_{nj} v_j \\ &= 2A\,\overset{v}{T}_n. \end{aligned} \tag{6}$$

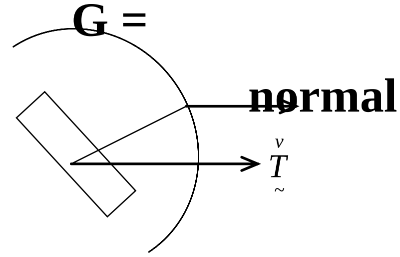


Fig. (2.9)

Equation (6) shows that vectors $\overset{v}{T}_n$ and $\frac{\partial G}{\partial x_n}$ are parallel. Hence the stress vector $\underset{\sim}{\overset{v}{T}}$ on the plane π at P^o is directed along the normal to the stress quadric at P, P being the end point of the radius vector $A_i = \overline{P^O P}$.

Remark 1: **Equation (6) can be rewritten as**

$$\underset{\sim}{\overset{v}{T}} = \frac{1}{2A}(\underline{\nabla G}). \qquad \textbf{(7)}$$

This relations gives an easy way of constructing the stress vector $\underset{\sim}{\overset{v}{T}}$ from the knowledge of the quadric surface $G(x_1, x_2, x_3) =$ constant and the magnitude A of the radius vector A_i.

Remark 2: **Taking principal axes along the coordinate axes , the stress quadric of Cauchy assumes the form**

$$\tau_1 x_1^2 + \tau_2 x_2^2 + \tau_3 x_3^2 = \pm k^2 \qquad \textbf{(8)}$$

Here the coefficients τ_1 , τ_2 , τ_3 are the principal stresses. Let the axes be so numbered that $\tau_1 \geq \tau_2 \geq \tau_3$.

If $\tau_1 > \tau_2 > \tau_3 > 0$, then equation (8) represents an ellipsoid with plus sign. Then , the relation $N = k^2/A^2$ implies that the force acting on every surface element through P^o is tensile (as $N < 0$).

If $0 > \tau_1 > \tau_2 > \tau_3$, then equation (8) represents an ellipsoid **with a negative sign on the right and $N = -k^2/A^2$ indicates that the normal stress is compressive ($N > 0$).If $\tau_1 = \tau_2 \neq \tau_3$ or $\tau_1 \neq \tau_2 = \tau_3$ or $\tau_1 = \tau_3 \neq \tau_2$, then the Cauchy's stress quadric is an** ellipsoid of revolution**.**

If $\tau_1 = \tau_2 = \tau_3$, then the stress quadric is a sphere.

2.10 PRINCIPAL STRESSES

In a general state of stress , the stress vector $\underset{\sim}{\overset{v}{T}}$ acting on a surface with outer normal $\hat{v}$ depends on the direction of $\hat{v}$.

Let us see in what direction $\hat{v}$ the stress vector $\underset{\sim}{\overset{v}{T}}$ becomes normal to the surface , on which the shearing stress is zero. Such a surface shall be called a principal plane **, its normal a** principal axis **, and the value of normal stress acting on the principal plane shall be called a** principal stress.

Let $\hat{v}$ define a principal axis at the point $P^o(x_i^o)$ and let τ be the corresponding principal stress and τ_{ij} be the stress tensor at that point. Let $\overset{v}{\underset{\sim}{T}}$ be the stress vector. Then

$$\overset{v}{\underset{\sim}{T}} = \tau\, \hat{v},$$

or

$$\overset{v}{T_i} = \tau\, v_i. \qquad (1)$$

Also $$\overset{v}{T_i} = \tau_{ij}\, v_j \qquad (2)$$

Therefore $$\tau_{ij}\, v_j = \tau v_i = \tau\, \delta_{ij}\, v_j$$

or $$(\tau_{ij} - \tau\, \delta_{ij})\, v_j = 0. \qquad (3)$$

The three equations , i = 1 , 2 , 3 , are to be solved for v_1 , v_2 , v_3. Since $\hat{v}$ is a unit vector , we must find a set of non – trivial solutions for which

$$v_1^2 + v_2^2 + v_3^2 = 1 .$$

Thus , equation (3) poses an eigenvalue problem. Equation (3) has a set of non – vanishing solutions v_1 , v_2 , v_3 iff the determinant of the coefficients vanishes , i.e. ,

$$|\, \tau_{ij} - \tau\, \delta_{ij} \,| = 0,$$

or

$$\begin{bmatrix} \tau_{11} - \tau & \tau_{12} & \tau_{13} \\ \tau_{12} & \tau_{22} - \tau & \tau_{23} \\ \tau_{13} & \tau_{23} & \tau_{33} - \tau \end{bmatrix} = 0. \qquad (3a)$$

On expanding (2) , we find

$$-\tau^3 + \theta_1\, \tau^2 - \theta_2\, \tau + \theta_3 = 0, \qquad (3b)$$

where

$$\theta_1 = \tau_{11} + \tau_{22} + \tau_{33}$$
(4a)

$$\theta_2 = \begin{vmatrix} \tau_{22} & \tau_{23} \\ \tau_{23} & \tau_{33} \end{vmatrix} + \begin{vmatrix} \tau_{11} & \tau_{13} \\ \tau_{31} & \tau_{33} \end{vmatrix} + \begin{vmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{vmatrix}, \qquad (4b)$$

$$\theta_3 = \in_{ijk}\, \tau_{1i}\, \tau_{2j}\, \tau_{3k} = \det.(\tau_{ij}). \tag{4c}$$

Equation (3) is a cubic equation in τ. Let its roots be τ_1 , τ_2 , τ_3. Since the matrix of stress , (τ_{ij}) is real and symmetric , the roots τ_i of (3) are all real. Thus , τ_1 , τ_2 , τ_3 are the principal stresses.

For each value of the principal stress , a unit normal vector $\hat{v}$ can be determined.

Case I : When $\tau_1 \neq \tau_2 \neq \tau_3$, **let $\overset{1}{v_i}$, $\overset{2}{v_i}$, $\overset{3}{v_i}$ be the unit principal axes corresponding to the principal stresses τ_1 , τ_2 , τ_3 , respectively. Then principal axes are mutually orthogonal to each other.**

Case II : **If $\tau_1 = \tau_2 \neq \tau_3$ are the principal stresses , then the direction $\overset{3}{v_i}$ corresponding to principal stress τ_3 is a principal direction and any two mutually perpendicular lines in a plane with normal $\overset{3}{v_i}$ may be chosen as the other two principal direction of stress.**

Case III : **If $\tau_1 = \tau_2 = \tau_3$, then any set of orthogonal axes through P^o may be taken as the principal axes.**

Remark : **Thus , for a symmetric real stress tensor τ_{ij} , there are three principal stresses which are real and a set of three mutually orthogonal principal directions.**

If the reference axes x_1 , x_2 , x_3 are chosen to coincide with the principal axes , then the matrix of stress components becomes

$$[\tau_{ij}] = \begin{bmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{bmatrix}. \tag{5}$$

Invariants of the stress – tensor :

Equation (3) can be written as

$$(\tau - \tau_1)\,(\tau - \tau_2)\,(\tau - \tau_3) = 0, \tag{6}$$

and we find

$$\left.\begin{aligned} \theta_1 &= \tau_1 + \tau_2 + \tau_3 \\ \theta_2 &= \tau_1\tau_2 + \tau_2\tau_3 + \tau_3\tau_1 \\ & \end{aligned}\right\} .$$

$$\theta_3 = \tau_1 \tau_2 \tau_3 \qquad (7)$$

Since the principal stress τ_1 , τ_2 , τ_3 characterize the physical state of stress at point , they are independent of any coordinates of reference.

Hence , coefficients θ_1 , θ_2 , θ_3 of equation (3) are invariant w.r.t. the coordinate transformation. Thus θ_1 , θ_2 , θ_3 are the three scalar invariants of the stress tensor τ_{ij}.

These scalar invariants are called the fundamental stress invariants.

Components of stress τ_{ij} in terms of τ_α's

Let X_α be the principal axes. The transformation law for axes is

$$X_\alpha = a_{i\alpha}\, x_i$$

or

$$x_i = a_{i\alpha}\, X_\alpha , \qquad (8)$$

where

$$a_{i\alpha} = \cos(x_i , X_\alpha). \qquad (9)$$

The stress – matrix relative to axes X_α is

$$\tau'_{\alpha\beta} = \mathrm{diag}(\tau_1 , \tau_2 , \tau_3). \qquad (10)$$

Let τ_{ij} be the stress – matrix relative to x_i – axis. Then transformation rule for second order tensor is

$$\tau_{ij} = a_{i\alpha}\, a_{i\beta}\, \tau'_{\alpha\beta}$$

$$= \sum_{\alpha=1}^{3} a_{i\alpha}\, (a_{j\alpha}\, \tau_\alpha).$$

This gives

$$\tau_{ij} = \sum_{\alpha=1}^{3} a_{i\alpha}\, a_{j\alpha}\, \tau_\alpha. \qquad (11)$$

Definition (Principal axes of Stress)

A system of coordinate axes chosen along the principal directions of stress is referred to as principal axes of stress.

Question: **Show that , as the orientation of a surface element at a point P varies , the normal stress on the surface element assumes an extreme value**

when the element is a principal plane of stress at P and that this extremum value is a principal stress.

Solution: **Let τ_{ij} be the stress tensor at the point P. Let τ be the normal stress on a surface element at P having normal in the direction of unit vector $\hat{v} = v_i$. Then , we know that**

$$\tau = \tau_{ij} v_i v_j \qquad (1)$$

we have to find $\hat{v} = v_i$ for which τ is an extremum. Since $\hat{v} = v_i$ is a unit vector , we have the restriction

$$v_k v_k - 1 = 0 \qquad (2)$$

We use the method of lagrange's multiplier to find the extremum values of τ. The extreme values are given by

$$\frac{\partial}{\partial v_i}\{\tau_{ij} v_i v_j - \lambda(v_k v_k - 1)\} = 0 \qquad (3)$$

where λ is a Lagrange's multiplier. From (3) , we find

$$\tau_{ij}\{v_j + \delta_{ij} v_i\} - \lambda\{2v_k \delta_{ik}\} = 0$$

$$\Rightarrow \quad 2\tau_{ij} v_j - 2\lambda v_i = 0$$

$$\Rightarrow \quad \tau_{ij} v_j - \lambda \delta_{ij} v_j = 0$$

$$\Rightarrow \quad (\tau_{ij} - \lambda \delta_{ij}) v_j = 0 . \qquad (4)$$

These conditions are satisfied iff $\hat{v} = v_j$ is a principal direction of stress and $\tau = \lambda$ is the corresponding principal stress.

Thus , τ assumes an extreme value on a principal plane of stress and a principal stress is an extreme value of τ given by (1).

2.11 MAXIMUM NORMAL AND SHEAR STRESSES

Let the co-ordinate axes at a point P^o be taken along the principle directions of stress. Let τ_1 , τ_2 , τ_3 be the principal stresses as P^o. Then

$$\tau_{11} = \tau_1 , \tau_{22} = \tau_2 , \tau_{33} = \tau_3 ,$$

$$\tau_{12} = \tau_{23} = \tau_{31} = 0.$$

Let $\overset{v}{\underset{\sim}{T}}$ be the stress vector on a planar element at P^o having the normal $\hat{v} = v_i$. Let N be the normal stress and S be the shearing stress. Then

$$|\overset{v}{\underset{\sim}{T}}|^2 = N^2 + S^2. \qquad (1)$$

The relation

$$\overset{v}{T_i} = \tau_{ij} v_j$$

gives

$$\overset{v}{T_1} = \tau_1 v_1 , \ \overset{v}{T_2} = \tau_2 v_2 = \overset{v}{T_3} = \tau_3 v_3 , \qquad (1a)$$

so that

$$N = \overset{v}{\underset{\sim}{T}} \ . \ \hat{v} = \overset{v}{T_i} \ v_i = \tau_1 v_1^2 + \tau_2 v_2^2 + \tau_3^2 \ . \qquad (1b)$$

N is a function of three variables v_1 , v_2 , v_3 connected by the relation

$$v_k v_k - 1 = 0. \qquad (2)$$

From (1) & (2) , we write

$$N = \tau_1(1 - v_2^2 - v_3^2) + \tau_2 v_2^2 + \tau_3 v_3^2$$

$$= \tau_1 + (\tau_2 - \tau_1)v_2^2 + (\tau_3 - \tau_1) v_3^2 \qquad (3)$$

The extreme value of N are given by

$$\frac{\partial N}{\partial v_2} = 0 \quad , \quad \frac{\partial N}{\partial v_3} = 0 ,$$

which yield

$$v_2 = 0 , v_3 = 0 \text{ for } \tau_2 \neq \tau_1 \ \& \ \tau_3 \neq \tau_1.$$

Hence

$$v_1 = \pm 1 , v_2 = v_3 = 0 \ \& \ N = \tau_1 .$$

Similarly , we can find other two directions

$$v_1 = 0 , v_2 = \pm 1 , v_1 = 0 , N = \tau_2$$

$$v_1 = 0 , v_2 = 0 , v_3 = \pm 1 , N = \tau_3$$

Thus , we find that the extreme values of the Normal stress N are along the principal directions of stress and the extreme values are themselves principal stresses. So , the absolute maximum normal stress is the maximum of the set {τ_1 , τ_2 , τ_3}. Along the principal directions , the shearing stress is zero (i.e. , the minimum).

Now $S^2 = (\tau_1^2 v_1^2 + \tau_2^2 v_2^2 + \tau_3^2 v_3^2) - (\tau_1 v_1^2 + \tau_2 v_2^2 + \tau_3 v_3^2)^2$ **(3a)**

To determine the directions associated with the maximum values of |S|. We maximize the function S(v_1 , v_2 , v_3) in (3) subject to the constraining relation $v_i v_i = 1$.

For this , we use the method of Lagrange multipliers to find the free extremum of the functions

$$F(v_1, v_2, v_3) = S^2 - \lambda(v_i v_i - 1) \qquad (4)$$

For extreme values of F. We must have

$$\frac{\partial F}{\partial v_1} = \frac{\partial F}{\partial v_2} = \frac{\partial F}{\partial v_3} = 0. \qquad (5)$$

The equation $\frac{\partial F}{\partial v_1} = 0$ **gives**

$$2\tau_1^2 v_1 - 4\tau_1 v_1(\tau_1 v_1^2 + \tau_2 v_2^2 + \tau_3 v_3^2) - 2\lambda v_1 = 0$$

or $$\lambda = \tau_1^2 - 2\tau_1(\tau_1 v_1^2 + \tau_2 v_2^2 + \tau_3 v_3^2). \qquad (6)$$

Similarly from other equations ,we obtain

$$\lambda = \tau_2^2 - 2\tau_2(\tau_1 v_1^2 + \tau_2 v_2^2 + \tau_3 v_3^2), \qquad (7)$$

$$\lambda = \tau_3^2 - 2\tau_3(\tau_1 v_1^2 + \tau_2 v_2^2 + \tau_3 v_3^2). \qquad (8)$$

Equations (6) & (7) yield

$$\tau_2^2 - \tau_1^2 = 2(\tau_2 - \tau_1)(\tau_1 v_1^2 + \tau_2 v_2^2 + \tau_3 v_3^2)$$

For $\tau_1 \neq \tau_2$, this leads to

$$\tau_2 + \tau_1 = 2(\tau_1 v_1^2 + \tau_2 v_2^2 + \tau_3 v_3^2)$$

or $$(2v_1^2 - 1)\tau_1 + (2v_2^2 - 1)\tau_2 + 2v_3^2\tau_3 = 0.$$

This relation is identically satisfied if

$$v_1 = \pm \frac{1}{\sqrt{2}}\ ,\ v_2 = \frac{1}{\sqrt{2}}\ ,\ v_3 = 0. \qquad (9)$$

From equations (1b) , (3a) and (9) , the corresponding maximum value of | S | is

$$\left.\begin{aligned} |\,S\,|_{max} &= \frac{1}{2}\,|\,\tau_2 - \tau_1\,| \\ \text{and} \qquad |\,N\,| &= \frac{1}{2}\,|\,\tau_1 + \tau_2\,| \end{aligned}\right\} \qquad (10)$$

For this direction

$$S^2 = \frac{1}{2}(\tau_1^2 + \tau_2^2) - \left(\frac{\tau_1 + \tau_2}{2}\right)^2$$

$$= \frac{1}{4}[2\tau_1^2 + 2\tau_2^2 - (\tau_1^2 + \tau_2^2 + 2\tau_1\,\tau_2)]$$

$$= \frac{1}{4}(\tau_1^2 + \tau_2^2 - 2\tau_1\,\tau_2)\ .$$

This implies

$$|\,S\,|_{max} = \frac{1}{2}\,|\,\tau_1 - \tau_2\,|.$$

Similarly , for the directions

$$v_1 = \pm\frac{1}{\sqrt{2}}\ ,\ v_2 = 0\ ,\ v_3 = \pm\frac{1}{\sqrt{2}}\ ,$$

we have

$$|\,S\,|_{max} = \frac{1}{2}\,|\,\tau_3 - \tau_1\,|\ ,$$

$$|\,N\,| = \frac{1}{2}\,|\,\tau_3 + \tau_1\,|.$$

Also, for the direction

$$v_1 = 0\ ,\ v_2 = \pm\frac{1}{\sqrt{2}}\ ,\ v_3 = \pm\frac{1}{\sqrt{2}}\ ,$$

the corresponding values of $| S |_{max}$ and $| N |$ are , respectively ,

$$\frac{1}{2} | \tau_2 - \tau_3 | \text{ and } \frac{1}{2} | \tau_2 + \tau_3 |.$$

These results can recorded in the following table

v_1	V_2	v_3	$\|S\|_{max/min}$	$\| N \|$
0	**0**	± 1	**Min S = 0**	$\| \tau_3 \|$ = **Max.**
0	± 1	**0**	**0 (Min.)**	$\| \tau_2 \|$ = **Max.**
± 1	**0**	**0**	**0(Min.)**	$\| \tau_1 \|$ = **Max.**
0	$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$	$\frac{1}{2} \| \tau_2 - \tau_3 \|$ **Max.**	$\frac{1}{2} \| \tau_2 + \tau_3 \|$ **Min.**
$\pm\frac{1}{\sqrt{2}}$	**0**	$\pm\frac{1}{\sqrt{2}}$	$\frac{1}{2} \| \tau_3 - \tau_1 \|$ **Max.**	$\frac{1}{2} \| \tau_3 + \tau_1 \|$ **Min.**
$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$	**0**	$\frac{1}{2} \| \tau_1 - \tau_2 \|$ **Max.**	$\frac{1}{2} \| \tau_1 + \tau_2 \|$ **Min.**

If $\tau_1 > \tau_2 > \tau_3$, then τ_1 is the absolute maximum **values of N and** τ_3 **is its** minimum **value, and the maximum value of | S | is**

$$| S |_{max} = \frac{1}{2} (\tau_3 - \tau_1).$$

and the maximum shearing stress acts on the surface element containing the x_2 principal axis and bisecting the angle between the x_1 – and x_3 – axes. Hence the following theorem is proved.

Theorem : **Show that the maximum shearing stress is equal to** one – half **the difference between the greatest and least normal stress and acts on the plane that bisects the angle between the directions of the largest and smallest principal stresses.**

2.12 MOHR'S CIRCLE (GEOMETRICAL PROOF OF THE THEOREM AS PROPOSED BY O.(OTTO) MOHR(1882))

We know that

$$\mathbf{N} = \tau_1 v_1^2 + \tau_2 v_2^2 + \tau_3 v_3^2 \qquad \textbf{(1)}$$

and $$S^2 + N^2 = \tau_1^2 v_1^2 + \tau_2^2 v_2^2 + \tau_3^2 v_3^2 . \qquad \textbf{(2)}$$

Also $$v_1^2 + v_2^2 + v_3^2 = 1 . \qquad \textbf{(3)}$$

Solving equations (1) – (3) , by Cramer's rule , for v_1^2 , v_2^2 , v_3^2 ; we find

$$v_1^2 = \frac{S^2 + (N-\tau_2)(N-\tau_3)}{(\tau_1-\tau_2)(\tau_1-\tau_3)} , \qquad \textbf{(4)}$$

$$v_2^2 = \frac{S^2 + (N-\tau_3)(N-\tau_1)}{(\tau_2-\tau_1)(\tau_2-\tau_3)} , \qquad \textbf{(5)}$$

$$v_3^2 = \frac{S^2 + (N-\tau_1)(N-\tau_2)}{(\tau_3-\tau_1)(\tau_3-\tau_2)} , \qquad \textbf{(6)}$$

Assume that $\tau_1 > \tau_2 > \tau_3$ so that $\tau_1 - \tau_2 > 0$ and $\tau_1 - \tau_3 > 0$. Since v_1^2 is non – negative. We conclude from equation (4) that

$$S^2 + (N - \tau_2)(N - \tau_3) \geq 0.$$

or $$S^2 + N^2 - N(\tau_2 + \tau_3) + \tau_2 \tau_3 \geq 0$$

or $$S^2 + \left(N - \frac{\tau_2+\tau_3}{2}\right)^2 \geq \left(\frac{\tau_2-\tau_3}{2}\right)^2 . \qquad \textbf{(7)}$$

This represents a region outside the circle

$$S^2 + \left(N - \frac{\tau_2+\tau_3}{2}\right)^2 = \left(\frac{\tau_2-\tau_3}{2}\right)^2 ,$$

in the (N , S) plane.

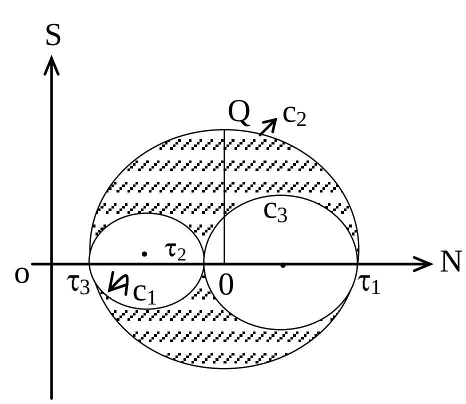


Fig. (2.10) Mohr's Circles

This circle , say C_1 , has centre $\left(\frac{\tau_2+\tau_3}{2},0\right)$ and radius $\frac{\tau_2-\tau_3}{2}$ in the cartesian SN-plane with the values of N as abscissas and those of S as ordinates.

Since $\tau_2 - \tau_3 > 0$ and $\tau_2 - \tau_1 < 0$, we conclude from (5) that

$$S^2 + (N - \tau_3)(N - \tau_1) \le 0 . \qquad (8)$$

Thus , the region defined by (8) is a closed region , interior to the circle c_2 , whose equation is

$$S^2 + (N - \tau_3)(N - \tau_1) = 0 . \qquad (8a)$$

The circle C_2 passed through the points $(\tau_3 , 0)$, $(\tau_1 , 0)$ and have centre on the N – axis.

Finally , equation (6) yields

$$S^2 + (N - \tau_1)(N - \tau_2) \ge 0 , \qquad (9)$$

since

$$\tau_3 - \tau_1 < 0 \text{ and } \tau_3 - \tau_2 < 0.$$

The region defined by (9) is exterior to the circle c_3 , with centre on the N –axis and passing through the points $(\tau_1 , 0)$, $(\tau_2 , 0)$.

It follows from inequalities (7) to (9) that the admissible values of S and N lie in the shaded region bounded by the circles as shown in the figure.

From figure , it is clear that the maximum value of shearing stress S is represented by the greatest ordinate O^1Q of the circle C_2.

Hence

$$S_{max} = \frac{\tau_1-\tau_3}{2} \qquad (10a)$$

The value of N , corresponding to S_{max} is OO′ where

$$OO' = \tau_3 + \frac{\tau_1-\tau_3}{2} = \frac{\tau_1+\tau_3}{2} \qquad (10b)$$

Putting the values of S & N from equations (10a , b) into equations (4) – (6). We find

$$v_1^2 = v_3^2 = \frac{1}{2}, \; v_2^2 = 0.$$

or

$$v_1 = \pm\frac{1}{\sqrt{2}}, \; v_3 = \pm\frac{1}{\sqrt{2}}, \; v_2 = 0. \qquad (11)$$

Equation (11) determines the direction of the maximum shearing stress. **Equation (11) shows that the maximum shearing stress acts on the plane that bisects the angle between the directions of the largest and smallest principal stresses.**

2.13 OCTAHEDRAL STRESSES

Consider a plane which is equally inclined to the principal directions of stress. Stresses acting on such a plane are known as octahedral stresses. Assume that coordinate axes coincide with the principal directions of stress. Let τ_1 , τ_2 , τ_3 be the principal stresses. Then the stress matrix is

$$\begin{bmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{bmatrix}.$$

A unit normal $\hat{v} = v_i$ to this plane is

$$v_1 = v_2 = v_3 = \frac{1}{\sqrt{3}}.$$

Then the stress vector $\overset{v}{\underset{\sim}{T}}$ on a plane element with normal $\hat{v}$ is given by

$$\overset{v}{T_i} = \tau_{ij} v_j .$$

This gives

$$\overset{v}{T_1} = \tau_1 v_1, \; \overset{v}{T_2} = \tau_2 v_2, \; \overset{v}{T_3} = \tau_3 v_3 .$$

Let N be the normal stress and S be the shear stress. Then

$$N = \overset{v}{\underset{\sim}{T}} \cdot \hat{v} = \tau_1 v_1^2 + \tau_2 v_2^2 + \tau_3 v_3^2 = \frac{1}{3}(\tau_1 + \tau_2 + \tau_3),$$

and

$$\mathbf{S^2} = |\overset{\nu}{\underset{\sim}{T}}|^2 - \mathbf{N}^2$$

$$= (\tau_1^2 \nu_1^2 + \tau_2^2 \nu_2^2 + \tau_3^2 \nu_3^2) - \frac{1}{9}(\tau_1 + \tau_2 + \tau_3)^2$$

$$= \frac{1}{3}(\tau_1^2 + \tau_2^2 + \tau_3^2) - \frac{1}{9}(\tau_1 + \tau_2 + \tau_3)^2$$

$$= \frac{1}{9}[3(\tau_1^2 + \tau_2^2 + \tau_3^2) - (\tau_1^2 + \tau_2^2 + \tau_3^2 + 2\,\tau_1\tau_2 + \tau_2\tau_3 + \tau_3\tau_1)]$$

$$= \frac{1}{9}[(\tau_1^2 + \tau_2^2 - 2\tau_1\tau_2) + (\tau_2^2 + \tau_3^2 - 2\tau_2\tau_3) + (\tau_3^2 + \tau_1^2 - 2\tau_1\tau_3)]$$

$$= \frac{1}{9}[(\tau_1 - \tau_2)^2 + (\tau_2 - \tau_3)^2 + (\tau_3 - \tau_1)^2],$$

giving

$$\mathbf{S} = \frac{1}{3}\sqrt{(\tau_1 - \tau_2)^2 + (\tau_2 - \tau_3)^2 + (\tau_3 - \tau_1)^2}\ .$$

Example : **At a point P , the principal stresses are $\tau_1 = 4$, $\tau_2 = 1$, $\tau_3 = -2$. Find the stress vector , the normal stress and the shear stress on the octahedral plane at P.**

[Hint : **$N = 1$, $S = \sqrt{6}$, $\overset{\hat{\nu}}{\underset{\sim}{T}} = \frac{1}{\sqrt{3}}(4\ \hat{e}_1 + \hat{e}_2 - 2\hat{e}_3)$.]**

2.14. STRESS DEVIATOR TENSOR

Let τ_{ij} be the stress tensor. Let

$$\sigma_0 = \frac{1}{3}(\tau_{11} + \tau_{22} + \tau_{33}) = \frac{1}{3}(\tau_1 + \tau_2 + \tau_3)$$

Then the tensor

$$\tau^{(d)}_{ij} = \tau_{ij} - \sigma_0\,\delta_{ij}$$

is called the stress deviator tensor. It specifies the deviation of the state of stress from the mean stress σ_0.

Chapter-3
Analysis of Strain

3.1 INTRODUCTION

Rigid Body

A rigid body is an ideal body such that the **distance between every pair of its points remains unchanged under the action of external forces.**

The possible displacements in a **rigid body** are **translation and rotation**. These displacements are called rigid displacements. In translation , each point of the rigid body moves a fixed distance in a fixed direction. In rotation about a line , every point of the body (rigid) moves in a circular path about the line in a plane perpendicular to the line.

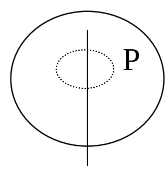


Fig. (3.1)

In a rigid body motion , there is a uniform motion throughout the body.

Elastic Body

A body is called **elastic** if it possesses the property of **recovering its original shape and size when the forces causing deformation are removed**.

Continuous Body

In a continuous body , the atomistic structure of matter can be **disregarded** and the body is replaced by a **continuous mathematical region of the space** whose geometrical points are **identified with material points** of the body.

The **mechanics of such continuous elastic bodies** is called mechanics of continuum. This branch covers a vast range of problems of **elasticity , hydromechanics , aerodynamics , plasticity ,** and **electrodynamics , seismology** , etc.

Deformation of Elastic Bodies

The change in the relative position of points in a continuous is called deformation **,** and the body itself is then called a strained body. The study of deformation of an elastic body is known as the **analysis of strain.** The deformation of the body is due to relative movements or distortions within the body.

3.2 TRANSFORMATION OF AN ELASTIC BODY

We consider the undeformed and deformed both positions of an elastic body. Let $Ox_1x_2x_3$ be mutually orthogonal cartesian coordinates fixed in space. Let a continuous body B , referred to system $Ox_1x_2x_3$, occupies the region R in the undeformed state. In the deformed state, the points of the body B will occupy some region ,say R′.

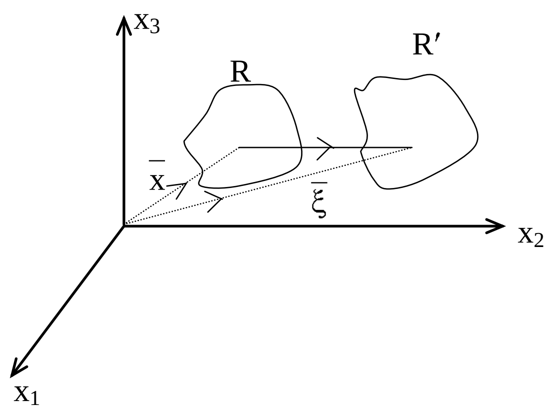


Fig. (3.2)

Let $P(x_1 , x_2 , x_3)$ be the coordinates of a material point P of the elastic body in the initial or unstained state. In the transformation or deformed state , let this material point occupies the geometric point $P'(\xi_1 , \xi_2 , \xi_3)$. We shall be concerned only with continuous deformations of the body from region R into the region R′ and we assume that the deformation is given by the equation

$$\xi_1 = \xi_1(x_1 , x_2 , x_3) ,$$
$$\xi_2 = \xi_2(x_1 , x_2 , x_3) ,$$
$$\xi_3 = \xi_3(x_1 , x_2 , x_3) . \tag{1}$$

The **vector** $\overline{PP'}$ **is called the displacement vector** of the point P and is denoted by u_i.

Thus

$$u_i = \xi_i - x_i \quad ; \quad i = 1 , 2 , 3 \tag{2}$$

or

$$\xi_i = x_i + u_i . \qquad i = 1 , 2 , 3 \qquad (3)$$

Equation (1) expresses the coordinates of the points of the body in the transformed state in terms of their coordinates in the initial undeformed state. This type of description of deformation is known as the **Lagrangian method** of describing the transformation of a continuous medium.

Another method , known as **Euler's method** expresses the coordinates in the undeformed state in terms of the coordinates in the deformed state.

The transformation (1) is invertible when

$J \neq 0$.

Then , we may write

$$x_i = x_i(\xi_1 , \xi_2 , \xi_3) \quad , \qquad i = 1 , 2 , 3. \qquad (4)$$

In this case , the transformation from the region R into region R′ is one – to – one.

Each of the above description of deformation of the body has its own advantages. It is however, more convenient in the study of the mechanics of solids to use **Lagrangian approach because the undeformed state of the body often possesses certain symmetries which make it convenient to use a simple system of coordinates.**

A part of the transformation defined by equation (1) may represent rigid body motions

(i.e. , translations and rotations) of the body as a whole. This part of the deformation leaves unchanged the length of every vector joining a pair of points within the body and is of no interest in the analysis of strain.

The remaining part of transformation (1) will be called **pure deformation.**

Now , we shall learn how to distinguish between pure deformation and rigid body motions when the latter are present in the transformation equations (1).

3.3 LINEAR TRANSFORMATION OR AFFINE TRANSFORMATION

Definition: The transformation

$$\xi_i = \xi_i(x_1 , x_2 , x_3)$$

is called a linear transformation or affine transformation when the functions ξ_i are **linear functions** of the coordinates x_1 , x_2 , x_3.

In order to distinguish between rigid motion and pure deformation , we consider the simple case in which the transformation (1) is linear.

We assume that the general form of the linear transformation (1) is of the type

$$\left.\begin{aligned} \xi_1 &= \alpha_{10} + (\alpha_{11}+1)\, x_1 + \alpha_{12}\, x_2 + \alpha_{13}\, x_3 , \\ \xi_2 &= \alpha_{20} + \alpha_{21}\, x_1 + (1+\alpha_{22})\, x_2 + \alpha_{23}\, x_3 , \\ \xi_3 &= \alpha_{30} + \alpha_{31}\, x_1 + \alpha_{32}\, x_2 + (1+\alpha_{33})\, x_3 , \end{aligned}\right\} \qquad (5)$$

or

$$\xi_i = \alpha_{i0} + (\alpha_{ij} + \delta_{ij})\, x_j , \qquad (6)$$

where the coefficients α_{ij} are **constants** and are well known.

Equation (5) can written in the matrix form as

$$\begin{bmatrix} \xi_1-\alpha_{10} \\ \xi_2-\alpha_{20} \\ \xi_3-\alpha_{30} \end{bmatrix} = \begin{bmatrix} 1+\alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & 1+\alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & 1+\alpha_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} , \qquad (7)$$

or

$$\begin{bmatrix} u_1-\alpha_{10} \\ u_2-\alpha_{20} \\ u_3-\alpha_{30} \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \qquad (8)$$

We can look upon the matrix $(\alpha_{ij} + \delta_{ij})$ as an operator acting on the vector $\bar{x} = x_i$ to give the vector α_{i0}.

If the matrix $(\alpha_{ij} + \delta_{ij})$ is non – singular , then we obtain

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (\alpha_{ij} + \delta_{ij})^{-1} \begin{bmatrix} \xi_1-\alpha_{10} \\ \xi_2-\alpha_{20} \\ \xi_3-\alpha_{30} \end{bmatrix} , \qquad (9)$$

which is also linear as inverse of a linear transformation is linear.

Infact, matrix algebra was developed basically to express linear transformations in a concise and lucid manner.

Result (1) : Sum of two linear transformations is a linear transformation.

Result (2) : Product of two linear transformations is a linear transformation which is not commutative.

Result (3) : Under a linear transformation , a plane is transformed into a plane.

Proof of (3) : Let

$$a x_1 + b x_2 + c x_3 + d = 0 ,$$

be an equation of a plane in the undeformed state. Let

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} ,$$

be the linear transformation of points. Let its inverse be

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} .$$

Then the equation of the plane is transformed to

$$a(A_1 \xi_1 + B_1 \xi_2 + C_1 \xi_3) + b(A_2\xi_1 + B_2 \xi_2 + C_2 \xi_3)$$
$$+ c(A_3 \xi_1 + B_3 \xi_2 + C_3 \xi_3) + d = 0$$

or

$$\alpha_1 \xi_1 + \beta_1 \xi_2 + \gamma_1 \xi_3 + d = 0 ,$$

which is again an equation of a plane in terms of new coordinates (ξ_1 , ξ_2 , ξ_3).

Hence the result.

Result (4) : A linear transformation carries line segments into line segments.

Thus , it is the linear transformation that allows us to assume that a line segment is transformed to a line segment and not to a curve.

3.4 SMALL/ INFINITESIMAL LINEAR DEFORMATIONS

Definition (Small / Infinitesimal Deformations)

A linear transformation of the type

$$\xi_i = \alpha_{i0} + (\alpha_{ij} + \delta_{ij})\, x_j$$

is said to be a small linear transformation of the coefficients α_{ij} are so small that their products can be neglected in comparison with the linear terms.

Result (1) : The product of two small linear transformations is a small linear transformation which is **COMMUTATIVE** and the product transformation is obtained by superposition of the original transformations and the result is independent of the order in which the transformations are performed.

Result (2) : In the study of finite deformations (as compared to the infinitesimal affine deformation), the principle of superposition of effects and the independence of the order of transformations are no longer valid.

Note : If a body is subjected to large linear transformations , a straight line element seldom remains straight. A curved element is more likely to result. The linear transformation then expresses the transformation of element $P_1\, P_2$ to the tangent $P_1'\, T_1'$ to the curve at P_1' for the curve itself.

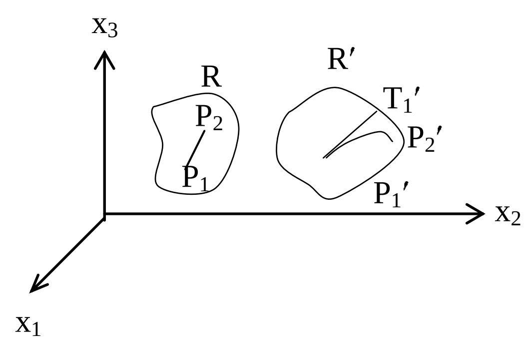


Fig. (3.3)

For this reason , a linear transformation is sometimes called **linear tangent transformation.**

It is obvious that the smaller the element $P_1\, P_2$, the better approximation of $P_1'P_2'$ by its tangent $P_1'T_1'$.

3.5 HOMOGENEOUS DEFORMATION

Suppose that a body B , occupying the region R in the undeformed state , is transformed to the region R′ under the linear transformation.

$$\xi_i = \alpha_{i0} + (\alpha_{ij} + \delta_{ij})\, x_j \qquad (1)$$

referred to orthogonal cartesian system $Ox_1x_2x_3$. Let $\hat{e}_1, \hat{e}_2, \hat{e}_3$ be the unit base vectors directed along the coordinate axes x_1 , x_2 , x_3 (Fig. 3.4)

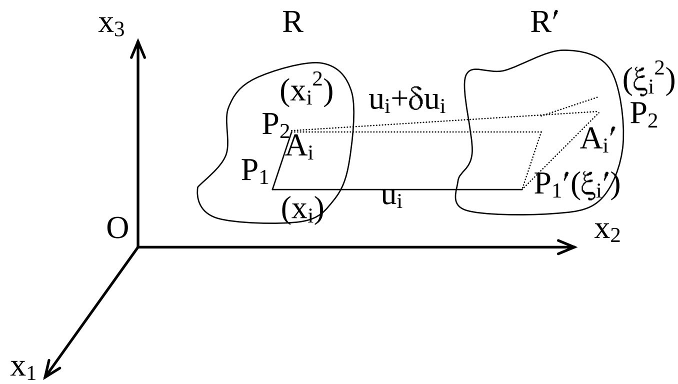


Fig. (3.5)

Let $P_1(\overset{1}{x}_1,\overset{1}{x}_2,\overset{1}{x}_3)$ and $P_2(\overset{2}{x}_1,\overset{2}{x}_2,\overset{2}{x}_3)$ be two points of the elastic body in the initial state. Let the positions of these points in the deformed state , due to linear transformation (1) , be $P_1'(\overset{1}{\xi}_1,\overset{1}{\xi}_2,\overset{1}{\xi}_3)$ and $P_2'(\overset{2}{\xi}_1,\overset{2}{\xi}_2,\overset{2}{\xi}_3)$.

Since the transformation (1) is linear , so the line segment $\overline{P_1P_2}$ is transformed into a line segment $\overline{P_1'P_2'}$.

Let the vector $\overline{P_1P_2}$ has components A_i and vector $\overline{P_1'P_2'}$ has components A_i'. Then

$$\overline{P_1P_2} = A_i\,\hat{e}_i \quad , \qquad A_i = \overset{2}{x}_i - \overset{1}{x}_i \, , \tag{2}$$

and

$$\overline{P_1'P_2'} = A_i'\,\hat{e}_i \quad , \qquad A_i' = \overset{2}{\xi}_i - \overset{1}{\xi}_i \, . \tag{3}$$

Let $$\delta A_i = A_i' - A_i \, , \tag{4}$$

be the change in vector A_i .

The vectors A_i and A_i' , in general , differ in direction and magnitude.

From equations (1) , (2) and (3) , we write

$$A_i' = \overset{2}{\xi}_i - \overset{1}{\xi}_i$$

$$= [\alpha_{i0} + (\alpha_{ij} + \delta_{ij})\ \overset{2}{x}_j\ ] - [\alpha_{i0} + (\alpha_{ij} + \delta_{ij})\ \overset{1}{x}_j\ ]$$

$$= (\overset{2}{x}_i - \overset{1}{x}_i) + \alpha_{ij}(\overset{2}{x}_j - \overset{1}{x}_j)$$

$$= A_i + \alpha_{ij} A_j .$$

This implies

$$A_i' - A_i = \alpha_{ij} A_j$$

$$\boldsymbol{\delta A_i = \alpha_{ij} A_j} \; . \qquad (5)$$

Thus , the linear transformation (1) changes the vector A_i into vector A_i' where

$$\begin{bmatrix} A_1' \\ A_2' \\ A_3' \end{bmatrix} = \begin{bmatrix} 1+\alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & 1+\alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & 1+\alpha_{33} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}, \qquad (6)$$

or

$$\begin{bmatrix} \delta A_1 \\ \delta A_2 \\ \delta A_3 \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}. \qquad (7)$$

Thus , the linear transformation (1) or (6) or (7) are all equivalent.

From equation (6) , it is clear that two vectors A_i and B_i whose components are equal transform into two vectors A_i' and B_i' whose components are again equal. Also two parallel vectors transformation into parallel vectors.

Hence,two equal and similarly oriented rectilinear polygons located in different parts of the region R will be transformed into equal and similarly oriented polygons in the transformed region R′ under the linear transformation (1).

Thus , the different parts of the body B , when the latter is subjected to the linear transformation (1) , **experience the same deformation independent of the position of the parts of the body.**

For this reason , the linear deformation (1) is called a homogeneous deformation.

Theorem: Prove that the necessary and sufficient condition for an infinitesimal affine transform

$$\xi_{ij} = \alpha_{i0} + (\alpha_{ij} + \delta_{ij})\, x_j$$

to represent a rigid body motion is that the matrix α_{ij} is skew – symmetric

Proof: With reference to an orthogonal cartesian system o x_1 x_2 x_3 fixed in space , let the line segment $\overrightarrow{P^oP}$ of the body in the undeformed state be transferred to the line segment $\overline{P^{0\prime}P'}$ in the deformed state due to infinitesimal affine transformation

$$\xi_i = \alpha_{i0} + (\alpha_{ij} + \delta_{ij})\, x_j \ , \tag{1}$$

in which α_{ij} are known constants. Let A_i be vector $\overline{P^oP}$ and A_i' be the vector $\overline{P^{0\prime}P'}$

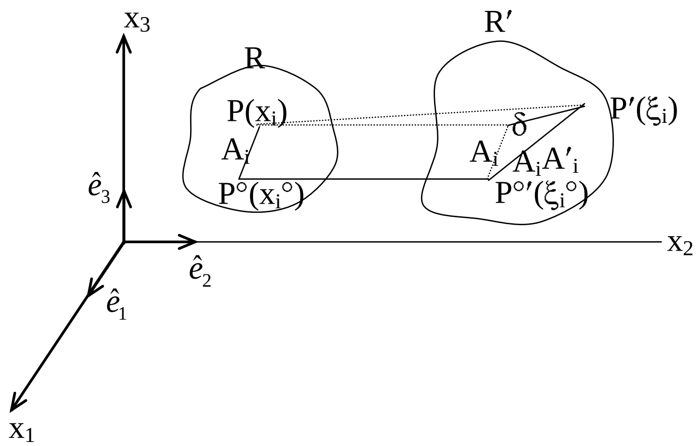


Fig. (3.6)

Then

$$A_i = x_i - x_i^\circ \quad , \quad A_i' = \xi_i - \xi_i^\circ \quad . \tag{2}$$

Let $$\delta A_i = A_i' - A_i \quad . \tag{3}$$

From (1) and (2) , we find

$$\begin{aligned} A_i' &= \xi_i - \xi_i^\circ \\ &= (\alpha_{i0} + \alpha_{ij}\, x_j + x_i) - (\alpha_{i0} + \alpha_{ij}\, x_j^\circ + x_i^\circ) \\ &= (x_i - x_i^\circ) + \alpha_{ij}\,(x_j - x_j^\circ) \\ &= A_i + \alpha_{ij}\, A_j \,. \end{aligned}$$

This gives

$$\delta A_i = A_i' - A_i = \alpha_{ij}\, A_j \ . \tag{4}$$

Let A denote the length of the vector. Then

$$\mathbf{A} = |\,\mathbf{A_i}\,| = \sqrt{A_i\, A_i} = \sqrt{A_1^2 + A_2^2 + A_3^2} \ . \tag{5}$$

Let δA denote the change in length A due to deformation. Then

$$\delta \mathbf{A} = | A_i' | - | A_i | . \tag{6}$$

It is obvious hat $\delta A \neq | \delta A_i |$, but

$$\delta A = \sqrt{(A_i + \delta A_i)(A_i + \delta A_i)} - \sqrt{A_i A_i} .$$

This imply

$$(A + \delta A)^2 = (A_i + \delta A_i)\,(A_i + \delta A_i) ,$$

or

$$(\delta A)^2 + 2\,A\,\delta A = (\delta A_i)\,(\delta A_i) + 2\,A_i\,(\delta A_i) . \tag{7}$$

Since the linear transformation (1) or (4) is small , the terms $(\delta A)^2$ and $(\delta A_i)(\delta A_i)$ are to be neglected in (7). Therefore , after neglecting these terms in (7) , we write

$$2\,A\,\delta A = 2\,A_i\,\delta A_i ,$$

or

$$\mathbf{A\,\delta A = A_i\,\delta A_i} = A_1\,\delta A_1 + A_2\,\delta A_2 + A_3\,\delta A_3 . \tag{8}$$

Using (4) , equation (8) becomes

$$A\,\delta A = A_i\,(\alpha_{ij}\,A_j)$$

$$= \alpha_{ij}\,A_i\,A_j$$

$$= \alpha_{11}\,A_1^2 + \alpha_{22}\,A_2^2 + \alpha_{33}\,A_3^2 + (\alpha_{12} + \alpha_{21})\,A_1\,A_2$$

$$+ (\alpha_{13} + \alpha_{31}) A_3\,A_1 + (\alpha_{23} + \alpha_{32})\,A_2\,A_3. \tag{9}$$

Case 1: Suppose that the infinitesimal linear transformation (1) represents a rigid body motion.

Then , the length of the vector A_i before deformation and after deformation remains unchanged.

That is

$$\delta A = 0 , \tag{10}$$

for all vectors A_i.

Using (9) , we then get

$$\alpha_{11}\,A_1^2 + \alpha_{22}\,A_2^2 + \alpha_{33}\,A_3^2 + (\alpha_{12} + \alpha_{21})\,A_1 A_2 + (\alpha_{23} + \alpha_{32})\,A_2\,A_3$$

$$+ (\alpha_{13} + \alpha_{31})A_1A_3 = 0 , \qquad (11)$$

for all vectors A_i.

This is possible only when

$$\alpha_{11} = \alpha_{22} = \alpha_{33} = 0 ,$$

$$\alpha_{12} + \alpha_{21} = \alpha_{13} + \alpha_{31} = \alpha_{23} + \alpha_{32} = 0 ,$$

i.e. , $$\alpha_{ij} = -\alpha_{ji} , \quad \text{for all } i \ \& \ j \qquad (12)$$

i.e. , the matrix α_{ij} is skew – symmetric.

Case 2: Suppose α_{ij} is skew – symmetric. Then , equation (9) shows that

$$A\,\delta A = 0 , \qquad (13)$$

for all vectors A_i. This implies

$$\delta A = 0 \qquad (14)$$

for all vectors A_i

This shows that the transformation (1) represents a rigid body linear small transformation.

This completes the proof of the theorem.

Remark: When the quantities α_{ij} are skew – symmetric , then the linear infinitesimal transformation

$$\delta A_i = \alpha_{ij} A_j \ ,$$

equation (11) takes the form

$$\delta A_1 = -\alpha_{21} A_2 + \alpha_{13} A_3 ,$$

$$\delta A_2 = \alpha_{21} A_1 - \alpha_{32} A_3 ,$$

$$\delta A_3 = -\alpha_{13} A_1 + \alpha_{32} A_2 . \qquad (15)$$

Let $$w_1 = \alpha_{32} = -\alpha_{23} ,$$

$$w_2 = \alpha_{13} = -\alpha_{31} ,$$

$$w_3 = \alpha_{21} = -\alpha_{12} . \qquad (16)$$

Then , the transformation (15) can be written as the vector product

$$\overline{\delta A} = \overline{w} \times \overline{A} \ , \tag{17}$$

where $\overline{w} = w_i$ is the infinitesimal rotation vector. Further

$$\begin{aligned} \delta A_i &= A_i' - A_i \\ &= (\xi_i - \xi_i^\circ) - (x_i - x_i^\circ) \\ &= (\xi_i - x_i) - (\xi_i^\circ - x_i^\circ) \\ &= \delta x_i - \delta x_i^\circ. \end{aligned} \tag{18}$$

This yields

$$\delta x_i = \delta x_i^\circ + \delta A_i \ ,$$

or

$$\delta x_i = \delta x_i^\circ + (\overline{w} \times \overline{A}) \ . \tag{19}$$

Here , the quantities

$$\delta x_i^\circ = \xi_i^\circ - x_i^\circ$$

are the components of the displacement vector representing the **translation** of the point P° and the remaining terms of (19) represent **rotation of the body** about the point P°.

3.6 PURE DEFORMATION AND COMPONENTS OF STRAIN TENSOR

We consider the infinitesimal linear transformation

$$\delta A_i = \alpha_{ij} A_j \quad . \tag{1}$$

Let

$$w_{ij} = \frac{1}{2}(\alpha_{ij} - \alpha_{ji}) \ , \tag{2}$$

and

$$e_{ij} = \frac{1}{2}(\alpha_{ij} + \alpha_{ji}) \ . \tag{3}$$

Then the matrix w_{ij} is **antisymmetric** while e_{ij} is **symmetric** .

Moreover ,

$$\alpha_{ij} = e_{ij} + w_{ij} , \quad (4)$$

and this decomposition of α_{ij} as a sum of a symmetric and skew – symmetric matrices is unique.

From (1) and (4) , we write

$$\delta A_i = e_{ij} A_j + w_{ij} A_j \quad . \quad (5)$$

This shows that the transformation of the components of a vector A_i given by

$$\delta A_i = w_{ij} A_j \quad , \quad (6)$$

represents rigid body motion with the components of rotation vector w_i given by

$$w_1 = w_{32} , w_2 = w_{13} , w_3 = w_{21} , \quad (7)$$

and the transformation

$$\boldsymbol{\delta A_i = e_{ij} A_j} , \quad (8)$$

with $$\boldsymbol{e_{ij} = e_{ji}} , \quad (9)$$

represents a **pure deformation**.

Strain Components

The symmetric coefficients , e_{ij} , in the pure deformation

$$\delta A_i = e_{ij} A_j$$

are called the **strain components.**

Note (1) : These components of strain characterize **pure deformation** of the elastic body. Since A_j and δA_i are vectors (each is a tensor of order 1) , therefore , by quotient law , the strain components $\boldsymbol{e_{ij}}$ **form a tensor of order 2.**

Note (2) : For most materials / structures , the strains are of the order **of $\boldsymbol{10^{-3}}$**. Such strains certainly **deserve to be called small**.

Note (3) : The strain components e_{11} , e_{22} , e_{33} are called **normal strain components** while e_{12} , e_{13} , e_{23} , e_{21} , e_{31} , e_{32} are called **shear strain components**.

Example : For the deformation defined by the linear transformation

$$\xi_1 = x_1 + x_2 , \xi_2 = x_1 - 2x_2 , \xi_3 = x_1 + x_2 - x_3 ,$$

find the inverse transformation , components of rotation and strain tensor , and axis of rotation.

Solution : The given transformation is expresses as

$$\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix} = \begin{bmatrix}1 & 1 & 0\\ 1 & -2 & 0\\ 1 & 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}, \qquad (1)$$

and its inverse transformation is

$$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}1 & 1 & 0\\ 1 & -2 & 0\\ 1 & 1 & -1\end{bmatrix}^{-1}\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix}2 & 1 & 0\\ 1 & -1 & 0\\ 1 & 0 & -1\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix}, \qquad (2)$$

giving

$$x_1 = \frac{1}{3}(2\,\xi_1 + \xi_2)\,,$$

$$x_2 = \frac{1}{3}(\xi_1 - \xi_2)\,,$$

$$x_3 = \xi_1 - \xi_3 \quad . \qquad (3)$$

Comparing (1) with

$$\xi_i = (\alpha_{ij} + \delta_{ij})\, x_j \quad , \qquad (4)$$

we find

$$(\alpha_{ij}) = \begin{bmatrix}0 & 1 & 0\\ 1 & -3 & 0\\ 1 & 1 & -2\end{bmatrix}. \qquad (5)$$

Then

$$w_{ij} = \frac{1}{2}(\alpha_{ij} - \alpha_{ji}) = \frac{1}{2}\begin{bmatrix}0 & 0 & -1\\ 0 & 0 & -1\\ 1 & 1 & 0\end{bmatrix}, \qquad (6)$$

and

$$e_{ij} = \frac{1}{2}(\alpha_{ij} + \alpha_{ji})$$

$$= \begin{bmatrix} 0 & 1 & \frac{1}{2} \\ 1 & -3 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -2 \end{bmatrix}, \quad (7)$$

and

$$\alpha_{ij} = w_{ij} + e_{ij} \ , \quad (8)$$

The axis of rotation is

$$\bar{w} = w_i \ \hat{e}_i$$

where

$$w_1 = w_{32} = \frac{1}{2} \ ,$$

$$w_2 = w_{13} = -\frac{1}{2} \ ,$$

$$w_3 = w_{21} = 0. \quad (9)$$

3.7 GEOMETRICAL INTERPRETATION OF THE COMPONENTS OF STRAIN

Normal Strain Component e_{11}

Let e_{ij} be the components of strains. The pure infinitesimal linear deformation of a vector A_i is given by

$$\delta A_i = e_{ij} A_j \quad , \quad (1)$$

with $e_{ij} = e_{ji}$.

Let e denote the extension (or change) in length per unit length of the vector A_i with magnitude A. Then , by definition ,

$$e = \frac{\delta A}{A} \ . \quad (2)$$

we note that e is positive or negative depending upon whether the material line element A_i experiences an **extension** or a **contraction**.

Also e = 0 iff the vector A retains its length during a deformation.

This number e is referred to **as the normal strain of the vector A_i**.

Since the deformation is linear and infinitesimal , we have (proved earlier)

$$A\,\delta A = A_i\,\delta A_i \tag{3}$$

or
$$\frac{\delta A}{A} = \frac{A_i\,\delta A_i}{A^2}\ .$$

Now from (1) – (3) , we write

$$e = \frac{\delta A}{A} = \frac{A_i\,\delta A_i}{A^2}$$

$$= \frac{1}{A^2}\ A_i\, e_{ij}\, A_j\ .$$

This implies

$$e = \frac{1}{A^2}\ [e_{11} A_1^2 + e_{22} A_2^2 + e_{33} A_3^2 + 2e_{12} A_1 A_2 + 2e_{13} A_1 A_3 + 2e_{23} A_2 A_3]\backslash \tag{4}$$

since $e_{ij} = e_{ji}$.

In particular , we consider the case in which the vector A_i in the undeformed state is parallel to the x_1- axis. Then

$$A_1 = A\ \ , A_2 = A_3 = 0 \tag{5}$$

Using (5) , equation (4) gives

$$e = e_{11}\ . \tag{6}$$

Thus , the component e_{11} of the strain tensor represents , to a good approximation the **extension or change in length per unit initial length** of a material line **segment** (or fibre of the material) **originally placed parallel to the x_1 – axis** in the **undeformed state**.

Similarly , normal strains e_{22} and e_{33} are to be interpreted.

Illustration : Let $e_{ij} = \begin{bmatrix} e_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Then all unit vectors parallel to the x_1 – axis will be extended by an amount e_{11}. In this case , one has a homogeneous deformation of material in the direction of the x_1 – axis. A cube of material whose edges before deformation are 'l' units long will become (after deformation due to e_{ij}) a rectangular parallelopiped whose dimension in the x_1 – direction is $l\,(1 + e_{11})$ units and whose dimensions in the direction of the x_2 – and x_3 – axes are unchanged.

Remark: The vector

$$\bar{A} = A_i = (A\ ,\ 0\ ,\ 0)$$

is changed to (due to deformation)

$$\bar{A}' = (A + \delta A_1)\ \hat{e}_1 + \delta A_2 \hat{e}_2 + \delta A_3 \hat{e}_3$$

in which

$$\delta A_i = e_{ij}\ A_j = e_{il}\ A_1$$

give

$$\delta A_1 = e_{11}\ A\ ,\ \delta A_2 = e_{12}\ A\ ,\ \delta A_3 = e_{13}\ A.$$

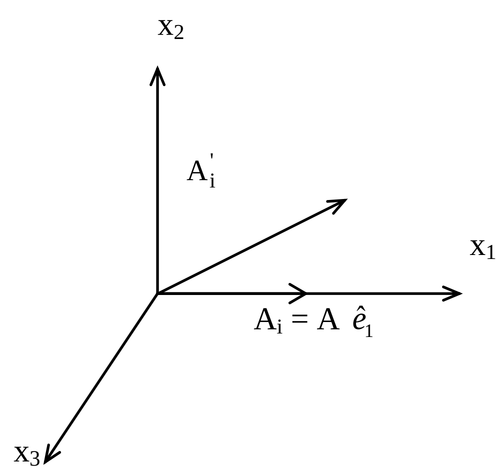


Fig. (3.7)

Thus

$$\bar{A}' = (A + e_{11}\ A\ ,\ e_{12}\ A\ ,\ e_{13}\ A)\ .$$

This indicates that vector A_i = (A , 0 , 0) upon deformation , in general , **changes its orientation also**.

The length of the vector due to deformation becomes $(1 + e_{11})$ A .

Question : From the relation $\delta A_i = e_{ij} A_j$, find δA and δA_i for a vector lying initially along the x – axis (i.e. , $\bar{A} = A\,\hat{e}_1$) and justify the fact that $\frac{\delta A}{A} = e_{11}$. Does δA_i lie along the x – axis ?

Answer : It is given that A_i = (A , 0 , 0). The given relation

$$\delta A_i = e_{ij} A_j \qquad (1)$$

gives

$$\delta A_1 = e_{11} A ,\ \delta A_2 = e_{12} A ,\ \delta A_3 = e_{13} A . \qquad (2)$$

Thus , in general , **the vector δA_i does not lie along the x – axis.**

Further

$$(A + \delta A) = \sqrt{[A(1+e_{11})^2 + (e_{12}A)^2 + (e_{13}A)^2}$$

$$= A\ \sqrt{1+2e_{11}+e_{11}{}^2+e_{12}{}^2+e_{13}{}^2}\ . \qquad (3)$$

Neglecting square terms as deformation is small, equation (3) gives

$$(A + \delta A)^2 = A^2 (1 + 2 e_{11}) \ ,$$

$$\Rightarrow \qquad \cancel{A^2} + 2 A \delta A = \cancel{A^2} + 2 A^2 e_{11} \ ,$$

$$2 A \delta A = 2 A^2 e_{11}$$

$$\frac{\delta A}{A} = e_{11}. \qquad (4)$$

This shows that e_{11} gives the extension of a vector (A , 0 , 0) per unit length due to deformation.

Remark : The strain components e_{ij} refer to the chosen set of coordinate axes. If the axes are changed , the strain components e_{ij} will , in general , change as per tensor transformation laws.

Geometrical Interpretation of Shearing Stress e_{23}

The shearing strain component e_{23} may be interpreted by considering **intersecting vectors initially parallel to two coordinate** axes – x_2 – and x_3 – axes.

Now , we consider in the undeformed state two vectors.

$$\bar{A} = A_2 \, \hat{e}_2 \, ,$$

$$\bar{B} = B_3 \, \hat{e}_3 \, , \qquad (1)$$

directed along x_2 – and x_3 – axis , respectively.

The relations of small linear deformation are

$$\boldsymbol{\delta A_i = e_{ij} A_j \, ,}$$

$$\boldsymbol{\delta B_i = e_{ij} B_3 \, ,} \qquad \textbf{(2)}$$

Further , the vectors A_i and B_i due to deformation become (Figure)

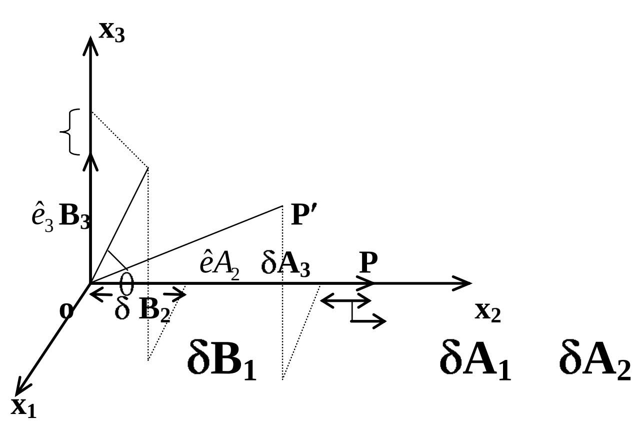


Fig. (3.8)

$$\left.\begin{aligned} \bar{A}' &= \delta A_1 \, \hat{e}_1 + (A_2 + \delta A_2) \, \hat{e}_2 + \delta A_3 \, \hat{e}_3 \, , \\ \bar{B}' &= \delta B_1 \, \hat{e}_1 + \delta B_2 \, \hat{e}_2 + (B_3 + \delta B_3) \, \hat{e}_3 \, . \end{aligned}\right\} \qquad \textbf{(3)}$$

Deformed vectors $\bar{A}'$ and $\bar{B}'$ need not lie in the $x_2 \, x_3$ – plane. Let θ be the angle between $\bar{A}'$ and $\bar{B}'$. Then

$$\cos\theta = \frac{\bar{A}'.\bar{B}'}{A'B'} = \frac{\delta A_1\delta B_1 + (A_2+\delta A_2)\delta B_2 + \delta A_3(B_3+\delta B_3)}{\sqrt{(\delta A_1)^2+(A_2+\delta A_2)^2+(\delta A_3)^2}\sqrt{(\delta B_1)^2+(\delta B_2)^2+(B_3+\delta B_3)^2}}. \tag{4}$$

Since , the deformation is small , we may neglect the products of the changes in the components of the vector A_i and B_i. Neglecting these products , equation (4) gives

$$\cos\theta = (A_2\,\delta B_2 + B_3\,\delta A_3)\,(A_2+\delta A_2)^{-1}\,(B_3+\delta B_3)^{-1}$$

$$= \frac{A_2\delta B_2 + B_3\,\delta A_3}{A_2\,B_3}\left(1+\frac{\delta A_2}{A_2}\right)^{-1}\left(1+\frac{\delta B_3}{B_3}\right)^{-1}$$

$$= \left(\frac{\delta B_2}{B_3}+\frac{\delta A_3}{A_2}\right)\left(1-\frac{\delta A_2}{A_2}\right)\left(1-\frac{\delta B_3}{B_3}\right) \quad ,$$

neglecting other terms. This gives

$$\cos\theta = \frac{\delta B_2}{B_3}+\frac{\delta A_3}{A_2} \quad , \tag{5}$$

neglecting the product terms involving changes in the components of the vectors A_i and B_i.

Since in formula (5) , all increments in the components of initial vectors A_i and B_i have been neglected except δA_3 and δB_2 , the deformation of these vectors on assuming (w.l.o.g)

$$\delta A_1 = \delta A_2 \equiv 0 \ ,$$

and

$$\delta B_1 = \delta B_3 \equiv 0 \ ,$$

can be represented as shown in the figure below (It shows that vectors A_i' and B_i' lie in the $x_2\,x_3$ – plane). We call that equations (3) now may be taken as

$$\bar{A}' = A_2\,\hat{e}_2 + \delta A_3\,\hat{e}_3 \ ,$$

$$\bar{B}' = \delta B_2\,\hat{e}_2 + B_3\,\hat{e}_3 . \tag{6}$$

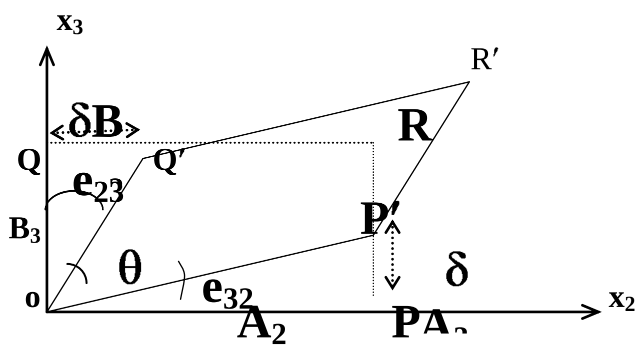


Fig. (3.9)

From equations (1) and (2) ,we obtain

$$\delta A_3 = e_{32} A_2 \quad ,$$

$$\delta B_2 = e_{23} B_3 \qquad (7)$$

This gives

$$e_{32} = \frac{\delta A_3}{A_2} = \tan \angle P'O P \qquad (8)$$

and

$$e_{23} = \frac{\delta B_2}{B_3} = \tan \angle Q'O Q \ . \qquad (9)$$

Since strains $e_{23} = e_{32}$ are small , so

$$\angle P'O P = \angle Q'O Q \cong e_{23} \quad ,$$

and hence

$$2e_{23} \cong 90° - \theta = \pi/2 - \theta \ . \qquad (10)$$

Thus , a positive value **of $2e_{23}$ represents a** decrease **in the right angle between the vectors A_i and B_i due to** small linear deformation **which were initially directed along the positive x_2 – and x_3 –axes. The quantity /strain component e_{23}** is called the shearing strain.

A similar interpretation can be made for the shear strain components e_{12} and e_{13}.

Shear strain components represent the changes in the relative orientations of material arcs.

Remark 1: **By rotating the parallelogram R′OP′Q′ through an angle e_{23} about the origin (in the x_2 x_3 – plane) , we obtain the following configurations (Figure)**

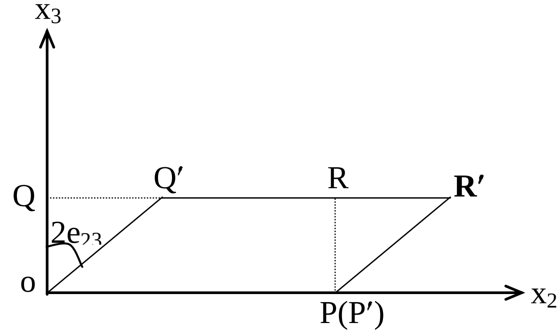


Fig. (3.10)

This figure shows a slide or a shear of planar elements parallel to the x_1 x_2 – plane.

Remark 2: **Figure shows that the areas of the rectangle OQRP and the parallelogram OQ′R′P′ are equal as they have the same height and same base in the x_2 x_3 – plane.**

Remark 3: **For the strain tensor** $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & e_{23} \\ 0 & e_{32} & 0 \end{bmatrix}$,

a cubical element is deformed into a parallelopiped and the volumes of the cube and parallelopiped remain the same.

Such a small linear deformation is called a pure shear.

3.8 NORMAL AND TANGENTIAL DISPLACEMENTS

Consider a point P(x_1 , x_2 , x_3) of the material. Let it be moved to Q under a small linear transformation. Let the components of the displacement vector $\overline{PQ}$ be u_1 , u_2 , u_3. In the plane OPQ , let $\overline{PN}$ = $\bar{n}$ be the projection of $\overline{PQ}$ on the line OPN and let $\overline{PT}$ = $\bar{t}$ be the tangential component of $\overline{PQ}$ in the plane of OPQ or PQN.

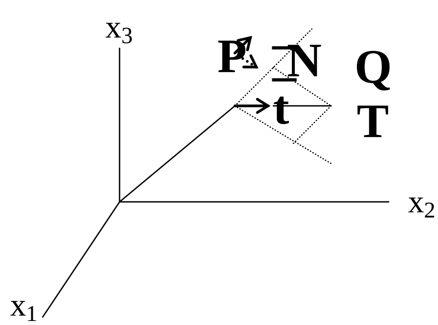


Fig. (3.11)

Definition: **Vectors $\bar{n}$ and $\bar{t}$ are , respectively , called** the normal and the tangential components of the displacement of P.

Note: **The magnitude n of normal displacement $\bar{n}$ is given by the dot product of vectors**

$$\overline{OP} = (x_1 , x_2 , x_3) \text{ and } \overline{PQ} = (u_1 , u_2 , u_3).$$

The magnitude t of tangential vector $\bar{t}$ is given the vector product of vectors $\overline{OP}$ and $\overline{PQ}$ (This does not give the direction of $\bar{t}$).

Thus

$$n = PQ \cos\underline{|NPQ} = \frac{\overline{OP}.\overline{PQ}}{|\overline{OP}|} \ ,$$

$$t = PQ \sin\underline{|NPQ} = \frac{(OP)(PQ)\sin(NPQ)}{OP} = \frac{|\overline{OP} \times \overline{PQ}|}{|\overline{OP}|} \ ,$$

and

$$n^2 + t^2 = u_1^2 + u_2^2 + u_3^2.$$

3.9 STRAIN QUADRIC CAUCHY

Let P°(x_1° , x_2° , x_3°) be any but fixed point of a continuous medium , with reference axes o x_1 x_2 x_3 fixed in space.

We introduce a local system of axes with origin at the point P° and with axes parallel to the fixed axes (figure).

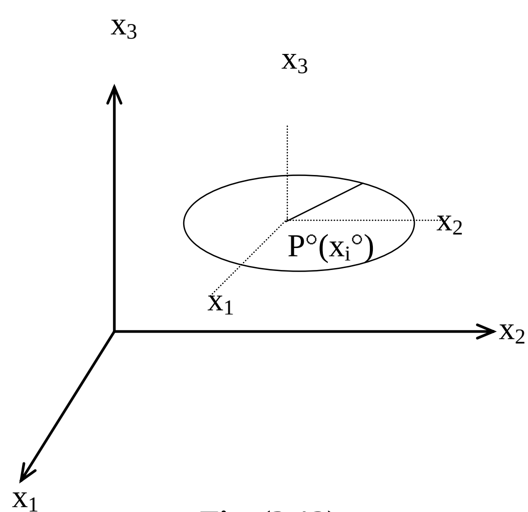


Fig. (3.12)

With reference to these local axes , consider the equation

$$e_{ij}\, x_i\, x_j = \pm k^2 \,, \qquad (1)$$

where k is a real constant and e_{ij} is the strain tensor at P°. This equation represents a quadric surface with its centre at P°.

This quadric is called the quadric surface of deformation **or** strain quadric **or** strain quadric of Cauchy.

The sign + or – in equation (1) be chosen so that the quadric surface (1) becomes a real one.

The nature of this quadric surface depends on the value of the strains e_{ij} .

If $|e_{ij}| \neq 0$, the quadric is either an ellipsoid or a hyperboloid.

If $|e_{ij}| = 0$, the quadric surface degenerates into a cylinder of the elliptic or hyperbolic type or else into two parallel planes symmetrically situated with respect to the origin P° of the quadric surface.

This strain quadric is completely determined once the strain components e_{ij} at point P° are known.

Let $\overline{P^\circ P}$ be the radius vector A_i of magnitude A to any point $P(x_1 , x_2 , x_3)$, referred to local axis , on the strain quadric surface (1). Let e be the extension of the vector A_i due to some linear deformation characterized by

$$\delta A_i = e_{ij}\, A_j \,. \qquad (2)$$

Then , by definition ,

$$e = \frac{\delta A}{A} = \frac{A\delta A}{A^2} = \frac{A_i \delta A_i}{A^2} \,.$$

This gives

$$e = \frac{e_{ij}\, A_i\, A_j}{A^2} \,, \qquad (3)$$

using (2).

Since $\overline{P^\circ P} = A_i$ and the coordinate of the point P , on the surface (1) , relative to P° are (x_1 , x_2 , x_3) , it follows that

$$A_i = x_i \,. \qquad (4)$$

From equations (1) , (3) and (4) ; we obtain

$$\mathbf{e\, A^2 = e_{ij}\, A_i\, A_j = e_{ij}\, x_i\, x_j = \pm\, k^2}$$

or $$\mathbf{e\, A^2 = \pm\, k^2}$$

or $$e = \pm\, \frac{k^2}{A^2}\,. \qquad \textbf{(5)}$$

Result (I) : **Relation (5) shows that the** extension or elongation of any radius vector A_i of the strain quadric of Cauchy **, given by equation (1) , is inversely proportional to the square of the length A of that radius vector.** This determines the elongation of any radius vector of the strain quadric at the point $P°(x_i°)$**.**

Result (II) : **We know that the length A of the radius vector A_i of strain quadric (1) at the point $P°(x_i°)$ has maximum and minimum values** along the axes of the quadric. **In general , axes of the strain quadric (1) differs from the coordinate axes through $P°(x_i°)$.**

Therefore , the maximum and minimum extensions /elongations of radius vectors of strain quadric (1) will be along its axes.

Result (III) : **Another interesting property of the strain quadric (1) is that** normal ν_i to this surface **at the end point P of the vector $\overline{P^\circ P}$ = A_i is parallel to the displacement vector δA_i.**

To prove this property , let us write equation (1) in the form

$$\mathbf{G = e_{ij}\, x_j\, x_i \mp k^2 = 0}\,. \qquad \textbf{(6)}$$

Then the direction of the normal $\hat{\nu}$ to the strain quadric (6) is given by the gradient of the scalar function G. The components of the gradient are

$$\frac{\partial G}{\partial x_k} = \mathbf{e_{ij}\, \delta_{ik}\, x_j + e_{ij}\, x_i\, \delta_{kj}}$$

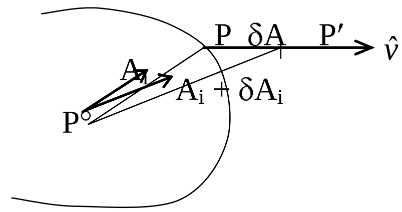
$$= \mathbf{e_{kj}\, x_j + e_{ik}\, x_i}$$

$$= \mathbf{2\, e_{kj}\, x_j}\;,$$

or

$$\frac{\partial G}{\partial x_k} = \mathbf{2\, \delta A_k}\,. \qquad \textbf{(7)}$$

This shows that vector $\frac{\delta G}{\delta x_k}$ and δA_k are parallel.



$\delta \mathbf{A_i} \;||\; \mathbf{v_i}$

Fig. (3.13)

Hence , the vector $\overline{\delta A}$ is directed along the normal at P to the strain quadric of Cauchy.

3.10 STRAIN COMPONENTS AT A POINT IN A ROTATION OF COORDINATE AXES

Let new axes o x_1' x_2' x_3' be obtained from the old reference system o x_1 x_2 x_3 by a rotation (figure).

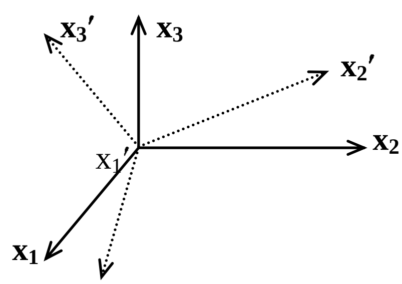


Fig. (3.14)

Let the directions of the new axes x_i' be specified relative to the old system x_i by the following table of direction cosines in which a_{pi} is the cosine of the angle between the x_p' - and x_i – axis.

That is ,

$$a_{pi} = \cos(x_p', x_i) .$$

Thus

	x_1	x_2	x_3
x_1'	a_{11}	a_{12}	a_{13}
x_2'	a_{21}	a_{22}	a_{23}
x_3'	a_{31}	a_{32}	a_{33}

Then the transformation law for coordinates is

$$x_i = a_{pi}\, x_p' ,$$

or

$$x_p' = a_{pi}\, x_i . \tag{2}$$

The well – known orthogonality relation are

$$a_{pi}\, a_{qi} = \delta_{pq} , \tag{3}$$

$$a_{pi}\, a_{pj} = \delta_{ij} , \tag{4}$$

with reference to new x_p' system , a new set of strain components e'_{pq} is determined at the point O while e_{ij} are the components of strain at O relative to old axes O $x_1\, x_2\, x_3$.

Let

$$e_{ij}\, x_i\, x_j = \pm\, k^2 , \tag{5}$$

be the equation of the strain quadric surface relative to old axis. The equation of quadric surface with reference to new prime system becomes

$$e'_{pq}\, x'_p\, x'_q = \pm\, k^2 , \tag{6}$$

as we know that quadric form is invariant w.r.t. an orthogonal transformation of coordinates.

Further , equations (2) to (6) together yield

$$\begin{aligned} e'_{pq}\, x'_p\, x'_q &= e_{ij}\, x_i\, x_j \\ &= e_{ij}\, (a_{pi}\, x'_p)\, (a_{qj}\, x'_q) \\ &= (e_{ij}\, a_{pi}\, a_{qj})\, x'_p\, x'_q , \end{aligned}$$

or

$$(e'_{pq} - a_{pi}\, a_{qj}\, e_{ij})\, x'_p\, x'_q = 0. \tag{7}$$

Since equation (7) is satisfied for arbitrary vector x'_p , we must have

$$e'_{pq} = a_{pi}\, a_{qj}\, e_{ij} . \tag{8}$$

Equation (8) is the law of transformation for a second order tensor.

We , therefore , conclude that the components of strain form a second order tensor.

Similarly , it can be verified that

$$e_{ij} = a_{pi}\, a_{qj}\, e'_{pq} . \qquad (9)$$

Question : **Assuming that e_{ij} is a tensor of order 2 , show that quadratic form** $e_{ij}\, x_i\, x_j$ is an variant.

Solution : **We have**

$$e_{ij} = a_{pi}\, a_{qj}\, e'_{pq} ,$$

so

$$\begin{aligned} e_{ij}\, x_i\, x_j &= a_{pi}\, a_{qj}\, e'_{pq}\, x_i\, x_j \\ &= e'_{pq}\, (a_{pi}\, x_i)\, (a_{qj}\, x_j) \\ &= e'_{pq}\, x'_p\, x'_q . \end{aligned}$$

Hence the result.

3.11 PRINCIPAL STRAINS AND INVARIANTS

From a material point P°(x_i°) there emerges infinitely many material arcs /filaments , and each of these arcs generally changes in length and orientation under a deformation.

We seek now the lines through P°(x_i°) whose orientation **is left unchanged by the small linear deformation given by**

$$\delta A_i = e_{ij}\, A_j \quad , \qquad (1)$$

where the strain components e_{ij} are small and constant.

In this situation , vectors A_i and δA_i are parallel **and , therefore ,**

$$\delta A_i = e\, A_i \quad , \qquad (2)$$

for some constant e.

Equation (2) shows that the constant e represents the extension

$$\left(e = \frac{|\delta A_i|}{|A_i|} = \frac{\delta A}{A} \right)$$

of vector A_i**.**

From equations (1) and (2) , we write

$$e_{ij}\, A_j = e\, A_i$$

$$= e\, \delta_{ij}\, A_j\,. \qquad (3)$$

This implies

$$(e_{ij} - e\, \delta_{ij})\, A_j = 0. \qquad (4)$$

We know that e_{ij} is a real symmetric tensor of order 2. The equation (3) shows that the scalar e is an eigenvalue of the real symmetric tensor e_{ij} with corresponding eigenvector A_i. Therefore , we conclude that there are precisely three mutually orthogonal directions whose orientations are not changed on account of deformation and these directions coincide with the three eigenvectors of the strain tensor e_{ij}.

These directions are known as principal directions or invariant directions of strain.

Equation (4) gives us a system of three homogeneous equations in the unknowns A_1 , A_2 , A_3. This system possesses a non – trivial solution iff the determinant of the coefficients of the A_1 , A_2 , A_3 is equal to zero , i.e. ,

$$\begin{vmatrix} e_{11}-e & e_{12} & e_{13} \\ e_{21} & e_{22}-e & e_{23} \\ e_{31} & e_{32} & e_{33}-e \end{vmatrix} = 0\,, \qquad (5)$$

which is a cubic equation in e.

Let e_1 , e_2 , e_3 be the three roots of equation (5). These are known as principal strains.

Evidently , the principal strains are the eigenvalues of the second order real symmetric strain tensor e_{ij} . Consequently , these principal strains are all real (not necessarily distinct).

Physically , the principal strains e_1 , e_2 , e_3 (all different) are the extensions of the vectors , say $\overset{i}{\overline{A}}$, in the principal /invariant directions of strain. So, vectors $\overset{i}{\underset{\sim}{A}}$, $\delta\overset{i}{\underset{\sim}{A}}$, $\overset{i}{\underset{\sim}{A}} + \delta\overset{i}{\underset{\sim}{A}}$ are collinear.

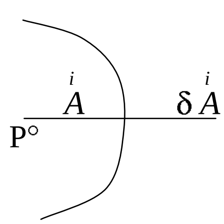


Fig. (3.15)

At the point P°, consider the strain quadric

$$e_{ij}\, x_i\, x_j = \pm k^2 . \tag{6}$$

For every principal direction of strain $\underset{\sim}{\overset{i}{A}}$ **, we know that** $\delta \overset{i}{A}$ **is normal to the quadric surface (6). Therefore , the principal directions of strain are also normal to the strain quadric of Cauchy. Hence, principal direction of strain must be the three principal axes of the strain quadric of Cauchy.**

If some of the principal strains e_i **are equal , then the associated directions become indeterminate but one can always select three directions that all mutually orthogonal (already proved).**

If $e_1 \neq e_2 = e_3$ **, then the quadric surface of Cauchy is a surface revolution and our principal direction,**

say $\underset{\sim}{\overset{1}{A}}$ **, will be directed along the axis of revolution.**

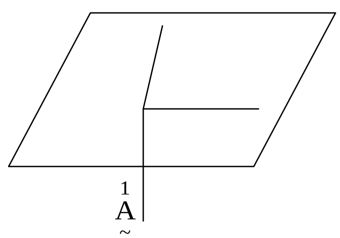


Fig. (3.16)

In this case , any two mutually perpendicular vectors lying in the plane normal to $\underset{\sim}{\overset{1}{A}}$ **may be taken as the other two principal directions of strain.**

If $e_1 = e_2 = e_3$ **, then strain quadric of Cauchy becomes a sphere and any three orthogonal directions may be chosen as the principal directions of strain.**

Result 1 : **If the principal directions of strain are taken as the coordinate axes , then**

$$e_{11} = e_1 \,,\ e_{22} = e_2 \,,\ e_{33} = e_3$$

and

$$e_{12} = e_{13} = e_{23} = 0 \,,$$

as a vector initially along an axis remains in the same direction after deformation (so changes in right angles are zero). In this case , the strain quadric of Cauchy has the equation

$$\mathbf{e_1 x_1^2 + e_2 x_2^2 + e_3^2 x_3^2 = \pm k^2.} \quad \mathbf{(7)}$$

Result 2: **Expanding the cubic equation (5) , we write**

$$-e^3 + \nu_1 e^2 - \nu_2 e + \nu_3 = 0.$$

where

$$\nu_1 = e_{11} + e_{22} + e_{33}$$

$$= e_{ii} = tr(E) \quad , \quad (8)$$

$$\nu_2 = e_{11} e_{22} + e_{22} e_{33} + e_{33} e_{11} - e_{23}^2 - e_{13}^2 - e_{12}^2$$

$$= tr(E^2) = \frac{1}{2}(e_{ii} e_{jj} - e_{ij} e_{ji}) \quad , \quad (9)$$

$$\nu_3 = \in_{ijk} e_{1i} e_{2j} e_{3k}$$

$$= | e_{ij} | = tr(E^3) . \quad (10)$$

Also , e_1 , e_2 , e_3 are roots of the cubic equation (8) , so

$$\left.\begin{aligned} \nu_1 &= e_1 + e_2 + e_3 \\ \nu_2 &= e_1 e_2 + e_2 e_3 + e_3 e_1 \\ \nu_3 &= e_1 e_2 e_3 \end{aligned}\right\} \quad (11)$$

We know that eigenvalues of a second order real symmetric tensor are independent of the choice of the coordinate system.

It follows that ν_1 , ν_2 , ν_3 are , as given by (10) , three invariants of the strain tensor e_{ij} with respect to an orthogonal transformation of coordinates**.**

Geometrical Meaning of the First Strain Invariant $\nu = e_{ii}$

The quantity $\nu = e_{ii}$ **has a simple geometrical meaning. Consider a volume element in the form of a rectangular parallelepiped whose edges of length l_1 , l_2 , l_3 are parallel to the principal directions of strain.**

Due to small linear transformation /deformation , this volume element becomes again a rectangular parallelepiped with edges of length $l_1(1+e_1)$, $l_2(1+e_2)$, $l_3(1+e_3)$, where e_1 , e_2 , e_3 are principal strains.

Hence , the change δV in the volume V of the element is

$$\delta V = l_1 \, l_2 \, l_3 (1 + e_1)(1 + e_2)(1 + e_3) - l_1 \, l_2 \, l_3$$

$$= l_1\, l_2\, l_3\, (1 + e_1 + e_2 + e_3) - l_1\, l_2\, l_3\,, \qquad \text{ignoring small strains } e_i.$$

$$= l_1\, l_2\, l_3\, (e_1 + e_2 + e_3)$$

This implies

$$\frac{\delta V}{V} = e_1 + e_2 + e_3 = \nu_1$$

Thus , the first strain invariant ν_1 represents the change in volume per unit initial volume due to strain produced in the medium**.**

The quantity ν_1 **is called the** cubical dilatation **or simply the** dilatation**.**

Note : **If** $e_1 > e_2 > e_3$ **then** e_3 **is called the** minor principal strain **,** e_2 **is called the** intermediate principal strain **, and** e_1 **is called the** major principal strain**.**

Question : **For small linear deformation , the strains** e_{ij} **are given by**

$$(e_{ij}) = \alpha \begin{bmatrix} x_2 & (x_1 + x_2)/2 & x_3 \\ (x_1 + x_2)/2 & x_1 & x_3 \\ x_3 & x_3 & 2(x_1 + x_2) \end{bmatrix}, \quad \alpha$$

= constant.

Find the strain invariants , principal strains and principal directions of strain at the point P(1 , 1 , 0).

Solution : **The strain matrix at the point P(1 , 1 , 0) becomes**

$$(e_{ij}) = \begin{bmatrix} \alpha & \alpha & 0 \\ \alpha & \alpha & 0 \\ 0 & 0 & 4\alpha \end{bmatrix},$$

whose characteristic equation becomes

$$e(e - 2\alpha)\,(e - 4\alpha) = 0\,.$$

Hence , the principle strains are

$$e_1 = 0\,,\ e_2 = 2\alpha\,,\ e_3 = 4\alpha.$$

The three scalar invariants are

$$\nu_1 = e_1 + e_2 + e_3 = 6\alpha\,,\ \nu_2 = 8\alpha^2\,,\ \nu_3 = 0$$

The three principal unit directions are found to be

$$\overset{1}{A_i} = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0 \right) \quad , \qquad \overset{2}{A_i} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \quad ,$$

$$\overset{3}{A_i} = (0, 0, 1)$$

Exercise : **The strain field at a point P(x , y , z) in an elastic body is given by**

$$\mathbf{e_{ij}} = \begin{bmatrix} 20 & 3 & 2 \\ 3 & -10 & 5 \\ 2 & 5 & -8 \end{bmatrix} \times \mathbf{10^{-6}} \, .$$

Determine the strain invariants and the principal strains.

Question : **Find the principal directions of strain by finding the extremal value of the extension e.**

OR

Find the directions in which the extension e is stationary.

Solution: **Let e be the extension of a vector A_i due to small linear deformation**

$$\delta A_i = e_{ij} A_j \quad . \qquad (1)$$

Then

$$e = \frac{\delta A}{A} . \qquad (2)$$

We know that for an infinitesimal linear deformation (1) , we have

$$A \, \delta A = A_i \, \delta A_i . \qquad (3)$$

Thus

$$e = \frac{A \, \delta A}{A^2} = \frac{A_i \, \delta A_i}{A^2} = \frac{e_{ij} A_i A_j}{A^2} . \qquad (4)$$

Let

$$\frac{A_i}{A} = a_i . \qquad (5)$$

Then

$$a_i a_i = 1 , \qquad (6)$$

and equation (4) then gives

$$e(a_1, a_2, a_3) = e_{ij}\, a_i\, a_j. \qquad (7)$$

Thus , the extension e is a function of a_1 , a_2 , a_3 which are not independent because of relation (6). The extreme/stationary(or Max/Min) values of the extension e are to be found by making use of Lagrange's method of multipliers.

For this purpose , we consider the auxiliary function

$$F(a_1, a_2, a_3) = e_{ij}\, a_i\, a_j - \lambda(a_i\, a_i - 1), \qquad (8)$$

where λ is a constant.

In order to find the values of a_1 , a_2 , a_3 for which the function (7) may have a maximum or minimum, we solve the equations

$$\frac{\partial F}{\partial a_k} = 0 \qquad , k = 1, 2, 3. \qquad (9)$$

Thus , the stationary values of e are given by

$$e_{ij}\,(\delta_{ik}\, a_j + a_i\, \delta_{jk}) - \lambda\, 2\, a_i\, \delta_{ik} = 0$$

or $$e_{kj}\, a_j + e_{ik}\, a_i - 2\,\lambda\, a_k = 0$$

or $$2\, e_{ki}\, a_i - 2\,\lambda\, a_k = 0$$

or $$e_{ki}\, a_i = \lambda\, a_k\,. \qquad (10)$$

This shows that λ is an eigenvalue of the strain tensor e_{ij} and a_i is the corresponding eigenvector. Therefore , equations in (10) determine the principal strains and principal directions of strain.

Thus , the extension e assumes the stationary values along the principal directions of strain and the stationary/extreme values are precisely the principal strains.

Remark : **Let M be the square matrix with eigenvectors of the strain tensor e_{ij} as columns. That is**

$$\mathbf{M} = \begin{bmatrix} \overset{1}{A}_1 & \overset{2}{A}_1 & \overset{3}{A}_1 \\ \overset{1}{A}_2 & \overset{2}{A}_2 & \overset{3}{A}_2 \\ \overset{1}{A}_3 & \overset{2}{A}_3 & \overset{3}{A}_3 \end{bmatrix}$$

Then $$e_{ij} \overset{1}{A_j} = e_1 \overset{1}{A_i} \,,$$

$$e_{ij} \overset{2}{A_j} = e_2 \overset{2}{A_i} \,,$$

$$e_{ij} \overset{3}{A_j} = e_3 \overset{3}{A_i}$$

The matrix M is called the modal matrix **of strain tensor e_{ij}.**

Let

$$E = (e_{ij}) \,,\; D = \text{dia}(e_1 , e_2 , e_3).$$

Then , we find

$$E\,M = M\,D$$

or $$M^{-1} E\,M = D\,.$$

This shows that the matrices E and D are similar.

We know that two similar matrices have the same eigenvalues. Therefore , the characteristic equation associated with $M^{-1}EM$ is the same as the one associated with E. Consequently , eigenvalues of E and D are identical.

Question : **Show that , in general , at any point of the elastic body there exists (at least) three mutually perpendicular principal directions of strain due to an infinitesimal linear deformation.**

Solution : **Let e_1 , e_2 , e_3 be the three principal strains of the strain tensor e_{ij}. Then , they are the roots of the cubic equation**

$$(e - e_1)\,(e - e_2)\,(e - e_3) = 0\,,$$

and $$e_1 + e_2 + e_3 = e_{11} + e_{22} + e_{33} = e_{ii}\,,$$

$$e_1\, e_2 + e_2\, e_3 + e_3\, e_1 = \frac{1}{2}\,(e_{ii}\, e_{jj} - e_{ij}\, e_{ji})\,,$$

$$e_1\, e_2\, e_3 = |\, e_{ij}\, | = \in_{ijk} e_{1i}\, e_{2j}\, e_{3k}\,.$$

We further assume that coordinate axes coincide with the principal directions of strain. Then , the strain components are given by

$$e_{11} = e_1 \,,\; e_{22} = e_2 \,,\; e_{33} = e_3 \,,$$

$$e_{12} = e_{13} = e_{23} = 0\,,$$

and the strain quadric of Cauchy becomes

$$e_1 x_1^2 + e_2 x_2^2 + e_3 x_3^2 = \pm k^2. \qquad (1)$$

Now , we consider the following three possible cases for principal strains.

Case 1 : **When $e_1 \neq e_2 \neq e_3$. In this case , it is obvious that there exists three mutually orthogonal eigenvectors of the second order real symmetric strain tensor e_{ij}. These eigenvectors are precisely the three principal directions that are mutually orthogonal.**

Case 2 : **When $e_1 \neq e_2 = e_3$.**

Let $\overset{1}{A}_i$ **and** $\overset{2}{A}_i$ **be the corresponding principal orthogonal directions corresponding to strains (distinct) e_1 and e_2 , respectively. Then**

$$e_{ij} \overset{1}{A}_j = e_1 \overset{1}{A}_i ,$$

$$e_{ij} \overset{2}{A}_j = e_2 \overset{2}{A}_i \qquad (2)$$

Let p_i be a vector orthogonal to both $\overset{1}{A}_i$ **and** $\overset{2}{A}_i$ **. Then**

$$p_i \overset{1}{A}_i = p_i \overset{2}{A}_i = 0. \qquad (3)$$

Let $$e_{ij} p_i = q_j . \qquad (4)$$

Then $$q_j \overset{1}{A}_j = (e_{ij} p_i) \overset{1}{A}_j = (e_{ij} \overset{1}{A}_j) p_i = e_1 \overset{1}{A}_i p_i = 0 \qquad (5a)$$

Similarly $$q_j \overset{2}{A}_j = 0 . \qquad (5b)$$

This shows that the vector q_j is orthogonal to both $\overset{1}{A}_j$ **and** $\overset{2}{A}_j$ **. Hence , the vectors q_i and p_i must be parallel. Let**

$$q_i = \alpha p_i , \qquad (6)$$

for some scalar α. From equations (4) and (6) , we write

$$e_{ij} p_j = q_i = \alpha p_i , \qquad (7)$$

which shows that the scalar α is an eigenvalue/principal strain of the strain tensor e_{ij} with corresponding principal direction p_i .

Since e_{ij} has only three principal strains e_1 , e_2 , α and two of these are equal , so α must be equal to $e_2 = e_3$.

We denote the normalized form of p_i by $\overset{3}{A_i}$.

This shows the existence of three mutually orthogonal principal directions in this case.

Further , let v_i be any vector normal to $\overset{1}{A_i}$. Then v_i lies in the plane containing principal directions $\overset{2}{A_i}$ and $\overset{3}{A_i}$. Let

$$v_i = k_1 \overset{2}{A_i} + k_2 \overset{3}{A_i} \qquad \text{for some scalars } k_1 \text{ and } k_2 \quad (8)$$

Now

$$\begin{aligned} e_{ij} v_j &= e_{ij} (k_1 \overset{2}{A_j} + k_2 \overset{3}{A_j}) \\ &= k_1(e_{ij} \overset{2}{A_j}) + k_2(e_{ij} \overset{3}{A_j}) \\ &= k_1(e_2 \overset{2}{A_i}) + k_2(e_3 \overset{3}{A_i}) \\ &= e_2[k_1 \overset{2}{A_i} + k_2 \overset{3}{A_i}] \qquad (\because e_2 = e_3) \\ &= e_2 v_i \end{aligned}$$

This shows that the direction v_i is also a principal direction corresponding to principal strain e_2. Thus, in this case , any two orthogonal(mutually) vectors lying on the plane normal to $\overset{1}{A_i}$ can be chosen as the other two principal directions. In this case , the strain quadric surface is a surface of revolution.

Case 3: **When $e_1 = e_2 = e_3$, then the strain quadric of Cauchy is a sphere with equation**

$$e_1(x_1^2 + x_2^2 + x_3^2) = \pm k^2 ,$$

or

$$x_1^2 + x_2^2 + x_3^2 = \pm \frac{k^2}{e_1}$$

and any three mutually orthogonal directions can be taken as the coordinate axes which are coincident with principal directions of strain.

Hence , the result.

3.12 GENERAL INFINITESIMAL DEFORMATION

Now we consider the general functional transformation and its relation to the linear deformation. Consider an arbitrary material point $P^\circ(x_i^\circ)$ in a continuous medium. Let the same material point assume after deformation the point $Q^\circ(\xi_i^\circ)$. Then

$$\xi_i^\circ = x_i^\circ + u_i(x_1^\circ , x_2^\circ , x_3^\circ) , \qquad (1)$$

where u_i are the components of the displacement vector $\overline{P^\circ Q^\circ}$. We assume that u_1 , u_2 , u_3 , as well as their partial derivatives are continuous functions.

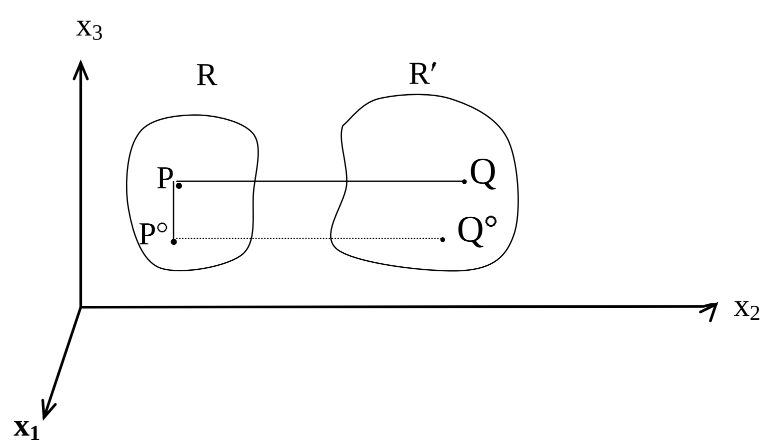


Fig. (3.17)

The nature of the deformation in the neighbourhood of the point P° can be determined by considering the change **in the vector $\overline{P^\circ P}$ = A_i in the undeformed state , where $P(x_1 , x_2 , x_3)$ is an arbitrary** neighbouring **point of P°.**

Let $Q(\xi_1 , \xi_2 , \xi_3)$ be the deformed position of P. Then the displacement u_i at the point P is

$$u_i(x_1 , x_2 , x_3) = \xi_i - x_i. \qquad (2)$$

The vector $$A_i = x_i - x_i^\circ , \qquad (3)$$

has now deformed to the vector

$$\xi_i - \xi_i^\circ = A_i' \text{ (say).} \tag{4}$$

Therefore ,

$$\delta A_i = A_i' - A_i$$

$$= (\xi_i - \xi_i^\circ) - (x_i - x_i^\circ)$$

$$= (\xi_i - x_i) - (\xi_i^\circ - x_i^\circ)$$

$$= u_i(x_1, x_2, x_3) - u_i(x_1^\circ, x_2^\circ, x_3^\circ)$$

$$= u_i(x_1^\circ + A_1, x_2^\circ + A_2, x_3^\circ + A_3) - u_i(x_1^\circ, x_2^\circ, x_3^\circ)$$

$$= \left(\frac{\partial u_i}{\partial x_j}\right)_\circ A_j \quad , \tag{5}$$

plus the higher order terms of Taylor's series. The subscript ∘ indicates that the derivative is to be evaluated at the point P°.

If the region in the nbd. of P° is chosen sufficiently small , i.e. , if the vector A_i is sufficiently small , then the product terms like $A_i A_j$ may be ignored.

Ignoring the product terms and dropping the subscript ∘ in (5) , we write

$$\delta A_i = u_{i,j} A_j \quad , \tag{6}$$

where the symbol $u_{i,j}$ has been used for $\frac{\partial u_i}{\partial x_j}$.

Result (6) holds for small vectors A_i.

If we further assume that the displacements u_i as well as their partial derivatives are so small that their products can be neglected , **then the transformation (which is linear) given by (4) becomes infinitesimal in the nbd of the point (P°) under consideration and**

$$\delta A_i = \alpha_{ij} A_j \quad , \tag{7}$$

with

$$\alpha_{ij} = u_{i,j}. \tag{8}$$

Hence , all results discussed earlier are immediately applicable.

The transformation (6) can be splitted into pure deformation and rigid body motion as

$$\delta \mathbf{A_i} = \mathbf{u_{i,j}}\, \mathbf{A_j} = \left(\frac{u_{i,j}+u_{j,i}}{2} + \frac{u_{i,j}-u_{j,i}}{2}\right) \mathbf{A_j}$$

$$= \mathbf{e_{ij}\, A_j + w_{ij}\, A_j}\ , \qquad \textbf{(9)}$$

where $\mathbf{e_{ij}} = \frac{1}{2}(\mathbf{u_{i,j} + u_{j,i}})$, **(10)**

$$\mathbf{w_{ij}} = \frac{1}{2}(\mathbf{u_{i,j} - u_{j,i}})\ . \qquad \textbf{(11)}$$

The transformation

$$\delta \mathbf{A_i} = \mathbf{e_{ij}\, A_j}\ , \qquad \textbf{(12)}$$

represents pure deformation and

$$\delta \mathbf{A_i} = \mathbf{w_{ij}\, A_j}\ , \qquad \textbf{(13)}$$

represents rotation . In general , the transformation (9) is no longer homogeneous as both the strain components e_{ij} and components of rotation w_{ij} are functions of the coordinates. We find

$$\nu = \mathbf{e_{ii}} = \frac{\partial u_i}{\partial x_i} = \mathbf{u_{i,i}} = \mathbf{div}\ \bar{\mathbf{u}}. \qquad \textbf{(14)}$$

That is , the cubic dilatation is the divergence of the displacement vector $\bar{u}$ and it differs , in general , from point to point of the body.

The rotation vector w_i is given by

$$\mathbf{w_1 = w_{32}\ ,\ w_2 = w_{13}\ ,\ w_3 = w_{21}}\ . \qquad \textbf{(15)}$$

Question : **For the small linear deformation given by**

$$\bar{\mathbf{u}} = \alpha\, \mathbf{x_1\, x_2}(\hat{e}_1 + \hat{e}_2) + 2\alpha(\mathbf{x_1 + x_2})\, \mathbf{x_3}\, \hat{e}_3\ ,\ \alpha = \textbf{constant.}$$

find the strain tensor , the rotation and the rotation vector.

Solution: **We find**

$$\mathbf{u_1} = \alpha\, \mathbf{x_1\, x_2}\ ,\ \mathbf{u_2} = \alpha\, \mathbf{x_1\, x_2}\ ,\ \mathbf{u_3} = 2\alpha(\mathbf{x_1 + x_2})\mathbf{x_3}\ ,$$

$$\mathbf{e_{11}} = \frac{\partial u_1}{\partial x_1} = \alpha \mathbf{x_2}\ ,\ \mathbf{e_{22}} = \frac{\partial u_2}{\partial x_2} = \alpha \mathbf{x_1}\ ,\ \mathbf{e_{33}} = \frac{\partial u_3}{\partial x_3} = 2\alpha(\mathbf{x_1 + x_2})$$

$$\mathbf{e_{12}} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) = \frac{\alpha}{2}\,(\mathbf{x_1 + x_2}),$$

$$\mathbf{e_{13}} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = \alpha\, \mathbf{x_3},\quad \mathbf{e_{23}} = \alpha\, \mathbf{x_3}$$

Hence

$$(\mathbf{e_{ij}}) = \alpha\begin{bmatrix} x_2 & (x_1+x_2)/2 & x_3 \\ (x_1+x_2)/2 & x_1 & x_3 \\ x_3 & x_3 & 2(x_1+x_2) \end{bmatrix}.$$

We know that

$$\mathbf{w_{ij}} = \frac{1}{2}\left(\frac{\partial \mathrm{u_i}}{\partial \mathrm{x_j}} - \frac{\partial \mathrm{u_j}}{\partial \mathrm{x_i}}\right)$$

We find

$$\mathbf{w_{11} = w_{22} = w_{33} = 0},$$

$$\mathbf{w_{12}} = \frac{\alpha}{2}\,[\mathbf{x_1 - x_2}] = -\mathbf{w_{21}},$$

$$\mathbf{w_{13}} = -\alpha \mathbf{x_3} = -\mathbf{w_{31}},$$

$$\mathbf{w_{23}} = -\alpha \mathbf{x_3} = -\mathbf{w_{32}},$$

Therefore

$$(\mathbf{w_{ij}}) = \alpha\begin{bmatrix} 0 & (x_1-x_2)/2 & -x_3 \\ -(x_1-x_2)/2 & 0 & -x_3 \\ x_3 & x_3 & 0 \end{bmatrix}. \qquad \mathbf{(2)}$$

The rotation vector $\bar{\mathbf{w}} = \mathbf{w_i}$ **is given by** $\mathbf{w_i} = \in_{\mathbf{ijk}} \mathbf{u_{k,j}}$. **We find**

$$\mathbf{w_1 = w_{32}} = \alpha\, \mathbf{x_3},\ \mathbf{w_2 = w_{13}} = -\alpha \mathbf{x_3},\ \mathbf{w_3 = w_{21}} = \frac{\alpha}{2}\,(\mathbf{x_2 - x_1}).$$

So

$$\bar{\mathbf{w}} = \alpha\, \mathbf{x_3}(\hat{e}_1 - \hat{e}_2) + \frac{\alpha}{2}\,(\mathbf{x_2 - x_1})\ \hat{e}_3.$$

Exercise 1: **For small deformation defined by the following displacements , find the strain tensor , rotation tensor and rotation vector.**

(i) $u_1 = -\alpha\, x_2\, x_3$, $u_2 = \alpha\, x_1\, x_2$, $u_3 = 0$.

(ii) $u_1 = \alpha^2(x_1 - x_3)^2$, $u_2 = \alpha^2(x_2 + x_3)^2$, $u_3 = -\alpha\, x_1\, x_2$, α **= constant.**

Exercise 2: **The displacement components are given by**

$$u = -yz\ ,\ v = xz\ ,\ w = \phi(x\ ,\ y)$$

calculate the strain components.

Exercise 3: **Given the displacements**

$$u = 3\,x^2\, y\ ,\ v = y^2 + 6\,xz\ ,\ w = 6z^2 + 2\,yz\ ,$$

calculate the strain components at the point (1 , 0 , 2). What is the extension of a line element (parallel to the x – axis) at this point ?

Exercise 4: **Find the strain components and rotation components for the small displacement components given below**

(a) Uniform dilatation – $u = e\,x$, $v = e\,y$, $w = e\,z$,

(b) Simple extension - $u = e\,x$, $v = w = 0$,

(c) Shearing strain - $u = 2\,sy$, $v = w = 0$,

(d) Plane strain - $u = u(x\ ,\ y)$, $v = v(x\ ,\ y)$, $w = 0$.

3.13 SAINT-VENANT'S EQUATIONS OF COMPATIBILITY

By definition , the strain components e_{ij} in terms of displacement components u_i are given by

$$e_{ij} = \frac{1}{2}[u_{i,j} + u_{j,i}] \qquad \textbf{(1)}$$

Equation (1) is used to find the components of strain if the components of displacement are given. However , if the components of strain , e_{ij} , are given then equation (1) is a set of 6 partial differential equations in the three unknowns u_1 , u_2 , u_3. Therefore , the system (1) will not have a single – valued solution for u_i unless given strains e_{ij} satisfy certain conditions which are known as the conditions of compatibility or equations of compatibility.

Geometrical meaning of Conditions of Compatibility

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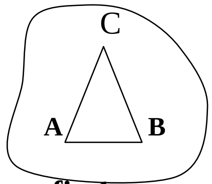
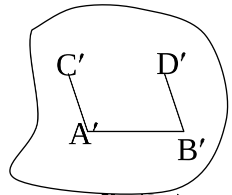


fig 1



fig(2a)

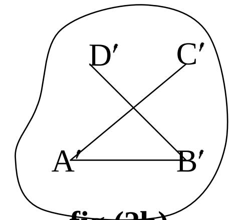


fig (2b)

Fig (1) shows a portion of the material of the body in the undeformed state in the form of a continuous triangle ABC. If we deform the body by an arbitrarily specified strain field then we might end up at the points C′ and D′ with a gap between them , after deformation , as shown in fig(2a) or with overlapping material as shown in fig(2b).

For a single valued continuous solution to exist the points C′ and D′ must be the same in the strained state. This cannot be guaranteed unless the specified strain components satisfy certain conditions , known as the conditions (or relations or equations) of compatibility.

Equations of Compatibility

We have

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \,. \tag{1}$$

So ,

$$e_{ij,kl} = \frac{1}{2}(u_{i,jkl} + u_{j,ikl}) \,. \tag{2}$$

Interchanging i with k and j with *l* in equation (2) , we write

$$e_{kl,ij} = \frac{1}{2}(u_{k,lij} + u_{l,kij}) \,. \tag{3}$$

Adding (2) and (3) , we get

$$e_{ij,kl} + e_{kl,ij} = \frac{1}{2}[u_{i,jkl} + u_{j,ikl} + u_{k,lij} + u_{l,kij}] \,. \tag{4}$$

Interchanging i and *l* in (4) , we get

$$e_{lj,ki} + e_{ki,lj} = \frac{1}{2}[u_{l,jki} + u_{j,lki} + u_{k,lij} + u_{i,ljk}] \,. \tag{5}$$

From (4) and (5) , we obtain

$$e_{ij,kl} + e_{kl,ij} = e_{lj,ki} + e_{ki,lj} \,,$$

or

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0 \,. \tag{6}$$

These equations are known as equations of compatibility**.**

These equations are necessary conditions **for the existence of a single valued continuous displacement field. These are 81 equations in number. Because of symmetry in indicies i , j, and k ,** *l*; **some of these equations are identically satisfied and some are repetitions.**

Only 6 out of these 81 equations are essential. These equations were first obtained by Saint – Venant in 1860**.**

A strain tensor e_{ij} that satisfies these conditions is referred to as a possible strain tensor.

Show that the conditions of compatibility are sufficient for the existence of a single valued continuous displacement field.

Let $P^\circ(x_i^\circ)$ be some point of a simply connected region at which the displacements u_i° and rotations w_{ij}° are known. The displacements u_i of an arbitrary point $P'(x_i')$ can be obtained in terms of the known functions e_{ij} by means of a line integral along a continuous curve C joining the points P_o and P'.

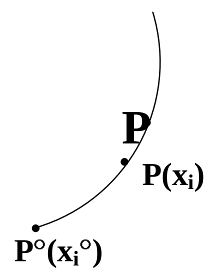


Fig. (3.18)

$$u_j(x_1', x_2', x_3') = u_j^\circ(x_1^\circ, x_2^\circ, x_3^\circ) + \int_{P^\circ}^{P'} du_j \quad . \qquad (7)$$

If the process of deformation does not create cracks or holes , i.e. , if the body remains continuous , the displacements u_j' should be independent of the path of integration.

That is , u_j' should have the same value regardless of whether the integration is along curve C or any other curve. We write

$$du_j = \frac{\partial u_j}{\partial x_k} dx_k = u_{j,k}\, dx_k = (e_{jk} + w_{jk})\, dx_k \,. \qquad (8)$$

Therefore ,

$$u_j' = u_j^\circ + \int_{P^\circ}^{P'} e_{jk}\, dx_k + \int_{P^\circ}^{P'} w_{jk}\, dx_k\,,$$ **(P(x_k) being point the joining curve).**

(9)

Integrating by parts the second integral , we write

$$\int_{P^\circ}^{P'} w_{jk}\, dx_k = \int_{P^\circ}^{P'} w_{jk}\, d(x_k - x_k') \qquad \text{(The point } P'(x_k') \text{ being fixed so } dx_k'=0)$$

$$= \{(x_k - x_k')\, w_{jk}\}_{P^\circ}^{P'} - \int_{P^\circ}^{P'} (x_k - x_k')\, w_{jk,l}\, dx_l$$

$$= (x_k' - x_k^\circ) w_{jk}^\circ + \int_{P^\circ}^{P'} (x_k' - x_k)\, w_{jk,l}\, dx_l \quad . \tag{10}$$

From equations (9) and (10) , we write

$$u_j(x_1', x_2', x_3') = u_j^\circ + (x_k' - x_k^\circ) w_{jk}^\circ + \int_{P^\circ}^{P'} e_{jk}\, dx_k + \int_{P^\circ}^{P'} (x_k' - x_k)\, w_{jk,l}\, dx_l$$

$$= u_j^\circ + (x_k' - x_k^\circ)\, w_{jk}^\circ + \int_{P^\circ}^{P'} [e_{jl} + (x_k' - x_k)\, w_{jk,l}]\, dx_l \quad , \tag{11}$$

where the dummy index k of e_{jk} has been changed to *l*.

But $$w_{jk,l} = \frac{1}{2} \frac{\partial}{\partial x_l} [u_{j,k} - u_{k,j}]$$

$$= \frac{1}{2} [u_{j,kl} - u_{k,jl}]$$

$$= \frac{1}{2} [u_{j,kl} + u_{l,jk}] - \frac{1}{2} [u_{l,jk} + u_{k,jl}]$$

$$= e_{jl,k} - e_{lk,j} \tag{12}$$

Using (12) , equation (11) becomes

$$u_j(x_1', x_2', x_3') = u_j^\circ + (x_k' - x_k^\circ)\, w_{jk}^\circ + \int_{P^\circ}^{P'} [e_{jl} + \{x_k' - x_k\}\{e_{jl,k} - e_{kl,j}\}]\, dx_l$$

$$= u_j^\circ + (x_k' - x_k^\circ)\, w_{jk}^\circ + \int_{P^\circ}^{P'} U_{jl}\, dx_l \quad , \qquad (13)$$

where for convenience we have set

$$U_{jl} = e_{jl} + (x_k' - x_k)\,(e_{jl,k} - e_{kl,j}) \quad , \qquad (14)$$

which is a known function as e_{ij} are known.

The first two terms in the right side of equation (13) are independent of the path of integration. From the theory of line integrals, the third term become independent of the path of integration when the integrands $U_{jl}\, dx_l$ must be exact differentials.

Therefore , if the displacements $u_i(x_1' , x_2' , x_3')$ are to be independent of the path of integration , we must have

$$\frac{\partial U_{jl}}{\partial x_l} = \frac{\partial U_{ji}}{\partial x_l} \ , \qquad \text{for } i , j , l = 1 , 2 , 3. \qquad (15)$$

Now

$$U_{jl,i} = e_{jl,i} + (x_k' - x_k)\,(e_{jl,ki} - e_{kl,ji}) - \delta_{ki}\,(e_{jl,k} - e_{kl,j})$$

$$= e_{jl,i} - e_{jl,i} + e_{il,j} + (x_k' - x_k)\,(e_{jl,ki} - e_{kl,ji}) \ , \qquad (16a)$$

and

$$U_{ji,l} = e_{ji,l} + (x_k' - x_k)\,(e_{ji,kl} - e_{ki,jl}) - \delta_{kl}(e_{ji,k} - e_{ki,j})$$

$$= e_{ji,l} - e_{ji,l} + e_{li,j} + (x_k' - x_k)\,(e_{ji,kl} - e_{ki,jl}) \quad . \qquad (16b)$$

Therefore , equations (15) and (16 a , b) yields

$$(x_k' - x_k)\,[e_{jl,ki} - e_{kl,ji} - e_{ji,kl} + e_{ki,jl}] = 0 \ .$$

Since this is true for an arbitrary choice of $x_k' - x_k$ (as P′ is arbitrary) , it follows that

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ki} = 0 \ , \qquad (17)$$

which is true as these are the compatibility relations.

Hence , the displacement (7) independent of the path of Integration. Thus , the compatibility conditions (6) are sufficient also.

Remark 1: **The compatibility conditions (6) are necessary and sufficient** for the existence of a single valued continuous displacement field **when the strain components are prescribed.**

In detailed form , these 6 conditions are

$$\frac{\partial^2 e_{11}}{\partial x_2 \partial x_3} = \frac{\partial}{\partial x_1}\left(\frac{-\partial e_{23}}{\partial x_1} + \frac{\partial e_{31}}{\partial x_2} + \frac{\partial e_{12}}{\partial x_3} \right),$$

$$\frac{\partial^2 e_{22}}{\partial x_3 \partial x_1} = \frac{\partial}{\partial x_2}\left(\frac{-\partial e_{31}}{\partial x_2} + \frac{\partial e_{12}}{\partial x_3} + \frac{\partial e_{23}}{\partial x_1} \right),$$

$$\frac{\partial^2 e_{33}}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_3}\left(\frac{-\partial e_{12}}{\partial x_3} + \frac{\partial e_{23}}{\partial x_1} + \frac{\partial e_{31}}{\partial x_2} \right),$$

$$\frac{2\partial^2 e_{12}}{\partial x_1 \partial x_2} = \frac{\partial^2 e_{11}}{\partial x_2^{\,2}} + \frac{\partial^2 e_{22}}{\partial x_1^{\,2}} ,$$

$$\frac{2\partial^2 e_{23}}{\partial x_2 \partial x_3} = \frac{\partial^2 e_{22}}{\partial x_3^{\,2}} + \frac{\partial^2 e_{33}}{\partial x_2^{\,2}} ,$$

$$\frac{2\partial^2 e_{31}}{\partial x_3 \partial x_1} = \frac{\partial^2 e_{33}}{\partial x_1^{\,2}} + \frac{\partial^2 e_{11}}{\partial x_3^{\,2}} .$$

These are the necessary and sufficient conditions for the strain components e_{ij} to give single valued displacements u_i for a simply connected region.

Definition : **A region space is said to be** simply connected **if an arbitrary closed curve lying in the region can be** shrunk to a point **,** by continuous deformation , without passing outside of the boundaries.

Remark 2: **The specification of the strains e_{ij} only does not determine the displacements u_i uniquely because the strains e_{ij} characterize only the pure deformation of an elastic medium in the neighbourhood of the point x_i.**

The displacements u_i may involve rigid body motions which do not affect e_{ij}.

Example 1 : **(i) Find the compatibility condition for the strain tensor e_{ij} if e_{11} , e_{22} , e_{12} are independent of x_3 and $e_{31} = e_{32} = e_{33} = 0$.**

(ii) Find the condition under which the following are possible strain components.

$$e_{11} = k(x_1^2 - x_2^2)\ ,\ e_{12} = k'\, x_1 x_2\ ,\ e_{22} = k\, x_1 x_2\ ,$$

$$e_{31} = e_{32} = e_{33} = 0\ ,\ k\ \&\ k' \text{ are constants}$$

(iii) when e_{ij} given above are possible strain components , find the corresponding displacements , given that $u_3 = 0$.

Solution : **(i) We verify that all the compatibility conditions except one are obviously satisfied. The only compatibility condition to be satisfied by e_{ij} is**

$$e_{11,22} + e_{22,11} = 2\, e_{12,12}\ . \qquad (1)$$

(ii) Five conditions are trivially satisfied. The remaining condition (1) is satisfied iff

$$k' = k$$

as $$e_{11\ 22} = -2k\ ,\ e_{12,12} = k'\ ,\ e_{22,11} = 0.$$

(iii) We find

$$e_{11} = u_{1,1} = k(x_1^2 - x_2^2)\ ,\ u_{2,2} = k\, x_1 x_2\ ,\ u_{1,2} + u_{2,1} = -2k\, x_1 x_2\ ,\ (\because k' = -k)$$

$$u_{2,3} = u_{1,3} = 0.$$

This shows that the displacement components u_1 and u_2 are independent of x_3. We find (exercise)

$$u_1 = \frac{1}{6}(2x_1^3 - 6x_1 x_2^2 + x_2^3) - c\, x_2 + c_1\ ,$$

$$u_2 = \frac{1}{2} k\, x_1 x_2^2 + c\, x_1 + c_2\ ,$$

where c_1 , c_2 and c are constants.

Example 2: **Show that the following are not possible strain components**

$$e_{11} = k(x_1^2 + x_2^2)\ ,\ e_{22} = k(x_2^2 + x_3^2)\ ,\ e_{33} = 0\ ,$$

$$e_{12} = k'\, x_1 x_2 x_3\ ,\ e_{13} = e_{23} = 0\ ,\ k\ \&\ k' \text{ being constants.}$$

Solution : T**he given components e_{ij} are possible strain components if each of the six compatibility conditions is satisfied. On substitution , we find**

$$2k = 2k'\, x_3.$$

This can't be satisfied for $x_3 \neq 0$.

For $x_3 = 0$, this gives k = 0 and then all e_{ij} vanish.

Hence , the given e_{ij} are not possible strain components.

Exercise 1: **Consider a linear strain field associated with a simply connected region R such that $e_{11} = A\, x_2^2$, $e_{22} = A\, x_1^2$, e_{12} B $x_1\, x_2$, $e_{13} = e_{23} = e_{33} = 0$. Find the relationship between constant A and B such that it is possible to obtain a single – valued continuous displacement field which corresponds to the given strain field.**

Exercise 2: **Show by differentiation of the strain displacement relations that the compatibility conditions are necessary conditions for the existence of continuous single – valued displacements.**

Exercise 3: **Is the following state of strain possible ? (c = constant)**

$e_{11} = c(x_1^2 + x_2^2)\, x_3$, $e_{22} = c\, x_2^2\, x_3$, $e_{12} = 2c\, x_1\, x_2\, x_3$, $e_{31} = e_{32} = e_{33} = 0$.

Exercise 4: **Show that the equations of compatibility represents a set of necessary and sufficient conditions for the existence of single – valued displacements. Derive the equations of compatibility for plane strain.**

Exercise 5: **If $e_{11} = e_{22} = e_{12} = e_{33} = 0$, $e_{13} = \phi_{,2}$ and $e_{23} = \phi_{,1}$; where ϕ is a function of x_1 and x_2 , show that ϕ must satisfy the equation**

$$\nabla^2 \phi = \text{constant}$$

Exercise 6: **If e_{13} and e_{23} are the only non – zero strain components and e_{13} and e_{23} are independent of x_3 , show that the compatibility conditions may be reduced to the following single condition**

$$e_{13,2} - e_{23,1} = \text{constant.}$$

Exercise 7: **Find which of the following values of e_{ij} are possible linear strains.**

(i) $e_{11} = \alpha(x_1^2 + x_2^2)$, $e_{22} = \alpha x_2^2$, $e_{12} = 2\alpha\, x_1\, x_2$, $e_{31} = e_{32} = e_{33} = 0$, α = constant.

(ii) $$(e_{ij}) = \begin{bmatrix} x_1 + x_2 & x_1 & x_2 \\ x_1 & x_2 + x_3 & x_3 \\ x_2 & x_3 & x_1 + x_3 \end{bmatrix}$$

compute the displacements in the case (i).

3.14 FINITE DEFORMATIONS

All the results reported in the preceding sections of this chapter were that of the classical theory of infinitesimal strains. Infinitesimal transformations permits the application of the principle of superposition of effects.

Finite deformations are those deformations in which the displacements u_i together with their derivatives are no longer small.

Consider an aggregate of particles in a continuous medium. We shall use the same reference frame for the location of particles in the deformed and undeformed states.

Let the coordinates of a particle lying on a curve C_o , before deformation , be denoted by (a_1 , a_2 , a_3), and let the coordinates of the same particle after deformation (now lying on some curve C) be (x_1 , x_2 , x_3).

Then the elements of arc of the curve C_o and C are given , respectively , by

$$d s_o^2 = d a_i \, d a_i , \qquad (1)$$

and

$$ds^2 = d x_i \, d x_i. \qquad (2)$$

We consider first the Eulerian description of the strain and write

$$a_i = a_i (x_1 , x_2 , x_3). \qquad (3)$$

Then

$$da_i = a_{i,j} \, d x_j = a_{i,k} \, dx_k . \qquad (4)$$

Substituting from (4) into (1) , we write

$$ds_o^2 = a_{i,j} \, a_{i,k} \, dx_j \, dx_k . \qquad (5)$$

Using the substitution tensor , equation (2) can be rewritten as

$$d s^2 = \delta_{jk} \, dx_j \, dx_k . \qquad (6)$$

we know that the measure of the strain is the difference $ds^2 – ds_o^2$.

From equations (5) and (6) , we get

$$ds^2 – ds_o^2 = (\delta_{jk} – a_{i,j} \, a_{i,k}) \, dx_j \, dx_k$$

$$= 2\eta_{jk}\, dx_j\, dx_k\,, \qquad (7)$$

where

$$2\eta_{jk} = \delta_{jk} - a_{i,j}\, a_{i,k}. \qquad (8)$$

We now write the strain components η_{jk} in terms of the displacement components u_i , where

$$u_i = x_i - a_i. \qquad (9)$$

This gives

$$a_i = x_i - u_i.$$

Hence

$$a_{i,j} = \delta_{ij} - u_{i,j}\,, \qquad (10)$$

$$a_{i,k} = \delta_{ik} - u_{i,k}\,. \qquad (11)$$

Equations (8) , (10) and (11) yield

$$\begin{aligned} 2\eta_{jk} &= \delta_{jk} - (\delta_{ij} - u_{i,j})(\delta_{ik} - u_{i,k}) \\ &= \delta_{jk} - [\delta_{jk} - u_{k,j} - u_{j,k} + u_{i,j}\, u_{i,k}] \\ &= (u_{j,k} + u_{k,j}) - u_{i,j}\, u_{i,k}. \qquad (12) \end{aligned}$$

The quantities η_{jk} are called the Eulerian strain components.

If , on the other hand , Lagrangian coordinates are used , and equations of transformation are of the form

$$x_i = x_i\,(a_1\,, a_2\,, a_3)\,, \qquad (13)$$

then

$$dx_i = x_{i,j}\, da_i = x_{i,k}\, da_k\,, \qquad (14)$$

and

$$ds^2 = x_{i,j}\, x_{i,k}\, da_j\, da_k\,, \qquad (15)$$

while

$$ds_o^2 = \delta_{jk}\, da_j\, da_k. \qquad (16)$$

The Lagrangian components of strain $\in_{jk}$ are defined by

$$ds^2 - ds_o^2 = 2\in_{jk}\, da_j\, da_k\,. \tag{17}$$

Since

$$x_i = a_i + u_i\,, \tag{18}$$

therefore ,

$$x_{i,j} = \delta_{ij} + u_{i,j}\ ,$$

$$x_{i,k} = \delta_{ik} + u_{i,k}\,.$$

Now

$$\begin{aligned} ds^2 - ds_o^2 &= (x_{i,j}\, x_{i,k} - \delta_{jk})\, da_j\, da_k \\ &= [(\delta_{ij} + u_{i,j})\,(\delta_{ik} + u_{i,k}) - \delta_{jk}]\, da_j\, da_{k=} \\ &= (u_{j,k} + u_{k,j} + u_{i,j}\, u_{i,k})\, da_j\, da_k. \end{aligned} \tag{19}$$

Equations (17) and (19) give

$$2\in_{jk} = u_{j,k} + u_{k,j} + u_{i,j}\, u_{i,k}. \tag{20}$$

It is mentioned here that the differentiation in (12) is carried out with respect to the variables x_i , while in (20) the a_i are regarded as the independent variables.

To make the difference explicitly clear , we write out the typical expressions η_{jk} and $\in_{jk}$ in unabridged notation,

$$\eta_{xx} = \frac{\partial u}{\partial x} - \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]. \tag{21}$$

$$2\eta_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right), \tag{22}$$

$$\in_{xx} = \frac{\partial u}{\partial a} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial a}\right)^2 + \left(\frac{\partial v}{\partial b}\right)^2 + \left(\frac{\partial w}{\partial a}\right)^2\right], \tag{23}$$

$$2\in_{xy} = \left(\frac{\partial u}{\partial b} + \frac{\partial v}{\partial a}\right) + \left(\frac{\partial u}{\partial a}\frac{\partial u}{\partial b} + \frac{\partial v}{\partial a}\frac{\partial v}{\partial b} + \frac{\partial w}{\partial a}\frac{\partial w}{\partial b}\right). \tag{24}$$

When the strain components are large , it is no longer possible to give simple geometrical interpretations of the strains $\in_{jk}$ and η_{jk}.

Now , we consider some particular cases.

Case I: **Consider a line element with**

$$ds_o = d\,a_1 ,\ d\,a_2 = 0 ,\ d\,a_3 = 0 . \tag{25}$$

Define the extension E_1 of this element by

$$E_1 = \frac{ds - ds_o}{ds_o} .$$

Then

$$ds = (1 + E_1)\, ds_o , \tag{26}$$

and consequently

$$ds^2 - ds_o^2 = 2 \in_{jk} d\,a_j\, d\,a_k$$

$$= 2 \in_{11} d\,a_j^2 . \tag{27}$$

Equations (25) to (27) yield

$$(1 + E_1)^2 - 1 = 2 \in_{11}$$

or

$$E_1 = \sqrt{1 + 2\in_{11}} - 1 . \tag{28}$$

When the strain $\in_{11}$ is small , (28) reduces to

$$E_1 \cong \in_{11} ,$$

as was shown in the discussion of infinitesimal strains.

Case II: **Consider next two line elements**

$$ds_o = da_2 ,\ da_1 = 0 ,\ da_3 = 0 , \tag{29}$$

and

$$d\bar{s}_o = d\,\bar{a}_3 ,\ d\,\bar{a}_1 = d\,\bar{a}_2 = 0. \tag{30}$$

These two elements lie initially along the a_2 – and a_3- axes.

Let θ denote the angle between the corresponding deformed elements dx_i and $d\bar{x}_i$, of lengths ds and $d\bar{s}$, respectively. Then

$$\mathbf{ds}\ d\bar{s}\ \mathbf{cos}\,\theta = \mathbf{d}x_i\ d\bar{x}_i$$
$$= x_{i,\alpha}\ \bar{x}_{i,\beta}\ \mathbf{da}_\alpha\ d\bar{a}_\beta$$
$$= x_{i,2}\ \bar{x}_{i,3}\ \mathbf{da}_2\ d\bar{a}_3$$
$$= 2 \in_{23} \mathbf{da}_2\ d\bar{a}_3 . \qquad \textbf{(31)}$$

Let

$$\alpha_{23} = \frac{\pi}{2} - \theta , \qquad \textbf{(32)}$$

denotes the change in the right angle between the line elements in the initial state. Then , we have

$$\sin \alpha_{23} = 2 \in_{23} \left(\frac{da_2}{ds}\right)\left(\frac{d\bar{a}_3}{d\bar{s}}\right). \qquad \textbf{(33)}$$

$$= \frac{2 \in_{23}}{\sqrt{1+2\in_{22}}\sqrt{1+2\in_{33}}} , \qquad \textbf{(34)}$$

using relations in (26) and (28).

Again , if the strains $\in_{ij}$ are so small that their products can be neglected , then

$$\alpha_{23} \cong 2 \in_{23} , \qquad \textbf{(35)}$$

as proved earlier for infinitesimal strains.

Remark: **If the displacements and their derivatives are small , then it is immaterial whether the derivatives of the displacements are calculated at the position of a point before or after deformation. In this case , we may neglect the nonlinear terms in the partial derivatives in (12) and (20) and reduce both sets of formulas to**

$$2\eta_{jk} = u_{j,k} + u_{k,j} = 2\in_{jk} ,$$

which were obtained for an infinitesimal transformation.

It should be emphasized that the transformations of finite homogeneous strain are not in general commutative and that the simple superposition of effects is no longer applicable to finite deformation.

Chapter-4
Constitutive Equations of Linear Elasticity

4.1. INTRODUCTION

It is a fact of experience that deformation of a solid body induces stresses within. The relationship between stress and deformation is expressed as a constitutive relation for the material and depends on the material properties and also on other physical observables like temperative and , perhaps , the electromagnetic field.

An elastic deformation is defined to be one in which the stress is determined by the current value of the strain only , and not on rate of strain or strain history : $\tau = \tau(e)$.

An elastic solid that undergoes only an infinitesimal **deformation and for which the governing material is** linear **is called** a linear elastic solid or Hookean solid**.**

From experimental observations , it is known that , under normal loadings , many structural materials such as metals , concrete , wood and rocks behave as linear elastic solids.

The classical theory of elasticity (or linear theory) serves as an excellent model for studying the mechanical behaviour of a wide variety of such solid materials.

Hook's law : **In 1678 , Robert Hook , on experimental grounds , stated that the extension is proportional to the force. Cauchy in 1822 generalized Hook law for the deformation of elastic solids. According to Cauchy , " Each component of stress at any point of an elastic body is a linear function of the components of strain at the point".**

This law is now known as Generalized Hooke's Law. **Here , linearity means that stress – strain relations are linear.**

4.2. GENERALIZED HOOKE'S LAW

In general , we write the following set of linear relations

$$\tau_{11} = c_{1111}\, e_{11} + c_{1112}\, e_{12} + \ldots\ldots\ldots. + c_{1133}\, e_{33} \quad ,$$

$$\tau_{12} = c_{1211}\, e_{11} + c_{1212}\, e_{12} + \ldots\ldots\ldots.. + c_{1233}\, e_{33} \quad ,$$

$$\tau_{33} = c_{3311}\, e_{11} + c_{3312}\, e_{12} + \ldots\ldots\ldots.. + c_{3333}\, e_{33} \quad ,$$

or

$$\tau_{ij} = c_{ijkl}\, e_{kl} \quad , \qquad (1)$$

where τ_{ij} is the stress tensor and e_{kl} is the strain tensor. The coefficients , which are $81 = 3^4$ in number , are called elastic moduli.

In general , these coefficients depend on the physical properties of the medium and are independent of the strain components e_{ij}.

We suppose that relations (1) hold at every point of the medium and at every instant of time and are solvable for e_{ij} in terms of τ_{ij}.

From (1) , it follows that τ_{ij} are all zero whenever all e_{ij} are 0.

It means that in the initial unstrained state the body is unstressed. From quotient law for tensors , relation (1) shows that c_{ijkl} are components of a fourth – order tensor.

This tensor is called elasticity tensor. **Since e_{ij} are dimensionless quantities , it follows that elastic moduli c_{ijkl} have the same dimensions as the stresses (force/Area).**

If , however , c_{ijkl} do not change throughout the medium for all time , we say that the medium is (elastically) homogeneous**.**

Thus , for a homogeneous elastic solid , the elastic moduli are constants so that the mechanical properties remain the same throughout the solid for all times. The tensor equation (1) represents the generalized Hooke's law in the x_i – system.

Since τ_{ij} is symmetric and e_{kl} is symmetric , there are left 6 independent equations in relation (1) and each equation contains 6 independent elastic moduli.

So, the number of independent elastic coefficients are , in fact , 36 for a generalized an anisotropic medium.

For simplicity , we introduce the following (engineering) notations

$$\left.\begin{aligned} &\tau_{11} = \tau_1 , \tau_{22} = \tau_2 , \tau_{33} = \tau_3 , \tau_{23} = \tau_4 , \tau_{31} = \tau_5 , \tau_{12} = \tau_6 , \\ &e_{11} = e_1 , e_{22} = e_2 , e_{33} = e_3 , 2e_{23} = e_4 , 2e_{31} = e_5 , 2e_{12} = e_6 . \end{aligned}\right\} \qquad (2)$$

Then , the generalized Hooke's law may be written in the form

$$\tau_i = c_{ij}\, e_j \quad ; \quad i , j = 1 , 2 , \ldots\ldots , 6 , \qquad (3)$$

or in the matrix form

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \tau_5 \\ \tau_6 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \ldots\ldots\ldots\ldots c_{16} \\ c_{21} & c_{22} \ldots\ldots\ldots\ldots c_{26} \\ c_{31} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ c_{41} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ c_{51} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ c_{61} & c_{62} \ldots\ldots\ldots\ldots c_{66} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{bmatrix} . \qquad (4)$$

If the elastic properties (or mechanical properties) of a medium at a point are independent of the orientation (i.e. $c'_{ij} = c_{ij}$) of the coordinate axes , then we say that the medium under consideration is isotropic.

If at a point of the medium , properties of medium (i.e. , c_{ij}'s) depend upon the orientation , then medium is called an Anisotropic **or** Aelotropic **medium.**

The 6 $\times$ 6 matrix (c_{ij}) in (4) is called stiffness matrix.

4.3. HOMOGENEOUS ISOTROPIC MEDIA

When the elastic coefficients c_{ijkl} in the generalized Hooke's law are constants throughout the medium and they are independent of the orientation of the coordinate axes , the elastic media is termed as homogeneous isotropic media.

We know that the generalized Hooke's law is

$$\tau_{ij} = c_{ijkl}\, e_{kl} \qquad (1)$$

where τ_{ij} and e_{kl} are symmetric strain and stress tensor , respectively , and c_{ijkl} are the components of a tensor of order 4.

Since the media is isotropic , therefore , the tensor c_{ijkl} is an isotropic tensor. Hence , it can be represented in the form

$$c_{ijkl} = \alpha\,\delta_{ij}\,\delta_{kl} + \beta\,\delta_{ik}\,\delta_{jl} + \gamma\,\delta_{il}\,\delta_{jk} \qquad \textbf{(2)}$$

where α , β , γ are some scalars. From equations (1) and (2) , we obtain

$$\begin{aligned}\tau_{ij} &= (\alpha\,\delta_{ij}\,\delta_{kl} + \beta\,\delta_{ik}\,\delta_{jl} + \gamma\,\delta_{il}\,\delta_{jk})\,e_{kl} \\ &= \alpha\,\delta_{ij}\,e_{kk} + \beta\,\delta_{ik}\,e_{kj} + \gamma\,\delta_{il}\,e_{jl} \\ &= \alpha\,\delta_{ij}\,e_{kk} + \beta e_{ij} + \gamma\,e_{ji} \\ &= \alpha\,\delta_{ij}\,e_{kk} + (\beta + \gamma)\,e_{ij} \quad , \qquad \textbf{(3)}\end{aligned}$$

since $e_{ij} = e_{ji}$. On redesignating α by λ and $(\beta + \gamma)$ by 2μ , relation (3) yields

$$\tau_{ij} = \lambda\,\delta_{ij}\,e_{kk} + 2\mu\,e_{ij} \quad . \qquad \textbf{(4)}$$

The two elastic coefficients λ and μ are known as Lame constants. δ_{ij} **is the substitution tensor.**

Let

$$v = e_{kk}\ ,\ \theta = \tau_{ii}\ . \qquad \textbf{(5)}$$

Taking j = i in (4) and using summation convention according , we find

$$\begin{aligned}\theta &= 3\lambda\,v + 2\,\mu\,v \\ &= (3\,\lambda + 2\,\mu)\,v \\ &= 3\,k\,v\ , \qquad \textbf{(6)}\end{aligned}$$

where

$$k = \lambda + \frac{2}{3}\,\mu\ , \qquad \textbf{(7)}$$

is the bulk modulus.

From (4) , we write

$$\begin{aligned}e_{ij} &= \frac{\lambda}{2\mu}\,\delta_{ij}\,v + \frac{1}{2\mu}\,\tau_{ij} \\ &= \frac{-\lambda}{2\mu(3\lambda + 2\mu)}\,\delta_{ij}\,\tau_{kk} + \frac{1}{2\mu}\,\tau_{ij}\ . \qquad \textbf{(8)}\end{aligned}$$

This relation expresses the strain components as a linear functions of components of stress tensor.

Question : **Show that if the medium is isotropic , the principal axes of stress are coincident with the principal axes of strain.**

Solution : **Let the x_i-axes be directed along the principal axes of strain.**

Then

$$e_{12} = e_{13} = e_{23} = 0. \quad (1)$$

The stress – strain relations for an isotropic medium are

$$\tau_{ij} = \lambda\, \delta_{ij}\, e_{kk} + 2\, \mu\, e_{ij}\ . \quad (2)$$

Combining (1) & (2) , we find

$$\tau_{12} = \tau_{13} = \tau_{23} = 0\ . \quad (3)$$

This shows that the coordinates axes x_i are also the principal axes of stress.

This proves the result. Thus, there is no distinction between the principal axes of stress and of strain for isotropic media.

4.4. PHYSICAL MEANINGS OF ELASTIC MODULI FOR AN ISOTROPIC MEDIUM

We have already introduced two elastic moduli λ and μ in the generalized Hooke's law for an isotropic medium. We introduce three more elastic moduli defined below

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}\ , \qquad \sigma = \frac{\lambda}{2(\lambda + \mu)}\ , \quad k = \lambda + \frac{2}{3}\mu\ . \quad (1)$$

The quantity σ is dimensionless and is called the Poisson ratio**. It was introduced by Simon D. Poisson in 1829.**

The quantity E is called Young's modulus **after Thomas Young who introduced it in the early 19th century , probably in 1807. Its dimension is that of a stress (force/area).**

The elastic modulus k is called the modulus of compression or the bulk modulus.

Solving the first two equations for λ and μ (in terms σ and E) , we find

$$\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)} \quad , \qquad \mu = \frac{E}{2(1+\sigma)} \quad . \tag{2}$$

From (2) , we find the following relations

$$\left.\begin{aligned} \lambda + 2\mu &= \frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)} \quad , \quad \frac{\lambda+\mu}{\mu} = \frac{1}{1-2\sigma} \quad , \\ \frac{\lambda+2\mu}{\mu} &= \frac{2(1-\sigma)}{1-2\sigma} \quad , \quad \frac{\lambda}{\lambda+2\mu} = \frac{\sigma}{1-\sigma} \quad . \end{aligned}\right\} \tag{3}$$

Generalized Hooke's Law in terms of Elastic Moduli σ and E

We know that the generalized Hooke's law (giving strain components in terms of stresses) in terms of Lame's constants λ and μ is

$$e_{ij} = \frac{-\lambda}{2\mu(3\lambda+2\mu)}\,\delta_{ij}\,\tau_{kk} + \frac{1}{2\mu}\,\tau_{ij} \quad . \tag{4}$$

Substituting the values of λ and μ in terms of E and σ from (2) into (4) , we find

$$e_{ij} = \frac{-\sigma}{E}\,\delta_{ij}\,\tau_{kk} + \frac{1+\sigma}{E}\,\tau_{ij} \quad . \tag{5}$$

Note : **(1) Out of five elastic moduli (namely ; λ , μ , E , σ , k) only two are independent.**

Note : **(2) The Hooke's law , given in (5) , is frequently used in engineering problems.**

Remark : **The following three experiments give some insight into the physical significance of various elastic moduli for isotropic media.**

(I) Simple Tension :

Consider a right cylinder with its axis parallel to the x_1 – axis which is subjected to longitudinal forces applied to the ends of the cylinder. These applied forces give rise to a uniform tension T in every cross – section of the cylinder so that the stress tensor τ_{ij} has only one non – zero component τ_{11} = T.

That is

$$\tau_{11} = T\ ,\ \tau_{22} = \tau_{33} = \tau_{12} = \tau_{23} = \tau_{31} = 0 \quad . \tag{1}$$

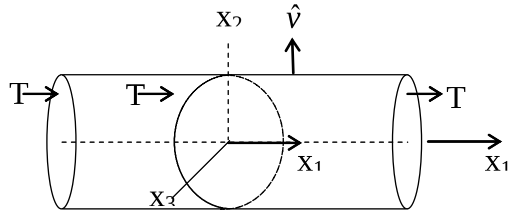


Fig. (4.1)

Since the body forces are absent (f_i = 0) , the state of stress given by (1) satisfies the equilibrium equation $\tau_{ij,j}$ = 0 in the interior of the cylinder.

A normal $\hat{v}$ to the lateral surface lies in the plane parallel to x_2 x_3 plane , so $\hat{v}$ = (0 , v_2 , v_3).

The relation $\overset{v}{T}_i = \tau_{ij} v_j$ implies that $\overset{v}{T}_1 = \overset{v}{T}_2 = \overset{v}{T}_3 = 0$.

Hence $\underset{\sim}{\overset{v}{T}} = 0$.

This shows that the lateral surface of the cylinder is free from tractions.

The generalized Hooke's law giving strains in terms of stresses is

$$e_{ij} = \frac{-\lambda}{2\mu(3\lambda+2\mu)}\,\delta_{ij}\,\tau_{kk} + \frac{1}{2\mu}\,\tau_{ij} \quad . \qquad (2)$$

We find from equations (1) & (2) that

$$e_{11} = \frac{\lambda+\mu}{\mu(3\lambda+2\mu)}T \quad ,$$

$$e_{22} = e_{33} = \frac{-\lambda}{2\mu(3\lambda+2\mu)}\,T \quad ,$$

$$e_{12} = e_{23} = e_{31} = 0 \quad . \qquad (3)$$

Since $\dfrac{\lambda+\mu}{\mu(3\lambda+2\mu)} = \dfrac{1}{E}$ **and** $\dfrac{\sigma}{E} = \dfrac{\lambda}{2\mu(3\lambda+2\mu)}$, **(4)**

Therefore

$$e_{11} = \frac{T}{E} \,,$$

$$e_{22} = e_{33} = -\frac{\sigma}{E}\,T = -\sigma\, e_{11} \quad , \qquad (5)$$

and $$e_{12} = e_{13} = e_{23} = 0 \quad .$$

These strain components obviously satisfies the compatibility equations

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0 \quad ,$$

and therefore , the state of stress given in (1) actually corresponds to one which can exist in a deformed elastic body. From equation (5) , we write

$$\frac{\tau_{11}}{e_{11}} = E \quad , \quad \frac{e_{22}}{e_{11}} = \frac{e_{33}}{e_{11}} = -\sigma \quad . \qquad (6)$$

Experiments conducted on most naturally occurring elastic media show that a tensile longitudinal stress produces a longitudinal extension together with a contraction in a transverse directions. According for τ_{11} = T > 0 , we take

$$e_{11} > 0 \text{ and } e_{22} < 0 \; , \; e_{33} < 0 \quad .$$

It then follows from (6) that

$$E > 0 \text{ and } \sigma > 0 \quad . \qquad (7)$$

From equation (6) , we see that E represents the ratio of the longitudinal stress τ_{11} to the corresponding longitudinal strain e_{11} produced by the stress τ_{11}.

From equation (6) , we get

$$\left|\frac{e_{22}}{e_{11}}\right| = \left|\frac{e_{33}}{e_{11}}\right| = \sigma \quad . \qquad (8)$$

Thus , the Poisson's ratio σ represents the numerical value of the ratio of the contraction e_{22} **(or e_{33})** in a transverse direction **to the corresponding** extension e_{11} in the longitudinal direction.

(II) Pure Shear

From generalized Hooke's law for an isotropic medium , we write

$$2\mu = \frac{\tau_{12}}{e_{12}} = \frac{\tau_{13}}{e_{13}} = \frac{\tau_{23}}{e_{23}} \quad . \qquad (9)$$

The constant 2μ is thus the ratio **of a shear stress component to the corresponding shear strain component. It is , therefore , related to the** rigidity **of the elastic material.**

For this reason , the coefficient μ is called the modulus of rigidity **or the** shear modulus**.**

The other lame constant λ has no direct physical meaning.

The value of μ in terms of Young's modulus E and Poisson ratio σ is given by

$$\mu = \frac{E}{2(1+\sigma)} \quad . \tag{10}$$

Since E > 0 , σ > 0 , it follows that $\mu > 0$ **(11)**

(III) Hydrostatic Pressure

Consider an elastic body of arbitrary shape which is put in a large vessel containing a liquid. A hydrostatic pressure p is exerted on it by the liquid and the elastic body experience all around pressure. The stress tensor is given by $\tau_{ij} = -p\,\delta_{ij}$**. That is ,**

$$\tau_{11} = \tau_{22} = \tau_{33} = -p\ ,\ \tau_{12} = \tau_{23} = \tau_{31} = 0.$$

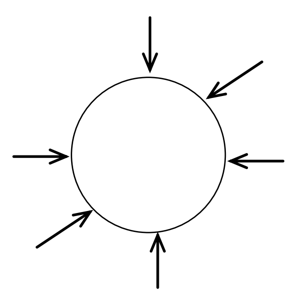


Fig. (4.2)

These stress components satisfy the equilibrium equations for zero body force. We find

$$\tau_{kk} = -3p\ ,$$

and the generalized Hooke's law giving strains in terms of stresses

$$e_{ij} = \frac{1}{2\mu}\left[\frac{-\lambda}{3\lambda+2\mu}\delta_{ij}\,\tau_{kk} + \tau_{ij}\right] \ , \tag{13}$$

gives

$$\mathbf{e_{12} = e_{23} = e_{31} = 0} \quad ,$$

$$\mathbf{e_{11} = e_{22} = e_{33}} = \frac{1}{2\mu}\left[\frac{3\lambda p}{3\lambda+2\mu}+(-p)\right] = -\frac{p}{3\lambda+2\mu} \quad , \qquad \textbf{(14)}$$

which obviously satisfy the compatibility equations. We find

$$\mathbf{e_{kk}} = \frac{-3p}{3\lambda+2\mu} = \frac{-p}{\lambda+\frac{2}{3}\mu} = \frac{-p}{k} \quad .$$

That is ,

$$v \text{ (cubical dilatation)} = \frac{-p}{k} \quad . \qquad \textbf{(15)}$$

From experiments , it has been found that a hydrostatic pressure tends to reduce the volume of the elastic material**. That is , if p > 0 , then**

$$\mathbf{e_{kk}} = v < 0.$$

Consequently , it follows from (15) that k > 0 .

Relation (15) also shows that the constant k represents the numerical value **of the** ratio **of the** compressive stress **to the dilatation.**

Substituting the value of λ and μ in terms of E and σ , we find

$$\mathbf{k} = \frac{E}{3(1-2\sigma)} \quad . \qquad \textbf{(16)}$$

Since k > 0 and E > 0 , it follows that $o < \sigma < \frac{1}{2}$ **for all physical substances.**

Since

$$\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)} \quad , \qquad \textbf{(17)}$$

and E > 0 , 0 < $\sigma < \frac{1}{2}$ **, it follows that** $\lambda > 0$.

Remark : **The solutions of many problems in elasticity are either exactly or approximately independent of the value chosen for Poisson's ratio. This**

fact suggests that approximate solutions may be found by so choosing Poisson's ratio as to simplify the problem. Show that , if one take $\sigma = 0$, then

$$\lambda = 0 \, , \, \mu = \frac{E}{2} \, , \, k = \frac{1}{3}E \, ,$$

and Hooke's law is expressed by

$$\tau_{ij} = E\, e_{ij} = \frac{1}{2} E\,(u_{i,j} + u_{j,i}) \quad .$$

Note 1: **The elastic constants μ , E , σ , k have definite physical meanings. These constants are called engineering elastic modulus.**

Note 2: **The material such as steel , brass , copper , lead , glass , etc. are isotropic elastic medium.**

Note 3: **We find**

$$e_{kk} = \frac{\tau_{kk}}{3k} = \frac{1-2\sigma}{E}\tau_{kk} \; .$$

Thus $e_{kk} = 0$ iff $\sigma = \frac{1}{2}$, provided E and τ_{kk} remain finite.

When $\sigma \to \frac{1}{2}$, $\lambda \to \infty$, $k \to \infty$, $\mu = \frac{E}{3}$, $v = e_{ii} = u_{i,i} = 0$,

This limiting case corresponds to which is called an incompressible elastic body.

Question : **In an elastic beam placed along the x_3 – axis and bent by a couple about the x_2–axis , the stresses are found to be**

$$\tau_{33} = -\frac{E}{R}\, x_1 \, , \, \tau_{11} = \tau_{22} = \tau_{12} = \tau_{13} = \tau_{23} = 0 \, , \, R = \text{constant}$$

Find the corresponding strains.

Solution : **The strains in terms of stresses & elastic moduli E and σ are given by the Hooke's law**

$$e_{ij} = -\frac{\sigma}{E}\,\delta_{ij}\,\tau_{kk} + \frac{1+\sigma}{E}\,\tau_{ij} \quad . \qquad (1)$$

Here

$$\tau_{kk} = -\frac{E}{R}\ x_1.$$

Hence , (1) becomes

$$e_{ij} = \frac{\sigma}{R} x_1\,\delta_{ij} + \frac{1+\sigma}{E}\tau_{ij} \quad . \qquad (2)$$

This gives $e_{11} = e_{22} = \frac{\sigma}{R}\ x_1$, $e_{33} = -\frac{1}{R}\ x_1$, $e_{12} = e_{23} = e_{13} = 0.$

Question : **A beam placed along the x_1 – axis and subjected to a longitudinal stress τ_{11} at every point is so constrained that $e_{22} = e_{33} = 0$ at every point. Show that $\tau_{22} = \sigma\tau_{11}$, $e_{11} = \frac{1-\sigma^2}{E}\tau_{11}$, $e_{33} = \frac{-\sigma(1+\sigma)}{E}\tau_{11}$.**

Solution : **The Hooke's law giving the strains in terms of stresses is**

$$e_{ij} = -\frac{\sigma}{E}\ \delta_{ij}\ \tau_{kk} + \frac{1+\sigma}{E}\tau_{ij} \quad . \qquad (1)$$

It gives

$$e_{22} = -\frac{\sigma}{E}(\tau_{11} + \tau_{22} + \tau_{33}) + \frac{1+\sigma}{E}\ \tau_{22}$$

$$= \frac{1}{E}\ \tau_{22} - \frac{\sigma}{E}(\tau_{11} + \tau_{33}) \quad . \qquad (2)$$

Putting $e_{22} = e_{33} = 0$ in (2) , we get

$$\tau_{22} = \sigma\ \tau_{11} \quad . \qquad (3)$$

Also , from (1) , we find

$$e_{11} = -\frac{\sigma}{E}(\tau_{11} + \tau_{22} + \tau_{33}) + \frac{1+\sigma}{E}\ \tau_{11}$$

$$= -\frac{\sigma}{E}(\tau_{11} + \sigma\ \tau_{11}) + \frac{1+\sigma}{E}\ \tau_{11}$$

$$= \frac{1}{E}[-\sigma - \sigma^2 + 1 + \sigma]\ \tau_{11} = \frac{1-\sigma^2}{E}\tau_{11} \quad . \qquad (4)$$

Also , from (1) , we get

$$e_{33} = -\frac{\sigma}{E}(\tau_{11} + \tau_{22}) + \frac{1+\sigma}{E}\tau_{33}$$

$$= -\frac{\sigma}{E}(\tau_{11} + \sigma\,\tau_{11}) = \frac{-\sigma(1+\sigma)}{E}\tau_{11} \quad . \qquad (5)$$

Exercise 1: **Find the stresses with the following displacement fields :**

(i) $u = k\,y\,z$, $v = k\,z\,x$, $w = k\,x\,y$

(ii) $u = k\,y\,z$, $v = k\,z\,x$, $w = k\,(x^2 - y^2)$

where k = constant.

Exercise 2 : **A rod placed along the x_1 – axis and subjected to a longitudinal stress τ_{11} is so constrained that there is no lateral contraction. Show that**

$$\tau_{11} = \frac{(1-\sigma)E}{(1+\sigma)(1-2\sigma)}e_{11}$$

4.5. EQUILIBRIUM AND DYNAMIC EQUATIONS FOR AN ISOTROPIC ELASTIC SOLID

We know that Cauchy's equation's of equilibrium in term of stress components are

$$\tau_{ij,j} + F_i = 0 \quad , \qquad (1)$$

where F_i is the body force per unit volume and $i , j = 1, 2, 3$.

The generalized Hooke's law for a homogeneous isotropic elastic body is

$$\tau_{ij} = \lambda\,\delta_{ij}\,e_{kk} + 2\mu\,e_{ij}$$

$$= \lambda\,\delta_{ij}\,u_{k,k} + \mu(u_{i,j} + u_{j,i}) \quad , \qquad (2)$$

where λ and μ are Lame constants. Putting the value of τ_{ij} from (2) into equation (1) , we find

$$\lambda\,\delta_{ij}\,u_{k,kj} + \mu(u_{i,jj} + u_{j,ij}) + F_i = 0$$

$$\lambda\,u_{k,ki} + \mu\,\nabla^2 u_i + \mu\,u_{k,ki} + F_i = 0$$

$$(\lambda + \mu)\frac{\partial\theta}{\partial x_i} + \mu\,\nabla^2 u_i + F_i = 0 \quad , \qquad (3)$$

where $\theta = u_{k,k} = \text{div } \bar{u}$ = cubical dilatation and i = 1 , 2 , 3.

Equations in (3) form a synthesis of the analysis of strain, analysis of stress and the stress – strain relation.

These fundamental partial differential equations of the elasticity theory are known as Navier's equations of equilibrium , after Navier (1821).

Equation (3) can be put in several different forms.

Form (A) : **In vector form , equation (3) can be written as**

$$(\lambda + \mu) \text{ grad div } \bar{u} + \mu \nabla^2 \bar{u} + \bar{F} = \bar{0} \quad . \qquad (4)$$

Form (B) :**We know the following vector identity :**

$$\text{curl curl } \bar{u} = \text{grad div } \bar{u} - \nabla^2 \bar{u} \quad . \qquad (5)$$

Putting the value of $\nabla^2 \bar{u}$ from (5) into (4) , we obtain

$$(\lambda + \mu) \text{ grad div } \bar{u} + u [\text{grad div } \bar{u} - \text{curl curl } \bar{u}] + \bar{F} = \bar{0}$$

or

$$(\lambda + 2\mu) \text{ grad div } \bar{u} - \mu \text{ curl curl } \bar{u} + \bar{F} = 0. \qquad (6)$$

Form (C) : **Putting the value of grad div $\bar{u}$ from (5) into (4) , we get**

$$(\lambda + \mu) [\nabla^2 \bar{u} + \text{curl curl } \bar{u}] + \mu \nabla^2 \bar{u} + \bar{F} = \bar{0}$$

or

$$(\lambda + 2\mu) \nabla^2 \bar{u} + (\lambda + \mu) \text{ curl curl } \bar{u} + \bar{F} = \bar{0} \qquad (7)$$

Form D : **We know that**

$$\frac{\lambda + \mu}{\mu} = \frac{1}{1 - 2\sigma} \quad . \qquad (8)$$

From (8) and (4) , we find

$$\nabla^2 \bar{u} + \frac{1}{1 - 2\sigma} \text{ grad div } \bar{u} + \frac{1}{\mu} \bar{F} = \bar{0} \quad . \qquad (9)$$

Dynamical Equations for an Isotropic Elastic Solid

Let ρ be the density of the medium. The components of the force (mass × acceleration/volume) per unit volume are $\rho \frac{\partial^2 u_i}{\partial t^2}$.

Hence , the dynamical equations in terms of the displacements u_i become

$$(\lambda + \mu)\frac{\partial \theta}{\partial x_i} + \mu \nabla^2 u_i + F_i = \rho \frac{\partial^2 u_i}{\partial t^2} \quad ,$$

for i = 1 , 2 , 3.

Various form of it can be obtained as above for equilibrium equations.

Question : **In an isotropic elastic body in equilibrium under the body force $\underline{f} = a\, x_1 x_2\, \hat{e}_3$, where a is a constant , the displacements are of the form**

$$u_1 = A\, x_1^2 x_2 x_3 \;,\; u_2 = B\, x_1 x_2^2 x_3 \;,\; u_3 = C\, x_1 x_2 x_3^2$$

where A , B , C are constants. Find A , B , C. Evaluate the corresponding stresses.

4.6. BELTRAMI-MICHELL COMPATIBILITY EQUATIONS IN TERMS OF THE STRESSES FOR AN ISOTROPIC SOLID

The strain – stress relations for an isotropic elastic solid are

$$e_{ij} = \frac{1+\sigma}{E}\tau_{ij} - \frac{\sigma}{E}\delta_{ij}\theta \qquad , \qquad \theta = \tau_{ii} \quad , \tag{1}$$

in which σ is the Poisson's ratio and E is the Young's modulus.

The Saint – Venant's compatibility equations in terms of strain components are

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0 \;, \tag{2}$$

which impose restrictions on the strain components to ensure that given e_{ij} yield single – valued continuous displacements u_i .

When the region τ is simply connected, using (1) in (2) , we find

$$\frac{1+\sigma}{E}\{\tau_{ij,kl} + \tau_{kl,ij} - \tau_{ik,jl} - \tau_{jl,ki}\} = \frac{\sigma}{E}\{\delta_{ij}\,\theta_{,kl} + \delta_{kl}\,\theta_{,ij} - \delta_{ik}\,\theta_{,jl} - \delta_{jl}\,\theta_{,ik}\}$$

$$\tau_{ij,kl} + \tau_{kl,ij} - \tau_{ik,jl} - \tau_{jl,ki} = \frac{\sigma}{1+\sigma}(\delta_{ij}\,\theta_{,kl} + \delta_{kl}\,\theta_{,ij} - \delta_{ik}\,\theta_{,jl} - \delta_{jl}\,\theta_{,ik}) \;, \tag{3}$$

with $\tau_{ij,kl} = \dfrac{\partial^2 \tau_{ij}}{\partial x_k \, \partial x_l}$, $\theta_{,ij} = \dfrac{\partial^2 \theta}{\partial x_i \, \partial x_j}$.

These are equations of compatibility in stress components. These are 81(=3^4) in number but all of them are not independent. If i & j or k & *l* are interchanged , we get same equations. Similarly for i = j = k = *l* , equations are identically satisfied. Actually , the set of equations (3) contains only six independent equations obtained by setting

k = *l* = 1	**,**	**i = j = 2**
k = *l* = 2	**,**	**i = j = 3**
k = *l* = 3	**,**	**i = j = 1**
k = *l* = 1	**,**	**i = 2 , j = 3**
k = *l* = 2	**,**	**i = 3 , j = 1**
k = *l* = 3	**,**	**i = 1 , j = 2.**

Setting k = *l* in (3) and then taking summation over the common index , we get

$$\tau_{ij,kk} + \tau_{kk,ij} - \tau_{ik,jk} - \tau_{jk,ik} = \frac{\sigma}{1+\sigma}\left(\delta_{ij}\,\theta_{,kk} + \delta_{kk}\,\theta_{,ij} - \delta_{ik}\,\theta_{,jk} - \delta_{jk}\,\theta_{,ik}\right) ,$$

Since $\theta_{,kk} = \nabla^2 \theta$, $\tau_{ij,kk} = \nabla^2 \tau_{ij}$,

$\tau_{kk,ij} = \theta_{,ij}$, $\delta_{kk} = 3$,

therefore , above equations become

$$\nabla^2 \tau_{ij} + \theta_{,ij} - \tau_{ik,jk} - \tau_{jk,ik} = \frac{\sigma}{1+\sigma}\left[\delta_{ij}\nabla^2\theta + 3\theta_{,ij} - 2\theta_{,ij}\right]$$

or

$$\nabla^2 \tau_{ij} + \frac{1}{1+\sigma}\theta_{,ij} - \tau_{ik,jk} - \tau_{jk,ik} = \frac{\sigma}{1+\sigma}\,\delta_{ij}\,\nabla^2\theta. \qquad \textbf{(4)}$$

This is a set of 9 equations and out of which only 6 are independent due to the symmetry of i & j. In combining equations (3) linearly , the number of independent equations is not reduced.

Hence the resultant set of equations in (4) is equivalent to the original equations in (3).

Equilibrium equations are

$$\tau_{ik,k} + F_i = 0 \quad ,$$

where F_i is the body force per unit volume.

Differentiating these equations with respect to x_j , we get

$$\tau_{ik,kj} = -F_{i\,,\,j} \quad . \tag{5}$$

Using (5) , equation (4) can be rewritten in the form

$$\nabla^2 \tau_{ij} + \frac{1}{1+\sigma}\theta_{,\,ij} - \frac{\sigma}{1+\sigma}\delta_{ij}\,\nabla^2\theta = -(F_{i\,,\,j} + F_{j\,,\,i}) \quad . \tag{6}$$

Setting j = i in (6) and adding accordingly, we write

$$\nabla^2\,\theta + \frac{1}{1+\sigma}\,\nabla^2\theta - \frac{3\sigma}{1+\sigma}\,\nabla^2\theta = -\,2F_{i,i}$$

$$\left(1+\frac{1}{1+\sigma}-\frac{3\sigma}{1+\sigma}\right)\nabla^2\theta = -2F_{i,i}$$

$$\frac{2(1-\sigma)}{1+\sigma}\,\nabla^2\theta = -2F_{i,i} = -2\,div\vec{F} \quad ,$$

giving

$$\nabla^2\theta = -\frac{1+\sigma}{1-\sigma}\,div\vec{F} \quad . \tag{7}$$

Using (7) in (6) , we find the final form of the compatibility equations in terms of stresses.

We get

$$\nabla^2\tau_{ij} + \frac{1}{1+\sigma}\theta_{,\,ij} = -\frac{\sigma}{1-\sigma}\delta_{ij}\,div\vec{F} - (F_{i\,,\,j} + F_{j\,,\,i}) \quad . \tag{8}$$

These equations in cartesian coordinates (x , y , z) can be written as

$$\left.\nabla^2\tau_{xx} + \frac{1}{1+\sigma}\frac{\partial^2\theta}{\partial x^2} = -\frac{\sigma}{1-\sigma}\,div\vec{F} - 2\frac{\partial F_x}{\partial x}\quad\right\}$$

$$\nabla^2 \tau_{yy} + \frac{1}{1+\sigma}\frac{\partial^2\theta}{\partial y^2} = -\frac{\sigma}{1-\sigma}\, div\vec{F} - 2\frac{\partial F_y}{\partial y}$$

$$\nabla^2 \tau_{zz} + \frac{1}{1+\sigma}\frac{\partial^2\theta}{\partial z^2} = -\frac{\sigma}{1-\sigma}\, div\vec{F} - 2\frac{\partial F_z}{\partial z}$$

$$\left.\begin{aligned}
\nabla^2 \tau_{yz} + \frac{1}{1+\sigma}\frac{\partial^2\theta}{\partial y\,\partial z} &= -\left(\frac{\partial F_y}{\partial z} + \frac{\partial F_z}{\partial y}\right)\\
\nabla^2 \tau_{zx} + \frac{1}{1+\sigma}\frac{\partial^2\theta}{\partial z\,\partial x} &= -\left(\frac{\partial F_z}{\partial x} + \frac{\partial F_x}{\partial z}\right)\\
\nabla^2 \tau_{xy} + \frac{1}{1+\sigma}\frac{\partial^2\theta}{\partial x\,\partial y} &= -\left(\frac{\partial F_x}{\partial y} + \frac{\partial F_y}{\partial x}\right)
\end{aligned}\right\} \quad (9)$$

In 1892 , **Beltrami obtained these equations for** $\vec{F} = \vec{0}$ **and in 1900 Michell obtained then in form as given in (9).**

These equations in (9) are called the Beltrami – Michell **compatibility equations.**

Definition : **A function V of class C^4 is called a biharmonic function when**

$$\nabla^2\, \nabla^2 \mathbf{V} = \mathbf{0}\ .$$

Theorem : **When the components of the body** $\vec{F}$ **are constants , show that the stress and strain invariants** θ **and** v **are harmonic functions and the stress components** τ_{ij} **and strain components** e_{ij} **are biharmonic functions.**

Proof : **The Beltrami – Michell compatibility equations in terms of stress are**

$$\nabla^2 \tau_{ij} + \frac{1}{1+\sigma}\theta_{,ij} = -\frac{\sigma}{1-\sigma}\delta_{ij}\, div\vec{F} - (F_{i,j} + F_{j,i}) \ , \qquad (1)$$

in which $\vec{F}$ **is the body force per unit volume.**

It is given that the vector $\vec{F}$ **is constant. In this case, equations in (1) reduce to**

$$\nabla^2 \tau_{ij} + \frac{1}{1+\sigma}\theta_{,ij} = 0\ . \qquad (2)$$

Setting j = i in (2) and taking summation accordingly , we get

$$\nabla^2 \tau_{ii} + \frac{1}{1+\sigma}\ \theta_{,ii} = 0$$

$$\nabla^2 \theta + \frac{1}{1+\sigma}\ \nabla^2\ \theta = 0$$

$$\left(1+\frac{1}{1+\sigma}\right)\nabla^2 \theta = 0$$

$$\nabla^2 \theta = 0\ . \qquad (3)$$

This shows that the stress invariant $\theta = \tau_{kk}$ is a harmonic function.

The standard relation between the invariants θ and v is

$$\theta = (3\lambda + 2\mu)\, v\ , \qquad (4)$$

and equation (3) implies that

$$\nabla^2 v = 0\ , \qquad (5)$$

showing that the strain invariant $v = e_{kk}$ is also a harmonic function.

Again

$$\nabla^2\ \nabla^2 \tau_{ij} = \nabla^2\left(-\frac{1}{1+\sigma}\theta_{,ij}\right)\ ,$$

$$= -\frac{1}{1+\sigma}\ \nabla^2 (\theta_{,ij})$$

$$= -\frac{1}{1+\sigma}(\nabla^2 \theta)_{,ij}\ ,$$

giving

$$\nabla^2\ \nabla^2 \tau_{ij} = 0\ . \qquad (6)$$

This shows that the stress components τ_{ij} are biharmonic functions.

The following strain – stress relations

$$e_{ij} = \frac{-\lambda}{2\mu(3\lambda+2\mu)}\delta_{ij}\,\theta + \frac{1}{2\mu}\tau_{ij}$$

give

$$\nabla^2 \nabla^2 e_{ij} = \frac{-\lambda}{2\mu(3\lambda+2\mu)} \delta_{ij} \nabla^2 \nabla^2 \theta + \frac{1}{2\mu} \nabla^2 \nabla^2 \tau_{ij}$$

$$\nabla^2 \nabla^2 e_{ij} = 0 \ . \qquad (7)$$

Equation (7) shows that the strain components e_{ij} are also biharmonic functions.

Theorem 2: **If the body force $\vec{F}$ is derived from a** harmonic potential **function , show that the strain and stress invariants e_{kk} & τ_{kk} are harmonic functions and the** strain and stress **components are** biharmonic function**.**

Proof : **Let ϕ be the potential function and $\vec{F}$ is derived from ϕ so that**

$$\vec{F} = \underline{\nabla} \phi \qquad \text{or} \qquad F_j = \phi_{,j} \ . \qquad (1)$$

Then

$$\text{div} \vec{F} = \phi_{,jj} = \nabla^2 \phi = 0 \ , \qquad (2)$$

since ϕ is a harmonic function (given). Further

$$F_{i,j} = F_{j,i} = \phi_{,ij} \qquad (3)$$

The Beltrami – Michell compatibility equations in terms of stresses , in this case , reduce to

$$\nabla^2 \tau_{ij} + \frac{1}{1+\sigma} \theta_{,ij} = -2\phi_{,ij} \ . \qquad (4)$$

Putting j = i in (4) and taking the summation accordingly , we obtain

$$\nabla^2 \tau_{ii} + \frac{1}{1+\sigma} \theta_{,ii} = -2\phi_{,ii}$$

$$= -2\nabla^2 \phi$$

$$= 0 \ ,$$

giving

$$\nabla^2 \theta = 0 \ , \qquad (5)$$

This shows that θ is harmonic.

The relation $\theta = (3\lambda + 2\mu)\, v$ immediately shows that v is also harmonic.

From equation (4) , we write

$$\nabla^2 \nabla^2 \tau_{ij} + \frac{1}{1+\sigma} \nabla^2 \theta_{,ij} = -2(\nabla^2 \phi)_{,ij}$$

$$= 0$$

This gives

$$\nabla^2 \nabla^2 \tau_{ij} = 0 \quad , \tag{6}$$

as $\nabla^2 \theta = \nabla^2 \phi = 0.$

It shows that the components τ_{ij} are biharmonic.

The strain – stress relations yields that the strain components are also biharmonic function.

Question : **Find whether the following stress system can be a solution of an elastostatic problem in the absence of body forces :**

$$\tau_{11} = x_2 x_3 , \tau_{22} = x_3 x_1 , \tau_{12} = x_3^2 , \tau_{13} = \tau_{33} = \tau_{32} = 0.$$

Solution : **In order that the given stress system can be a solution of an elastostatic problem in the absence of body forces , the following equations are to be satisfied :**

(i) Cauchy's equations of equilibrium with $f_i = 0$, i.e. ,

$$\tau_{11,1} + \tau_{12,2} + \tau_{13,3} = 0 \quad ,$$

$$\tau_{12,1} + \tau_{22,2} + \tau_{23,3} = 0 \quad ,$$

$$\tau_{13,1} + \tau_{23,2} + \tau_{33,3} = 0 \quad . \tag{1}$$

(ii) Beltrami – Michell equations with $f_i = 0$, i.e. ,

$$\nabla^2 \tau_{11} + \frac{1}{1+\sigma} (\tau_{11} + \tau_{22} + \tau_{33})_{,11} = 0 ,$$

$$\nabla^2 \tau_{22} + \frac{1}{1+\sigma} (\tau_{11} + \tau_{22} + \tau_{33})_{,22} = 0 \quad ,$$

$$\nabla^2 \tau_{33} + \frac{1}{1+\sigma} (\tau_{11} + \tau_{22} + \tau_{33})_{,33} = 0 \quad ,$$

$$\nabla^2 \tau_{12} + \frac{1}{1+\sigma}(\tau_{11} + \tau_{22} + \tau_{33})_{,12} = 0 \quad ,$$

$$\nabla^2 \tau_{13} + \frac{1}{1+\sigma}(\tau_{11} + \tau_{22} + \tau_{33})_{,13} = 0 \quad ,$$

$$\nabla^2 \tau_{23} + \frac{1}{1+\sigma}(\tau_{11} + \tau_{22} + \tau_{33})_{,23} = 0 \quad . \qquad (2)$$

It is easy to check that all the equilibrium equations in (1) are satisfied.

Moreover, all except the fourth one in (2) are satisfied by the given stress system.

Since the given system does not satisfy the Beltrami – Michell equations fully , it can not form a solution of an elastostatic problem.

Remark : **The example illustrates the important fact that a stress system may not be a solution of an elasticity problem even though it satisfies Cauchy's equilibrium equations.**

Exercise 1: **Show that the stress – system $\tau_{11} = \tau_{22} = \tau_{13} = \tau_{23} = \tau_{12} = = 0$, $\tau_{33} = \rho g\ x_3$, where ρ and g are constants ,satisfies that equations of equilibrium and the equations of compatibility for a suitable body force.**

Exercise 2: **Show that the following stress system can not be a solution of an elastostatic problem although it satisfies cauchy's equations of equilibrium with zero body forces :**

$$\tau_{11} = x_2^2 + \sigma(x_1^2 - x_2^2)\ ,\ \tau_{22} = x_1^2 + \sigma(x_2^2 - x_1^2)\ ,\ \tau_{33} = \sigma(x_1^2 + x_2^2)$$

$$\tau_{12} = -2\sigma\, x_1\, x_2\ ,\ \tau_{23} = \tau_{31} = 0$$

where σ is a constant of elasticity.

Exercise 3: **Determine whether or not the following stress components are a possible solution in elastostatics in the absence of body forces :**

$$\tau_{11} = a\, x_2\, x_3\ ,\ \tau_{22} = b\, x_3\, x_1\ ,\ \tau_{33} = c\, x_1\, x_2\ ,\ \tau_{12} = d\, x_3^2$$
$$\tau_{13} = e\, x_2^2\ ,\ \tau_{23} = f\, x_1^2\ ,$$

where a , b , c , d , e , f are all constants.

Exercise 4: **In an elastic body in equilibrium under the body force $\underline{f} = a\, x_1\, x_2\, \hat{e}_3$, where a is a constant , the stresses are of the form**

$$\tau_{11} = a\, x_1\, x_2\, x_3\ ,\ \tau_{22} = b\, x_1\, x_2\, x_3\ ,\ \tau_{33} = c\, x_1\, x_2\, x_3$$
$$\tau_{12} = (a\, x_1^2 + b\, x_2^2)\, x_3\ ,\ \tau_{23} = (b\, x_2^2 + c\, x_3^2)\, x_1\ ,\ \tau_{13} = (c\, x_3^2 + a\, x_1^2)\, x_2$$

where a , b , c are constants. Find these constants.

Exercise 5: **Define the stress function S by**

$$\tau_{ij} = S_{,ij} = \frac{\partial^2 S}{\partial x_i \, \partial x_j}$$

and consider the case of zero body force. Show that , if $\sigma = 0$, then the equilibrium and compatibility equations reduce to

$$\nabla^2 S = \text{Constant.}$$

4.7. UNIQUENESS OF SOLUTION

The most general problem of the elasticity theory is to determine the distribution of stresses **and** strains **as well as displacements at all points of a body when certain boundary conditions and certain initial conditions are specified (under the assumption that the body force $\underline{f}$ is known before hand).**

In the linear elasticity , the displacements , strains and stresses are governed by the following equations

(I) $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ **strain – displacement relations** **(1)**

(II) $\tau_{ij} = \lambda\delta_{ij}\, e_{kk} + 2\mu\, e_{ij} = \lambda\delta_{ij}\, u_{k,k} + \mu(u_{i,j} + u_{j,i})$ **material law** **(2)**

or

$$e_{ij} = -\frac{\sigma}{E}\delta_{ij}\,\tau_{kk} + \frac{1+\sigma}{E}\tau_{ij}$$

(III) $\tau_{ij,j} + f_i = 0$ **Cauchy's equation of equilibrium**

or **(3)**

$(\lambda + \mu)$ **grad div** $\bar{u} + \mu\, \nabla^2\, \bar{u} + \bar{f} = \bar{0}$ **Navier equation of equilibrium**

Accordingly , solving a problem in linear elasticity generally amounts to solving these equations for u_i , e_{ij} and τ_{ij} **under certain specified boundary conditions and initial conditions.**

Let a body occupying a region V has the boundary S.

Initially , **It is assumed that the body is in the** undeformed state. **That is,**

$u_i = 0$ for $e_{ij} = 0$ in V . **(at time t = 0)** **(4)**

The boundary conditions specified are usually of one of following three kinds :

(i) The stress vector is specified at every point of boundary S for all times**, i.e. ,**

$$\underset{\sim}{\overset{v}{T}} = \underset{\sim}{s} \qquad \textbf{on S}$$

where $\underset{\sim}{s}$ is a known vector point function.

(ii) the displacement vector is specified at every point of S and for all times , i.e. ,

$$\underset{\sim}{u} = \underset{\sim}{u}^* \qquad \textbf{on S} \qquad \textbf{(6)}$$

when $\underset{\sim}{u}^*$ **is a known function.**

(ii) **the stress vector is specified at every point of a part S_τ of S and the displacement vector is specified at every point of the remaining part $S_u = S - S_\tau$; i.e. ,**

$$\left.\begin{aligned} \overset{v}{\underset{\sim}{T}} &= \underset{\sim}{s} && \textbf{on } S_\tau \\ \underset{\sim}{u} &= \underset{\sim}{u}^* && \textbf{on } S_u = S - S_\tau \end{aligned}\right\} \quad . \qquad \textbf{(7)}$$

The problem of solving equations (1) , (2) , (3) under the initial condition(4) and one of the boundary **conditions in (5) – (7) to determine u_i , e_{ij} , τ_{ij} is known as** boundary value problem in elastostatics.

A set {u_i , e_{ij} , τ_{ij}} so obtained / determined , if it exists , is called a solution of the problem**.**

When the boundary condition is of the form (5) , the problem is referred to as the traction (or stress) boundary value problem **; and when the boundary condition of the form (6) , the problem is referred to as the** displacement BV problem **; and when the boundary condition is of the form (7) , the problem is referred to** as mixed BV problem**.**

The three problems are together called the fundamental boundary value problems**. The boundary conditions valid for all the three problems can be written down in the form of (7). For the traction problem $S_\tau = S$ and $S_u = \phi$, for the displacement problem $S = S_u$ and $S_\tau \neq \phi$, and for mixed problem $S_u \neq S \neq S_\tau$.**

Uniqueness : **The solutions of an elastostatic problem governed by equations (1) – (3) and the boundary condition (7) is** unique **within a rigid body displacement.**

Remark 1: **The displacement boundary value problem is completely solved if one obtains a solution of the Navier equation subject to the boundary condition (6). Note that we need not adjoin the compatibility equations**

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0 \, , \qquad \textbf{(8)}$$

for the only purpose of the latter is to impose restrictions on the strain components that shall ensure that the e_{ij} yield single – valued continuous displacements u_i , when the region is simply connected. From the knowledge of functions u_i , one can determine the strains , and hence stresses by making use of Hooke's law in (2).

Remark 2: **The** stress boundary value problem **suggests the desirability of expressing all the differential equations entirely in terms of** stress**. The compatibility equations (Betrami – Michell compatibility equations) in terms of stresses are**

$$\nabla^2 \tau_{ij} + \frac{1}{1+\sigma} \tau_{kk,ij} = -\frac{\sigma}{1-\sigma} \delta_{ij} \, \mathrm{div} \, \overline{F} - (F_{i,j} + F_{j,i}) \quad . \qquad (9)$$

In order to determine the state of stress in the interior of an elastic body , one must solve the system of equations consisting of Cauchy's equations of equilibrium (3) and B–M compatibility equations subject to the boundary conditions in (5).

4.8. ST. VENANT'S PRINCIPAL

In the analysis of actual structures subjected to external loads , it can be invariably found that the distributions of surface forces are so complex as to define them more accurately for solving the appropriate governing equations. It is true that the solutions obtained for such problems using the elasticity equations are exact only if the external loads are applied in a specific manner. However , in many cases it is possible to predict the net effect of the external surface tractions without worrying about the precise manner in which they are distributed over the boundary.

In 1853 in his "Memoire Sur la Torsion des Prismes" , Saint – Vanant developed solutions for the torsion of prismatic bars which gave the same stress distribution for all cross – sections. He attempted to justify the usefulness of his formulation by the following : " The fact is that the means of application and distribution of the forces towards the extremities of the prisms is immaterial to the perceptible effects produced on the rest of the length , so that one can always , in a sufficiently similar manner , replace the forces applied with equivalent static forces or with those having the same total moments and the same resultant forces"**.**

St. Venant's Principal : **If a certain distribution of forces acting on a portion of the surface of a body is replaced by different distribution of forces acting on the same portion of the body ,then the effects of the two different distributions on the parts sufficiently far removed (large compared to linear dimensions of the body) from the region of application of forces , are essentially the same , provided that the two distributions of forces are statically equivalent (that is , the same resultant forces and the same resultant moment).**

St . Venant principal is profitable when solving problems in rigid – body mechanics to employ the concept of a point force when we had a force distribution over a small area. At other times , we employed the rigid –

body resultant force system of some distribution in the handling of a problem. Such replacements led to reasonably accurate and direct solutions. From the viewpoint of rigid – body mechanics , this principal states that **the stresses reasonably distant from an applied load on a boundary are not significantly altered if this load is changed to another load which is equivalent to it. We may call such a second load the** statically equivalent load.

This principal is actually summarized by one of its more detailed names, "The principal of the elastic equivalence of statically equipollent systems of load".

St. Venant's principal is very convenient and useful in obtaining solutions to various problems in elasticity. However , the statements are in general vague. They do not specifically state either the extent of the region within which the effects of two different statically equivalent force systems are not quite the same or the magnitude of the error.

Therefore , St. Venant's principal is only qualitative and expresses only a trend**.**

Nevertheless , St. Venant's principal has many important implications with respect to many practical problems. For instance , in many structures , the overall deflections are not unduly affected by the local changes in the distribution of forces or localized stress concentrations due the holes , cracks , etc. But it should be realized that the presence of defects in a region , or a non – uniform application of load will cause changes in stress distribution.

This principle is mainly used in elasticity to solve the problems of extension/bending/torsion of elastic beams. Under this technique , certain assumptions about the components of stress , strain and displacements are made , while leaving enough degree of freedom , so that the equations of equilibrium and compatibility are satisfied . The solution so obtained will be unique by the uniqueness of solution of the general boundary – value problems of linear elasticity.

Table for various elastic coefficients for an isotropic media

λ	$\frac{2\mu\sigma}{1-2\sigma}$	$\frac{\mu(E-2\mu)}{3\mu-E}$	$k-\frac{2}{3}\mu$	$\frac{E\sigma}{(1+\sigma)(1-2\sigma)}$	$\frac{3k\sigma}{1+\sigma}$	$\frac{3k(3k-E)}{(9k-E)}$	—	—	—
μ	—	—	—	$\frac{E}{2(1+\sigma)}$	$\frac{3k(1-2\sigma)}{2(1+\sigma)}$	$\frac{3kE}{9k-E}$	—	$\frac{\lambda(1-2\sigma)}{2\sigma}$	$\frac{3}{2}$ **(k–λ)**
σ	—	$\frac{E}{2\mu}-1$	$\frac{3k-2\mu}{2(3k+\mu)}$	—	—	$\frac{3k-E}{6k}$	$\frac{\lambda}{2(\lambda+\mu)}$	—	$\frac{\lambda}{3k-\lambda}$
E	**2μ(1+σ)**	—	$\frac{9k\mu}{3k+\mu}$	—	**3k(1–2σ)**	—	$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\frac{\lambda(1+\sigma)(1-2\sigma)}{\sigma}$	$\frac{9k(k-\lambda)}{3k-\lambda}$
k	$\frac{2\mu(1+\sigma)}{3(1-2\sigma)}$	$\frac{\mu E}{3(3\mu-E)}$	—	$\frac{E}{3(1-2\sigma)}$	—	—	$\lambda+\frac{2}{3}\mu$	$\frac{\lambda(1+\sigma)}{3\sigma}$	—

Chapter-5
Strain-Energy Function

5.1. INTRODUCTION

The energy stored in an elastic body by virtue of its deformation is called the **strain energy**. This energy is acquired by the body when the body forces and surface tractions do some work. This is also termed as **internal energy**. It depends upon the shape and temperature of the body.

5.2. STRAIN – ENERGY FUNCTION

Let τ_{ij} be the stress tensor and e_{ij} be the strain tensor for an infinitesimal affine deformation of an elastic body. We write

$$\left.\begin{array}{l} \tau_{11} = \tau_1 \ , \tau_{22} = \tau_2 \, , \tau_{33} = \tau_4 \\ \tau_{23} = \tau_4 \, , \tau_{31} = \tau_5 \, , \tau_{12} = \tau_6 \end{array}\right\} , \quad (1)$$

and

$$\left.\begin{array}{l} e_{11} = e_1 \, , e_{22} = e_2 \, , e_{33} = e_3 \\ 2e_{23} = e_4 \, , 2e_{13} = e_5 \, , 2e_{12} = e_6 \end{array}\right\} , \quad (2)$$

in terms of engineering notations.

We assume that the deformation of the elastic body is isothermal or adiabatic. **Love(1944)** has proved that , under this assumption there exists a function of strains

$$W = W(e_1 \, , e_2 \, , e_3 \, , \ldots\ldots, e_6) \ , \quad (3)$$

with the property

$$\frac{\partial W}{\partial e_i} = \tau_i \quad , \qquad \text{for } i = 1 \, , 2 \, , \ldots.. \, , 6. \quad (4)$$

This function W is called the strain energy function.

W represents strain energy , per unit of undeformed volume , stored up in the body by the strains e_i.

The units of W are $\frac{\text{force} \times L}{L^3} = \frac{\text{force}}{L^2}$, **that of a stress**.

The existence of W was first introduced by George Green (1839).

Expanding the strain energy function W , given in (3) in a power series in terms of strains e_i , we write

$$2W = d_0 + 2\, d_i\, e_i + d_{ij}\, e_i\, e_j \qquad , i , j = 1, 2 , \ldots ,6 \ . \qquad (5)$$

after discarding all terms of order 3 and higher in the strains e_i as strains e_i are assumed to be **small**. In second term , summation of i is to be taken and in 3rd term , summation over dummy sufficies i & j are to be taken.

In the **natural state , $e_i = 0$, consequently** W = 0 **for $e_i = 0$**.

This gives

$$d_0 = 0 \ . \qquad (6)$$

Even otherwise , the constant term in (5) can be neglected since we are interested only in the partial derivatives of W. Therefore ,equations (5) and (6) yield

$$2W = 2d_i\, e_i + d_{ij}\, e_i\, e_j \ . \qquad (7)$$

This gives

$$\frac{\partial W}{\partial e_k} = d_i\, \delta_{ik} + \frac{1}{2}\, \frac{\partial}{\partial e_k} \{d_{ij}\, e_i\, e_j\}$$

$$= d_k + \frac{1}{2} \{d_{ij}\, \delta_{ki}\, e_j + d_{ij}\, e_i\, \delta_{kj}]$$

$$= d_k + \frac{1}{2} [d_{kj}\, e_j + d_{ki}\, e_i]$$

$$= d_k + \frac{1}{2} (d_{kj} + d_{kj})\, e_j$$

$$= d_k + (d_{kj}\, e_j)\, e_j \ .$$

This gives

$$\tau_i = d_i + c_{ij}\, e_j \ , \qquad (8)$$

where

$$c_{ij} = \frac{1}{2} (d_{ij} + d_{ji}) = c_{ji} \ . \qquad (9)$$

We observe that c_{ij} **is symmetric.**

We further assume that the stresses $\tau_i = 0$ in the undeformed state , when $e_i = 0$.

This assumption , using equation (8) , gives

$$\mathbf{d_i = 0} \quad , \qquad i = 1 , 2 , \ldots , 6 \quad . \tag{10}$$

Equations (7) , (8) and (10) give

$$\tau_i = c_{ij}\, e_j \tag{11}$$

and

$$W = \frac{1}{2}\, c_{ij}\, e_i\, e_j = \frac{1}{2}\, e_i\, \tau_i \quad , \tag{12}$$

since , two quadric homogeneous forms for W are equal as

$$d_{ij}\, e_i\, e_j = c_{ij}\, e_i\, e_j \quad . \tag{13}$$

Equation (12) shows that the strain energy function W **is a homogeneous function of degree 2** in strains e_i , $i = 1 , 2 , \ldots, 6$, and coefficients c_{ij} are symmetric.

The generalized Hooke's law under the conditions of existence of strain energy function is given in equations (9) and (11).

In matrix form , it can be expressed as

$$\begin{bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{23} \\ 2e_{13} \\ 2e_{12} \end{bmatrix} . \tag{14}$$

This law contains 21 independent elastic constants.

Result 1: From equation (2) ; we write

$$W = \frac{1}{2}\,[\tau_1 e_1 + \tau_2 e_2 + \tau_3 e_3 + \tau_4 e_4 + \tau_5 e_5 + \tau_6 e_6]$$

$$= \frac{1}{2}[\tau_{11}\, e_{11} + \tau_{22}\, e_{22} + \tau_{33}\, e_{33} + 2\,\tau_{23}\, e_{23} + 2\tau_{13}\, e_{13} + 2\tau_{12}\, e_{12}]$$

$$= \frac{1}{2}\tau_{ij}\, e_{ij} \qquad , \; i , j = 1 , 2 , 3 . \qquad (15)$$

This **result in (14) is called Claperon formula.**

Result II : For an isotropic elastic medium , the Hooke's law gives

$$\tau_{ij} = \lambda\, \delta_{ij}\, e_{kk} + 2\,\mu\, e_{ij} \quad , \quad i , j = 1 , 2 , 3. \qquad (16)$$

This gives

$$W = \frac{1}{2}\, e_{ij}[\lambda\delta_{ij}\, e_{kk} + 2\mu\, e_{ij}]$$

$$= \frac{1}{2}\lambda\, e_{kk}\, e_{kk} + \mu\, e_{ij}\, e_{ij} .$$

$$= \frac{1}{2}\lambda\, e_{kk}^{\,2} + \mu\, e_{ij}^{\,2}$$

$$= \frac{1}{2}\,\lambda(e_{11} + e_{22} + e_{33})^2 + \mu(e_{11}^{\,2} + e_{22}^{\,2} + e_{33}^{\,2} + 2e_{12}^{\,2} + 2e_{13}^{\,2} + 2\, e_{23}^{\,2}) . \; (17)$$

Result 3: Also , we have

$$e_{ij} = -\frac{\sigma}{E}\,\delta_{ij}\,\tau_{kk} + \frac{1+\sigma}{E}\tau_{ij} \; . \qquad (18)$$

Hence

$$W = \frac{1}{2}\,\tau_{ij}\,[-\frac{\sigma}{E}\delta_{ij}\,\tau_{kk} + \frac{1+\sigma}{E}\,\tau_{ij}]$$

$$= -\frac{\sigma}{2E}\tau_{ii}\,\tau_{kk} + \frac{1+\sigma}{2E}\,\tau_{ij}\,\tau_{ij} \; . \qquad (19)$$

Result 4: From equation (12) , we note that in the value of W , we may interchange e_i and τ_i . Consequently , interchanging e_i and τ_i in equation (4) , we obtain

$$\frac{\partial W}{\partial \tau_i} = e_i \qquad , \qquad \text{for } i = 1 , 2 , 3 , \ldots\ldots 6. \qquad (20)$$

This result is due to **Castigliano (1847 – 1884).**

It follows from the assumed linear assumed linear stress – strain relations.

Result 5: We know that the elastic moduli λ and μ are both positive for all physical elastic solids. The quadratic form on the right side of (17) takes only positive values for every set of values of the strains.

This shows that the strain energy function W is a positive definite form in the strain components e_{ij} , for an isotropic elastic solid.

Question : Show that the strain – energy function W for an isotropic solid **is independent** of the **choice of coordinate** axes.

Solution : We know that the strain energy function W is given by

$$W = \frac{1}{2}\ \tau_{ij}\, e_{ij}$$

$$= \frac{1}{2}\ e_{ij}\, (\lambda\, \delta_{ij}\, e_{kk} + 2\mu\, e_{ij})$$

$$= \frac{1}{2}\lambda(e_{11} + e_{22} + e_{33})^2 + \mu\ (e_{11}{}^2 + e_{22}{}^2 + e_{33}{}^2 + 2e_{12}{}^2 + 2e_{13}{}^2 + 2e_{23}{}^2]\ \ . (1)$$

Let

$$I_1 = e_{ii} = e_{11} + e_{22} + e_{33}\ , \qquad (2)$$

$$I_2 = e_{ii}\, e_{jj} – e_{ij}\, e_{ji}\ \ . \qquad (3)$$

be the first and second invariants of the strain tensor e_{ij}. As the given medium is isotropic , the elastic moduli λ and μ are also independent of the choice of coordinate axes. We write

$$W = \frac{1}{2}\lambda\ I_1{}^2 + \mu\ [(e_{11} + e_{22} + e_{33})^2 – 2\ e_{11}\, e_{22} – 2\ e_{22}\, e_{33}$$

$$–\ \ 2\ e_{33}\, e_{11} + 2e_{12}{}^2 + 2e_{13}{}^2 + 2e_{23}{}^2]$$

$$= \frac{1}{2}\ \lambda\ I^2 + \mu\ [I_1{}^2 – 2\{(e_{11}\, e_{22} – e_{12}{}^2) + (e_{22}\, e_{33} – e_{23}{}^2) + (e_{11}\, e_{33} – e_{13}{}^2\}]$$

$$= \frac{1}{2}\ \lambda\ I_1{}^2 + \mu\ I_1{}^2 – 2\ \mu\ I_2$$

$$= \left(\frac{\lambda}{2} + \mu\right) I_1^2 - 2\mu\, I_2 . \qquad (4)$$

Hence , equation (4) shows that the strain energy function W is invariant relative to all rotations of cartesian axes.

Question : Evaluate W for the stress field (for an isotropic solid)

$$\tau_{11} = \tau_{22} = \tau_{33} = \tau_{12} = 0$$

$\tau_{13} = -\mu\, \alpha\, x_2$, $\tau_{23} = \mu\, \alpha\, x_1$, $\alpha \neq 0$ is a constant and μ is the Lame's constant

Solution : We find

$$\tau_{kk} = \tau_{11} + \tau_{22} + \tau_{33} = 0.$$

Hence , the relation

$$e_{ij} = \frac{1}{2\mu} [\tau_{ij} - \frac{\lambda}{3\lambda + 2\mu} \delta_{ij}\, \tau_{kk}] \qquad ; \quad i , j = 1 , 2 , 3.$$

gives

$$e_{ij} = \frac{1}{2\mu} \tau_{ij}$$

That is , $e_{11} = e_{22} = e_{33} = e_{12} = 0$, (1)

$$e_{13} = -\frac{1}{2} \alpha\, x_2 \ , \ e_{12} = \frac{1}{2} \alpha\, x_1 \ . \qquad (2)$$

The energy function W is given by

$$W = \frac{1}{2} \tau_{ij}\, e_{ij}$$

$$= \frac{1}{4\mu} \tau_{ij}\, \tau_{ij}$$

$$= \frac{1}{4\mu} [\tau_{13}^2 + \tau_{23}^2]$$

$$= \frac{1}{4} \mu\alpha^2 (x_1^2 + x_2^2).$$

Exercise : Show that the strain energy function W is given by

$$W = W_1 + W_2 \ ,$$

where $W_1 = \frac{1}{2}\, k\, e_{ii}\, e_{ii} = \frac{1}{18k}\, \tau_{ii}\, \tau_{ii}$, $\quad$ k = bulk modulus ,

and $W_2 = \frac{1}{3}\mu[(e_{11} - e_{22})^2 + (e_{22} - e_{33})^2 + (e_{33} - e_{11})^2 + 6(e_{12}{}^2 + e_{23}{}^2 + e_{31}{}^2)]$

$$= \frac{1}{12\mu}[(\tau_{11} - \tau_{22})^2 + (\tau_{22} - \tau_{33})^2 + (\tau_{33} - \tau_{11})^2 + 6(\tau_{12}{}^2 + \tau_{23}{}^2 + \tau_{31}{}^2)] \ .$$

Question : If $W = \frac{1}{2}[\lambda\, e_{kk}{}^2 + 2\mu\, e_{ij}\, e_{ij}]$, prove the following ,

(i) $\frac{\partial W}{\partial e_{ij}} = \tau_{ij}$

(ii) $W = \frac{1}{2}\tau_{ij}\, e_{ij}$

(iii) W is a scalar invariant.

(iv) $W \geq 0$ and $W = 0$ iff $e_{ij} = 0$

(v) $\frac{\partial W}{\partial \tau_{ij}} = e_{ij}$.

Solution : We note that W is a function of e_{ij}. Partial differentiation of this function w.r.t. e_{ij} gives

$$\frac{\partial W}{\partial e_{ij}} = \frac{1}{2}[\lambda.2e_{kk}\frac{\partial e_{kk}}{\partial e_{ij}} + 4\mu\, e_{ij}]$$

$$= [\lambda\, e_{kk}\, \delta_{ij} + 2\,\mu\, e_{ij}]$$

$$= \tau_{ij} \ . \qquad (1)$$

(ii) We write

$$W = \frac{1}{2}[\lambda\, e_{kk}\, e_{kk} + 2\,\mu\, e_{ij}\, e_{ij}]$$

$$= \frac{1}{2}[\lambda\, e_{kk}\{\delta_{ij}\, e_{ij}\} + 2\,\mu\, e_{ij}\, e_{ij}]$$

$$= \frac{1}{2}(\lambda\, \delta_{ij}\, e_{kk} + 2\,\mu\, e_{ij})\, e_{ij}$$

$$W = \frac{1}{2}\, \tau_{ij}\, e_{ij}\ . \qquad (2)$$

(iii) Since τ_{ij} and e_{ij} are components of tensors , each of order 2 , respectively. So by contraction rule, $W = \frac{1}{2}\tau_{ij}\, e_{ij}$ is a scalar invariant.

(iv) Since $\lambda > 0$, $\mu > 0$, $e_{kk}^2 \geq 0$ and $e_{ij} . e_{ij} \geq 0$, if follows that $W \geq 0$.

Moreover $W = 0$ iff $e_{kk} = 0$ and $e_{ij} = 0$. Since $e_{ij} = 0$ automatically implies that $e_{kk} = 0$. Hence $W = 0$ holds iff $e_{ij} = 0$.

(v) Putting

$$e_{ij} = \frac{1+\sigma}{E}\, \tau_{ij} - \frac{\sigma}{E}\tau_{kk}\, \delta_{ij}$$

into (2) , we find

$$W = \frac{1}{2}\left[\frac{1+\sigma}{E}\, \tau_{ij}\, \tau_{ij} - \frac{\sigma}{E}\tau_{kk}\, \delta_{ij}\, \tau_{ij}\right] = \frac{1}{2}\left[\frac{1+\sigma}{E}\, \tau_{ij}\, \tau_{ij} - \frac{\sigma}{E}\tau_{kk}^2\right]\ .$$

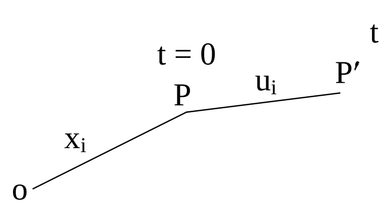
This implies

$$\frac{\partial W}{\partial \tau_{ij}} = \frac{1+\sigma}{E}\tau_{ij} - \frac{\sigma}{E}\, \tau_{kk}\, \frac{\partial \tau_{kk}}{\partial \tau_{ij}} \Rightarrow \frac{\partial W}{\partial \tau_{ij}} = \frac{1+\sigma}{E}\tau_{ij} - \frac{\sigma}{E}\tau_{kk}\, \delta_{ij} = e_{ij}\ . \quad (3)$$

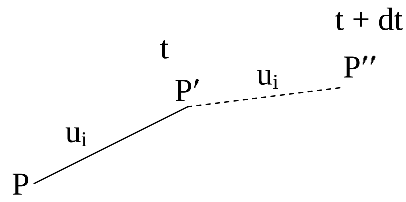
Theorem : Show that the total work done by the external forces in altering (changing) the configuration of the **natural state** to the state at **time t** is equal to the sum of the **kinetic energy** and the **strain energy**.

Proof : the natural / unstrained state of an elastic body is one in which there is a uniform temperature and zero displacement with reference to which all strains will be specified.

Let the body be in the natural state when t = 0. Let (x_1 , x_2 , x_3) denote the coordinate of an arbitrary material point of the elastic body in the undeformed / unstrained state.



If the elastic body is subjected to the action of external forces , then it may produce a deformation of the body and at any time ‘t’ , the coordinate of the same material point will be $x_i + u_i(x_1 , x_2 , x_3 , t)$.



The displacement of the point P in the interval of time (t , t + dt) is given by

$$\frac{\partial u_i}{\partial t}\, dt = \dot{u}_i\, dt \quad ,$$

where

$$\dot{u}_i = \frac{\partial u_i}{\partial t} \quad .$$

The work done by the body forces F_i acting on the volume element dτ , in time ‘dt’ sec , located at the material point P is

$$(F_i\, d\tau)\, (\dot{u}_i\, dt) = F_i\, \dot{u}_i\, d\tau\, dt \quad ,$$

and the work performed by the external surface forces $\overset{v}{T_i}$ in time interval (t , t + dt) is

$$\overset{v}{T_i}\, \dot{u}_i\, d\sigma\, dt \quad ,$$

where dσ is the element of surface.

Let E denote the work done by the body and surface forces acting on the elastic body.

Then , the rate of doing work on the body originally occupying some region τ (by external forces) is

$$\frac{dE}{dt} = \int_\tau F_i\, \dot{u}_i\, d\tau + \int_\Sigma \overset{v}{T_i}\, \dot{u}_i\, d\sigma \quad , \qquad (1)$$

where Σ denotes the original surface of the elastic body.

Now $$\int_{\Sigma} \overset{v}{T_i}\ \dot{u}_i\ d\sigma = \int_{\Sigma} (\tau_{ij}\, v_j)\ \dot{u}_i\ d\sigma$$

$$= \int_{\Sigma} (\tau_{ij}\ \dot{u}_i)\ v_j\ d\sigma$$

$$= \int_{\tau} (\tau_{ij}\ \dot{u}_i)_{,j}\ d\tau \quad ,$$

$$= \int_{\tau} [\ \tau_{ij,j}\ \dot{u}_i + \tau_{ij}\ \dot{u}_{i,j}\ ]\ d\tau$$

$$= \int_{\tau} \tau_{ij,j}\ \dot{u}_i\ d\tau + \int_{\tau} \tau_{ij}\ \dot{e}_{ij}\ d\tau + \int_{\tau} \tau_{ij}\ \dot{w}_{ij}\, d\tau \quad , \qquad (2)$$

where

$$\dot{e}_{ij} = (\dot{u}_{i,j} + \dot{u}_{j,i})/2 \quad ,$$

$$\dot{w}_{ij} = (\dot{u}_{i,j} - \dot{u}_{j,i})/2 \quad . \qquad (3)$$

Since

$$\dot{w}_{ij} = -\dot{w}_{ji} \ \text{ and } \ \tau_{ij} = \tau_{ji} ,$$

so

$$\tau_{ij}\ \dot{w}_{ij} = 0 \ , \qquad (4a)$$

From dynamical equations of motion for an isotropic body , we write

$$\tau_{ij,j} = \rho\ \ddot{u}_i - F_i \quad .$$

Therefore, $$\tau_{ij,j}\ \dot{u}_i = \rho\ \ddot{u}_i\ \dot{u}_i - F_i\ \dot{u}_i \quad . \qquad (4b)$$

Using results (4a ,b) ; we write from equations (3) and (1) ,

$$\frac{dE}{dt} = \int_{\tau} F_i\ \dot{u}_i\ d\tau + \int_{\tau} [\rho\, \ddot{u}_i\ \dot{u}_i - F_i\ \dot{u}_i]\ d\tau + \int_{\tau} \tau_{ij}\ \dot{e}_{ij}\ d\tau$$

$$= \int_{\tau} \rho\ \ddot{u}_i\ \dot{u}_i\ d\tau + \int_{\tau} \tau_{ij}\ \dot{e}_{ij}\ d\tau \ . \qquad (5)$$

The kinetic energy K of the body in motion is given by

$$K = \frac{1}{2} \int_\tau \rho \, \dot{u}_i \, \dot{u}_i \, d\tau \qquad (6)$$

Then

$$\frac{dK}{dt} = \int_\tau \rho \, \ddot{u}_i \, \dot{u}_i \, d\tau \, . \qquad (7)$$

We define the engineering notation

$$\left.\begin{array}{l} \tau_{11} = \tau_1 \, , \tau_{22} = \tau_2 \, , \tau_{33} = \tau_3 \\ \tau_{23} = \tau_4 \, , \tau_{13} = \tau_5 \, , \tau_{12} = \tau_6 \\ e_{11} = e_1 \, , e_{22} = e_2 \, , e_{33} = e_3 \\ 2e_{23} = e_4 \, , 2e_{13} = e_5 \, , 2e_{12} = e_6 \end{array}\right\} \quad . \qquad (8)$$

Then $$\int_\tau \tau_{ij} \, \dot{e}_{ij} \, d\tau = \int_\tau \tau_i \, \frac{\partial e_i}{\partial t} d\tau \quad , \qquad (9)$$

for i = 1 , 2 , 3, … , 6, and under isothermal condition , there exists a energy function

$$W = W(e_1 \, , e_2 \, , \ldots\ldots \, , e_6) \quad ,$$

with the property that

$$\frac{\partial W}{\partial e_i} = \tau_i \quad , \qquad (10)$$

$1 \le i \le 6$. From equations (9) and (10) , we write

$$\int_\tau \tau_{ij} \, \dot{e}_{ij} \, d\tau = \int_\tau \left(\frac{\partial W}{\partial e_i} \frac{\partial e_i}{\partial t} \right) d\tau = \frac{d}{dt} \int_\tau W \, d\tau$$

$$= \frac{dU}{dt} \quad , \qquad (11)$$

where

$$U = \int_\tau W \, d\tau \quad . \qquad (12)$$

From equations (5) , (7) and (11) , we write

$$\frac{dE}{dt} = \frac{dK}{dt} + \frac{dU}{dt} \quad . \qquad (13)$$

Integrating equation (13) w.r.t. ‘t’ between the limits t = 0 and t = t , we obtain

$$E = K + U \ , \qquad (14)$$

since both E and K are zero at t = 0.

The equation (14) proves the required result.

Note 1: If the elastic body is in equilibrium instead of in motion , then K = 0 and

consequently E = U.

Note 2: U is called the total strain energy of the deformation.

5.3. CLAPEYRON’S THEOREM

Statement. If an elastic body **is in equilibrium** under a given system of body forces F_i and surface forces $\overset{v}{T_i}$, then the **strain energy of deformation** is equal to **one – half** the work that would be **done by the external forces** (of the equilibrium state) acting through the **displacements u_i** form the **unstressed state** to the **state of equilibrium**.

Proof : We are required to prove that

$$\int_\tau F_i u_i \, d\tau + \int_\Sigma \overset{v}{T_i} \, u_i \, d\sigma = 2\int_\tau W \, d\tau \quad , \qquad (1)$$

where Σ denotes the original surface of the unstressed region τ of the body and W is the energy density function representing the strain every per unit volume. Now

$$\int_\Sigma \overset{v}{T_i} \, u_i \, d\sigma = \int_\Sigma \tau_{ij} u_i v_j \, d\sigma$$

$$= \int_\tau (\tau_{ij} u_i)_{,j} \, d\tau$$

$$= \int_\tau \{\tau_{ij,j} u_i + \tau_{ij} u_{i,j}\} \, d\tau$$

$$= \int_\tau \tau_{ij,j}\, u_i\, d\tau + \int_\tau \tau_{ij} \left\{ \frac{u_{i,j}+u_{j,i}}{2} + \frac{u_{i,j}-u_{j,i}}{2} \right\} d\tau$$

$$= \int_\tau \tau_{ij,j}\, u_i\, d\tau + \int_\tau \tau_{ij}\,(e_{ij} + w_{ij})\, d\tau$$

$$= \int_\tau (\tau_{ij,j}\, u_i + \tau_{ij}\, e_{ij})\, d\tau \qquad (2)$$

since

$$w_{ij} = -\,w_{ji} \text{ and } \tau_{ij} = \tau_{ji}\,.$$

Again from (2)

$$\int_\Sigma \overset{v}{T_i}\, u_i\, d\sigma = \int_\tau (-F_i\, u_i + 2W)\, d\tau\ , \qquad (3)$$

since

$$\tau_{ij,j} + F_i = 0\ ,$$

being the equilibrium equations and

$$W = \frac{1}{2}\tau_{ij}\, e_{ij}\ .$$

From (3) , we can write

$$\int_\tau F_i\, u_i\, d\tau + \int_\Sigma \overset{v}{T_i}\, u_i\, d\sigma = 2\int_\tau W\, d\tau\ , \qquad (4)$$

proving the theorem.

5.4 RECIPROCAL THEOREM OF BETTI AND RAYLEIGH

Statement : If an elastic body is subjected to two systems of body and surface forces producing two equilibrium states , show that the work done by the system of forces in acting through the displacements of the second system is equal to the work done by the second system of forces in acting through the displacements of the first system.

Proof : Let the first system of body and surface forces $\{F_i\,,\ \overset{v}{T_i}\}$ produces the displacement u_i and the second system $\{F_i'\,,\ \overset{v}{T_i}{}'\}$ produces displacements u_i'. Let

W_1 = work done by the first system of forces in acting through the displacement of the second system.

Then

$$W_1 = \int_V F_i\, u_i'\, dv + \int_S \overset{v}{T_i}\, u_i'\, ds$$

$$= \int_V F_i\, u_i'\, dv + \int_S \tau_{ij}\, v_j\, u_i'\, ds$$

$$= \int_V F_i\, u_i'\, dv + \int_V (\tau_{ij}\, u_i')_{,j}\, dv$$

$$= \int_V F_i\, u_i'\, dv + \int_V \tau_{ij,j}\, u_i'\, dv + \int_V \tau_{ij}\, u'_{i,j}\, dv$$

$$= \int_V (\tau_{ij,j} + F_i) u_i'\, dv + \int_V \tau_{ij}\, e'_{ij}\, dv \quad ,$$

$$= \int_V \tau_{ij}\, e'_{ij}\, dv \quad , \qquad (1)$$

using equations of equilibrium

$$\tau_{ij,j} + F_i = 0 \quad .$$

Hence

$$W_1 = \int_V [\lambda\, \delta_{ij}\, e_{kk} + 2\mu\, e_{ij}]\, e'_{ij}\, dv$$

$$= \int_V [\lambda\, e_{kk}\, e'_{kk} + 2\, \mu\, e_{ij}\, e'_{ij}]\, dv. \qquad (2)$$

This expression is symmetric in primed and unprimed quantities.

We conclude that $W_1 = W_2$ where W_2 is the workdone by the forces of the second system in acting through the displacements u_i of the first system.

This completes the proof of the theorem.

Corollary : Let $\overset{(1)}{\tau_{ij}}$ be the stresses corresponding to the strains $\overset{(1)}{e_{ij}}$ and $\overset{(2)}{\tau_{ij}}$ be the stresses corresponding to the strains $\overset{(2)}{e_{ij}}$, in an elastic body. Prove that

$$\overset{(1)}{\tau_{ij}}\ \overset{(2)}{e_{ij}} = \overset{(2)}{\tau_{ij}}\ \overset{(1)}{e_{ij}} \, .$$

Remark 1: Reciprocal theorem relates the equilibrium states of an elastic solid under the action of different applied loads.

Remark 2: An alternative form of the reciprocal theorem is

$$\int_{\Sigma} T_i u_i' \, d\sigma + \int_{\tau} F_i u_i' \, d\tau = \int_{\tau} \tau_{ij} e'_{ij} \, d\tau \, .$$

5.5. THEOREM OF MINIMUM POTENTIAL ENERGY

Now, we introduce an important functional , **called the potential energy of deformation ,** and prove that this functional attains an absolute minimum when the displacements of the elastic body are those of the equilibrium configuration.

Statement : Of all displacements satisfying the given boundary conditions, those which satisfy the equilibrium equations make the potential energy an absolute minimum.

Proof : Let a body τ be in equilibrium under the action of specified body and surface forces. Suppose that the surface forces T_i are prescribed only over a portion Σ_T of the surface Σ , and over the remaining surface Σ_u the displacements are known.

We denote the displacements of the equilibrium state by u_i. We consider a class of arbitrary displacements $u_i + \delta u_i$, consistent with constraints imposed on the elastic body. This means that

$$\delta u_i = 0 \quad , \qquad \text{on } \Sigma_u \qquad (1)$$

but δu_i are arbitrary over the part Σ_T , except for the condition that they belong to class C^3 and are of the order of magnitude of displacements admissible in linear elasticity.

Displacements δu_i are called **virtual displacements** .

We know that the strain energy U is given by the formula

$$U = \int_{\tau} W \, d\tau \ , \qquad (2)$$

where the strain energy function W is given by the formula

$$W = \frac{1}{2}\ \lambda\, e_{kk}\, e_{kk} + \mu\, e_{ij}\, e_{ij} \quad , \qquad (3)$$

λ and μ being Lame constants , and e_{ij} strain tensor.

The strain energy U is equal to the work done by the external forces on the elastic body in the process of bringing the body from the natural state to the equilibrium state characterized by the displacements u_i.

The **virtual work δU** performed by the external force F_i and T_i during the virtual displacements δu_i is defined by the equation

$$\delta U = \int_\tau F_i\, \delta u_i\, d\tau + \int_\Sigma T_i\, \delta u_i\, d\sigma \ . \qquad (4)$$

Since the volume τ is fixed and the forces F_i and T_i do not vary when the arbitrary variations δu_i are considered , equation (4) can be written in the form

$$\delta U = \delta\left(\int_\tau F_i\, u_i\, d\tau + \int_\Sigma T_i\, u_i\, d\sigma \right) \ . \qquad (5)$$

From equation (2) , we have

$$\delta U = \delta\left(\int_\tau W\, d\tau \right) \ . \qquad (6)$$

Equations (5) and (6) provide

$$\delta\left(\int_\tau W\, d\tau - \int_\tau F_i\, u_i\, d\tau - \int_\Sigma T_i\, u_i\, d\sigma \right) = 0 \ . \qquad (7)$$

The potential energy V is defined by the formula

$$V = \int_\tau W\, d\tau - \int_\tau F_i\, u_i\, d\tau - \int_\Sigma T_i\, u_i\, d\sigma \ . \qquad (8)$$

In view of equation (8) , relation (7) reads

$$\delta V = 0. \qquad (9)$$

This formula shows that the potential energy functional V has a stationary value in a class of admissible variations δu_i of the displacements u_i of the equilibrium state.

We shall finally show that the functional V assumes a minimum value when the displacements u_i are those of the equilibrium state.

To show this , we demonstrate that the **increment** ΔV produced in V by replacing the equilibrium displacements u_i by $u_i + \delta u_i$ is positive for all non – vanishing variations δu_i.

First , we calculate the increment ΔW in W. From (3)

$$\Delta W = \left(\frac{\lambda}{2}v^2 + \mu e_{ij}\, e_{ij}\right)\Bigg|_{u_i+\delta u_i} - \left(\frac{\lambda}{2}v^2 + \mu e_{ij}\, e_{ij}\right)\Bigg|_{u_i} . \qquad (10)$$

Now

$$e_{ij}\,\Big|_{u_i+\delta u_i} = \frac{1}{2}(u_{i,j} + u_{j,i})\,\Big|_{u_i+\delta u_i}$$

$$= \frac{1}{2}\,(u_{i,j} + u_{j,i}) + \frac{1}{2}\,[(\delta u_i)_{,j} + (\delta u_j)_{,i}]$$

$$= e_{ij} + \frac{1}{2}\,(\delta u_i)_{,j} + \frac{1}{2}\,(\delta u_j)_{,i} \qquad (11)$$

and

$$v\big|_{u_i+\delta u_i} = e_{kk}\,\big|_{u_i+\delta u_i}$$

$$= e_{ii} + (\delta u_i)_{,i}$$

$$= v + (\delta u_i)_{,i} \qquad (12)$$

Therefore , equations (10) to (12) yield

$$\Delta W = \left(\frac{\lambda}{2}\right)[v + (\delta u_i)_{,i}]\,[v + (\delta u_i)_{,i}] + \mu[e_{ij} + \frac{1}{2}(\delta u_i)_{,i} + \frac{1}{2}(\delta u_j)_{,i}]$$

$$\times\, [e_{ij} + \frac{1}{2}(\delta u_i)_{,j} + \frac{1}{2}(\delta u_i)_{,i}] - \left(\frac{\lambda}{2}\right)v^2 - \mu\, e_{ij}\, e_{ij}$$

$$= \lambda\, v\, (\delta u_i)_{,i} + 2\mu\, e_{ij}\, (\delta u_i)_{,j} + P \quad , \qquad (13)$$

where

$$P = \left(\frac{\lambda}{2}\right)[(\delta u_i)_{,i}]^2 + \left(\frac{\mu}{4}\right)[(\delta u_i)_{,j} + (\delta u_j)_{,i}]^2 \geq 0 \quad . \qquad (14)$$

Equation (13) can be rewritten in the form

$$\Delta W = \lambda\, v\, \delta_{ij}\, (\delta u_i)_{,j} + 2\,\mu\, e_{ij}\, (\delta u_i)_{,j} + P$$

$$= (\lambda\, v\, \delta_{ij} + 2\,\mu\, e_{ij})\, [(\delta u_i)_{,j}] + P$$

$$= \tau_{ij}\, [(\delta u_i)_{,j}] + P. \tag{15}$$

The increment ΔU in strain energy is , therefore ,

$$\Delta U = \int_\tau \Delta W\, d\tau$$

$$= \int_\tau \tau_{ij}\, (\delta u_i)_{,j}\, d\tau + \int_\tau P\, d\tau$$

$$= \int_\tau [(\tau_{ij}\, \delta u_i)_{,j} - \tau_{ij,j}\, \delta u_i]\, d\tau + Q$$

$$= \int_\tau (\tau_{ij}\, \delta u_i)_{,j}\, d\tau - \int_\tau \tau_{ij,j}\, \delta u_i\, d\tau + Q$$

$$= \int_\Sigma \tau_{ij}\, v_j\, \delta u_i\, d\sigma - \int_\tau \tau_{ij,j}\, \delta u_i\, d\tau + Q \quad . \tag{16}$$

In equation (16) , we have used divergence theorem and

$$Q = \int_\tau P\, d\tau \geq 0 \quad . \tag{17}$$

Since $P \geq 0$ by virtue of (14).

If the body is in equilibrium , then we have

$$\tau_{ij,j} = -F_i \quad , \quad \text{in } \tau \tag{18}$$

$$\tau_{ij}\, v_j = \overset{v}{T_i} \quad , \quad \text{on } \Sigma \tag{19}$$

and , therefore , equation (16) becomes

$$\Delta U = \int_\Sigma \overset{v}{T_i}\, \delta u_i\, d\sigma + \int_\tau F_i\, \delta u_i\, d\tau + Q. \tag{20}$$

Using the definition (8) for potential energy. we get

$$\Delta V = \Delta U - \int_\tau F_i \, \delta u_i \, d\tau - \int_\Sigma \overset{v}{T_i} \, \delta u_i \, d\sigma \ . \qquad (21)$$

Substituting (20) in (21) , we obtain

$$\Delta V = \left[\int_\Sigma \overset{v}{T}_i \, \delta u_i \, d\sigma + \int_\tau F_i \, \delta u_i \, d\tau + Q \right] - \int_\tau F_i \, \delta u_i \, d\tau - \int_\Sigma \overset{v}{T_i} \, \delta u_i \, d\sigma$$

$$= Q \ . \qquad (22)$$

Since

$$Q \geq 0 \ ,$$

(Q = 0 in the case of equilibrium only as P = 0 in this case), we find

$$\Delta V \geq 0. \qquad (23)$$

This completes the proof of the theorem.

Converse : Assume that there is a set of admissible functions $u_i + \delta u_i$ which satisfy the prescribed boundary conditions and such that

$$\Delta V = [\Delta U - \int_\Sigma \overset{v}{T_i} \, \delta u_i \, d\sigma - \int_\tau F_i \, \delta u_i \, d\tau \,] \geq 0 \ , \qquad (24)$$

on this set of functions.

From equation (16) , we write

$$\Delta U = \int_\Sigma \tau_{ij} \, v_j \, \delta u_i \, d\sigma - \int_\tau \tau_{ij\,,\,j} \, \delta u_i \, d\tau + Q \quad ,$$

where Q is given in (17). Inserting this value of ΔU in (24) , we obtain

$$\left[-\int_\tau (\tau_{ij\,,\,j} + F_i) \, \delta u_i \, d\tau + \int_\Sigma (\tau_{ij} \, v_j - \overset{v}{T_i}) \, \delta u_i \, d\sigma + Q \right] \geq 0 \ . \qquad (25)$$

On the part Σ_T of Σ , where T_i are assigned ,

$$\tau_{ij} \, v_j - T_i = 0 \quad , \qquad (26)$$

and over the remaining part Σ_u of Σ ,

$$\delta u_i = 0 \ . \tag{27}$$

Therefore ,

$$\int_{\Sigma} (\tau_{ij}\, \nu_j - \overset{\nu}{T_i})\, \delta u_i \, d\sigma = 0. \tag{28}$$

Hence , equation (25) reduces to

$$\left[- \int_{\tau} (\tau_{ij\,,\,j} + F_i)\, \delta u_i \, d\tau + Q \right] \geq 0. \tag{29}$$

Since Q is essentially positive and the displacements **δu_i are arbitrary** , the inequality (29) implies that

$$\tau_{ij,j} + F_i = 0 \ , \tag{30}$$

for every point interior to τ.

Thus , the equations of equilibrium are satisfied for every interior point in τ. This proves the converse part.

5.6. THEOREM OF MINIMUM COMPLEMENTARY ENERGY

Definition : The complementary energy V* is defined by the formula

$$V^* = U - \int_{\Sigma u} T_i\, u_i \, d\sigma = \int_{\tau} W \, d\tau - \int_{\Sigma u} T_i\, u_i \, d\sigma \ ,$$

where U is the strain energy and W is the strain energy function.

Statement : The complementary energy V* has an absolute minimum when the stress tensor τ_{ij} is that of the equilibrium state and the varied states of stress fulfill the following conditions :

(i) $(\delta\tau_{ij})_{,\,j} = 0$ in τ ,

(ii) $(\delta\, \tau_{ij})\nu_j = 0$ on Σ_T ,

(iii) $\delta\tau_{ij}$ are arbitrary on Σ_u .

Proof : Let a body τ be in equilibrium under the action of body forces F_i and surface forces T_i assigned over a part Σ_T of the surface Σ. On the remaining part Σ_u of Σ , the displacements u_i are assumed to be known.

If the τ_{ij} are the stress components of the equilibrium state , then we have

$$\left.\begin{aligned} &\tau_{ij\,,\,j} + F_i = 0 \quad \text{in } \tau\ , && (1)\\ &\tau_{ij}\, \nu_j = T_i \qquad \text{on } \Sigma_T\ , && (2)\\ &u_i = f_i \qquad \text{on } \Sigma_u\ . && (3)\end{aligned}\right\}$$

We introduce a set of functions $\overset{1}{\tau}_{ij}$ of class C^2 the body τ , which we shall also write as

$$\overset{1}{\tau}_{ij} = \tau_{ij} + \delta\tau_{ij}\ , \tag{4}$$

satisfying the conditions :

(i) $\overset{1}{\tau}_{ij\,,\,j} + F_i = 0$, in τ (5)

(ii) $\overset{1}{\tau}_{ij}\, \nu_j = T_i$, on Σ_T (6)

(iii) $\overset{1}{\tau}_{ij}$ are arbitrary on the surface Σ_u .

From equations (4) and (5) at each point of τ ; we write

$$(\tau_{ij} + \delta\,\tau_{ij})_{\,,\,j} + F_i = 0$$

$$(\tau_{ij\,,\,j} + F_i)\ \ (\delta\tau_{ij})_{\,,\,j} = 0$$

$$(\delta\,\tau_{ij})_{,j} = 0 \qquad \text{in } \tau\ . \tag{7}$$

Also from equations (4) and (6) , we have

$$(\tau_{ij} + \delta\,\tau_{ij})\, \nu_j = T_i \qquad \text{on } \Sigma_T$$

$$\tau_{ij}\, \nu_j + (\delta\tau_{ij})\, \nu_j = T_i \qquad \text{on } \Sigma_T\ ,$$

$$(\delta\tau_{ij})\, \nu_j = 0 \qquad \text{on } \Sigma_T\ . \tag{8}$$

Since $\overset{1}{\tau}_{ij}$ are arbitrary on Σ_u , so the variations $\delta\tau_{ij}$ are arbitrary on Σ_u.

As the stresses τ_{ij} are associated with the equilibrium state of the body , so τ_{ij} satisfy the Biltrami – Michell compatibility equations. Let W denote the strain – energy density function. It is given by the formula

$$W = \left(\frac{1+\sigma}{2E}\right)\tau_{ij}\ \tau_{ij} - \left(\frac{\sigma}{2E}\right)\tau_{ii}\ \tau_{ii}\ , \tag{9}$$

where σ = Poisson ration and E = Young's modulus.

The increment ΔU in the strain energy U is given by the formula

$$\Delta U = \int_\tau \Delta W \, d\tau \quad , \tag{10}$$

where the increment ΔW in W is produced by replacing τ_{ij} in (9) by $\overset{1}{\tau_{ij}} = \tau_{ij} + \delta\,\tau_{ij}$. That is ,

$$W + \Delta W = \left(\frac{1+\sigma}{2E}\right)(\tau_{ij} + \delta\tau_{ij})\,(\tau_{ij} + \delta\,\tau_{ij}) - \left(\frac{\sigma}{2E}\right)(\tau_{ii} + \delta\tau_{ii})^2$$

$$= W + \left(\frac{1+\sigma}{2E}\right)[2\tau_{ij}(\delta\,\tau_{ij}) + (\delta\,\tau_{ij})] - \left(\frac{\sigma}{2E}\right)[2\,\tau_{ii}\,(\delta\,\tau_{ii}) + (\delta\,\tau_{ii})^2] \;.$$

Hence ,

$$\Delta W = \left(\frac{1+\sigma}{E}\right)\tau_{ij}\,(\delta\,\tau_{ij}) - \frac{\sigma}{E}\tau_{ii}(\delta\,\tau_{ii}) + W(\delta\,\tau_{ij}) \;, \tag{11}$$

where

$$W(\delta\,\tau_{ij}) = \left[\left(\frac{1+\sigma}{2E}\right)(\delta\tau_{ij})(\delta\tau_{ij}) - \left(\frac{\sigma}{2E}\right)(\delta\tau_{ii})^2\right] \geq 0 \;, \tag{12}$$

since the strain energy function W is a positive definite quadric form in its variables.

From Hooke's law for isotropic solids , we have

$$e_{ij} = \left(\frac{1+\sigma}{E}\right)\tau_{ij} - \left(\frac{\sigma}{E}\right)\tau_{kk}\,\delta_{ij} \;. \tag{13}$$

From equations (11) to (12) , we write

$$\Delta W = \left[\left(\frac{1+\sigma}{2E}\right)\tau_{ij} - \left(\frac{\sigma}{E}\right)\tau_{kk}\,\delta_{ij}\right](\delta\,\tau_{ij}) + W\,(\delta\,\tau_{ij})$$

$$= (e_{ij})\,(\delta\,\tau_{ij}) + W(\delta\,\tau_{ij})$$

$$= \left(\frac{u_{i,j} + u_{j,i}}{2}\right)(\delta\,\tau_{ij}) + W(\delta\,\tau_{ij})$$

$$= (u_{i,j})\,(\delta\,\tau_{ij}) + W(\delta\,\tau_{ij})$$

$$= [(u_i\,\delta\,\tau_{ij})_{,j} - u_i(\delta\,\tau_{ij})_{,j}] + W(\delta\,\tau_{ij}). \qquad (14)$$

Since the stress components τ_{ij} were assumed to satisfy the Beltrami – Michell compatibility equations , therefore, the displacements u_i appearing in (14) are those of the actual equilibrium state of the body.

Using (14) in (10) , the increment ΔU in the strain energy becomes

$$\Delta U = \int_\tau [(u_i\,\delta\,\tau_{ij})_{,j} - u_i\,(\delta\,\tau_{ij})_{,j} + W(\delta\tau_{ij})]\,d\tau$$

$$= \int_\tau (u_i\,\delta\tau_{ij})_{,j}\,d\tau - \int_\tau u_i(\delta\tau_{ij})_{,j}\,d\tau + \int_\tau W(\delta\tau_{ij})\,d\tau$$

$$= \int_{\Sigma_u} (u_i\,\delta\tau_{ij})\,v_j\,d\sigma + P\ , \qquad (15)$$

using the Gauss divergence theorem and equations (7) and (8). In equation (15) ,

$$P = \int_\tau W(\delta\,\tau_{ij})\,d\tau \geq 0. \qquad (16)$$

As the variations $\delta\tau_{ij}$ are arbitrary on the surface Σ_u , we write

$$(\delta\tau_{ij})\,v_j = \Delta T_i\ , \qquad \text{on } \Sigma_u \qquad (17)$$

then , equation (15) reads as

$$\Delta U = \int_{\Sigma_u} u_i\,\Delta T_i\,d\sigma + P\ . \qquad (18)$$

Since the displacements u_i are assigned on the surface Σ_u , we can write (18) as

$$\Delta\Big(U - \int_{\Sigma_u} u_i\ T_i \Big) = P \geq 0\ ,$$

or

$$\Delta V^* \geq 0. \qquad (19)$$

That is , the increment ΔV^* in the complementary energy V^* (for the equilibrium state) is essentially positive.

Hence , the complementary energy functional V* has an absolute minimum in the case of an equilibrium state of the body.

This completes the proof.

5.7. THEOREM OF MINIMUM STRAIN ENERGY

Statement : The strain energy U of an elastic body in equilibrium under the action of prescribed surface forces is an absolute minimum on the set of all values of the functional U determined by the solutions of the system

$$\tau_{ij,j} + F_i = 0 \quad \text{in } \tau \quad , \qquad \tau_{ij}\, v_j = T_i \quad \text{on } \Sigma \ .$$

Proof : Continuing from the previous theorem on complementary energy , we write

$$(\delta\, \tau_{ij})\, v_j = 0 \quad \text{on } \Sigma = \Sigma_T \cup \Sigma_u \ ,$$

and equation (15) reduces to

$$\Delta U = P \geq 0 \ ,$$

showing that the increment ΔU in the strain energy U of a body in equilibrium state is positive. Therefore , U is an absolute minimum.

Hence the result.

Chapter-6
Two-Dimensional Problems

6.1 INTRODUCTION

Many physical problems regarding the deformation of elastic solids are reducible to two-dimensional elastostatic problems. This reduction facilitates an easy solution.

6.2 PLANE STRAIN DEFORMATION

An elastic body is said to be in the state of plane strain deformation, parallel to the x_1 x_2-plane, if the displacement component u_3 vanishes identically and the other two displacement components u_1 and u_2 are function of x_1 and x_2 coordinates only and independent of x_3 coordinate.

Thus, the state of plane strain deformation (parallel to x_1x_2-plane) is characterised by the displacement components of the following type

$$u_1 = u_1(x_1, x_2),\ u_2 = u_2(x_1, x_2),\ u_3 = 0. \qquad \ldots(1)$$

The plane strain deformation is a two-dimensional approximation.

A plane strain state is used for a body in which one dimension is much larger than the other two.

For example, a long pressurized pipe or a dam between two massive end walls is a suitable case of plane strain deformation.

The maintenance of a state of plane strain requires the application of tension or pressure over the terminal sections, adjusted so as to keep constant the lengths of all the longitudinal filaments.

The states of plane strain deformation can be maintained in bodies of cylindrical form by suitable forces. We take the generators of the cylindrical bounding surface to be parallel to the x_3-axis. We further suppose that the terminal sections are at right angles to this axis. The body force, if any, must be at right angles to the x_3-axis and independent of it.

The strain components, e_{ij} are given by the following strain-displacement relation

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})\ . \qquad \ldots(2a)$$

We find, for plane strain deformation parallel to x_1x_2-plane,

$$e_{13} = e_{23} = e_{33} = 0, \qquad \ldots(3)$$

and

$$e_{11} = \frac{\partial u_1}{\partial x_1},\ e_{22} = \frac{\partial u_2}{\partial x_2},$$

$$e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right), \qquad \ldots(4)$$

which are independent of x_3.

It shows that all non-zero strains are on the x_1x_2-plane and x_3-axis is strain-free/extension-free.

The strain matrix is, thus

$$(e_{ij}) = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & 0 \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The stress components τ_{ij} in terms of strains components e_{ij} are governed by generalized Hooke's law for isotropic elastic solids

$$\tau_{ij} = \lambda\, \delta_{ij}\, e_{kk} + 2\mu\, e_{ij}\,. \qquad \ldots(5a)$$

We find, for plane strain deformation parallel to x_1x_2-plane,

$$\tau_{11} = \lambda(e_{11} + e_{22}) + 2\mu\, e_{11} = (\lambda + 2\mu)\, e_{11} + \lambda\, e_{22}, \qquad \ldots(5b)$$

$$\tau_{22} = \lambda(e_{11} + e_{22}) + 2\mu\, e_{22} = \lambda\, e_{11} + (\lambda + 2\mu)e_{22}, \qquad \ldots(5c)$$

$$\tau_{12} = 2\mu\, e_{12}, \qquad \ldots(5d)$$

$$\tau_{13} = \tau_{23} = 0, \qquad \ldots(6a)$$

$$\tau_{33} = \lambda(e_{11} + e_{22}) = \sigma\,(\tau_{11} + \tau_{22})$$

$$= \frac{\lambda}{2(\lambda+\mu)}(\tau_{11} + \tau_{22}). \qquad \ldots(6b)$$

where

$$\sigma = \frac{\lambda}{2(\lambda+\mu)},$$

is the Poisson's ratio.

These relations shows all stress components are also independent of x_3 coordinate.

Since stress

$$\tau_{13} = \tau_{23} = 0, \text{ but } \tau_{33} \neq 0,$$

so the strain-free axis (x_3-axis) is not stress-free, in general.

The Cauchy's equilibrium equations for an elastic solid are

$$\tau_{ij,j} + f_i = 0, \qquad \ldots(7)$$

where $\underline{f} = f_i$ is the body force per unit volume.

In the case of plane strain deformation parallel to x_1x_2-plane, these equations reduce to, using equations (6),

$$\tau_{11,1} + \tau_{12,2} + f_1 = 0,$$

$$\tau_{12,1} + \tau_{22,2} + f_2 = 0,$$

$$f_3 = 0. \qquad \ldots(8)$$

It shows that, for a plane strain deformation parallel to x_1x_2-plane, the body force is also independent of x_3 coordinate and the body force must be perpendicular to x_3-direction.

In general, there are 6 Saint-Venant compatibility conditions for infinitesimal strain components. In the state of plane strain deformation five out of these 6 conditions are identically satisfied and the only compatibility condition to be considered further, for plane strain deformation parallel to x_1x_2-plane, is

$$e_{11,22} + e_{22,11} = 2\, e_{12,12}\,. \qquad \ldots(9)$$

Remark :- To distinguish plane strain case from the general case, we shall use subscripts α, β instead of i , j. We shall also assume that α, β vary form 1 to 2.

From equations (5a) and (8a, b); we write as follows:

$$\tau_{\alpha\beta,\beta} + f_\alpha = 0\,, \qquad \text{for } \alpha = 1, 2\,,$$

$$\lambda\,\delta_{\alpha\beta}(e_{11} + e_{22})_{,\beta} + \mu\,(u_{\alpha,\beta\beta} + u_{\beta,\alpha\beta}) + f_\alpha = 0,$$

$$\lambda(e_{11} + e_{22})_{,\alpha} + \mu\,[\nabla^2 u_\alpha + (e_{11} + e_{22})_{,\alpha}] + f_\alpha = 0$$

$$(\lambda + \mu)\,\frac{\partial}{\partial x_\alpha}(e_{11} + e_{22}) + \mu\,\nabla^2 u_\alpha + f_\alpha = 0 \qquad \ldots(10)$$

for $\alpha = 1, 2$ and

$$\nabla^2 = \frac{\partial^2}{\partial x_1^{\,2}} + \frac{\partial^2}{\partial x_2^{\,2}}\,. \qquad \ldots(10a)$$

These are the equations of equilibrium (or Navier equations) for plane strain deformation, parallel to x_1x_2-plne.

Beltrami – Michell Conditions of Compatibility for Plane Strain Deformation Parallel to x_1x_2-Plane.

Solving equations (5b, c, d) for strain components in terms of stresses, we write

$$e_{11} = \frac{(\lambda + 2\mu)\tau_{11} - \lambda\tau_{22}}{4\mu(\lambda + \mu)},$$

$$e_{22} = \frac{(\lambda + 2\mu)\tau_{22} - \lambda\tau_{11}}{4\mu(\lambda + \mu)},$$

$$2e_{12} = \frac{1}{\mu}\,\tau_{12}\,. \qquad \ldots(11)$$

Substituting the values of these strain components into Saint-Venant compatibility condition (9), we obtain

$$\frac{1}{4\mu(\lambda + \mu)}\,[(\lambda+2\mu)\,\tau_{11,22} - \lambda\,\tau_{22,22} + (\lambda+2\mu)\,\tau_{22,11} - \lambda\,\tau_{11,11}] = \frac{1}{\mu}\,\tau_{12,12}$$

$$(\lambda+2\mu)\,(\tau_{11,22} + \tau_{22,11}) - \lambda\,(\tau_{11,11} + \tau_{22,22}) = 4(\lambda + \mu)\,\tau_{12,12} \qquad \ldots(12)$$

Differentiating the equilibrium equations

$$\tau_{\alpha\beta,\beta} + f_\alpha = 0 ,$$

w.r.t. x_α and adding under summation convention, we find

$$\tau_{\alpha\beta,\alpha\beta} + f_{\alpha,\alpha} = 0 ,$$

$$\tau_{11,11} + \tau_{22,22} + 2\,\tau_{12,12} + f_{\alpha,\alpha} = 0 . \qquad \ldots(13)$$

Eliminating $\tau_{12,12}$ from (12) and (13), we get

$$(\lambda+2\mu)\,(\tau_{11,22} + \tau_{22,11}) - \lambda(\tau_{11,11} + \tau_{22,22}) + 2(\lambda+\mu)\,[\tau_{11,11} + \tau_{22,22} + f_{\alpha,\alpha}] = 0$$

$$(\lambda + 2\mu)\,[\tau_{11,22} + \tau_{22,11} + \tau_{11,11} \;\; \tau_{22,22}] + 2(\lambda+\mu)\, f_{\alpha,\alpha} = 0$$

$$\left(\frac{\partial^2}{\partial x_1^{\,2}} + \frac{\partial^2}{\partial x_2^{\,2}}\right)(\tau_{11} + \tau_{22}) + \frac{2(\lambda+\mu)}{\lambda+2\mu}(f_{1,1} + f_{2,2}) = 0$$

$$\left(\frac{\partial^2}{\partial x_1^{\,2}} + \frac{\partial^2}{\partial x_2^{\,2}}\right)(\tau_{11} + \tau_{22}) + \frac{1}{1-\sigma}\,\operatorname{div}\,\bar{f} = 0 . \qquad \ldots(14)$$

When the body force is constant or absent, then the Beltrami-Michell compatibility condition (14) for plane strain deformation (parallel to x_1x_2-plane) reduces to

$$\left(\frac{\partial^2}{\partial x_1^{\,2}} + \frac{\partial^2}{\partial x_2^{\,2}}\right)(\tau_{11} + \tau_{22}) = 0 . \qquad \ldots(15)$$

Equation (15) shows that the stress $\tau_{11} + \tau_{22}$ is harmonic, when the body force is either absent or constant, and consequently $(e_{11} + e_{22})$ is harmonic.

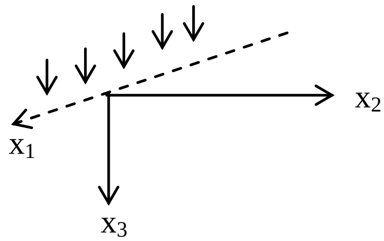
Note :- The generalized Hooke's law

$$\tau_{ij} = \frac{E}{1+\sigma}\left[e_{ij} + \frac{\sigma}{1-2\sigma}\delta_{ij}\, e_{kk}\right] ,$$

may also be used to calculate the stress components for plane strain deformation parallel to x_1x_2-plane is term of elastic modulli E, σ instead of λ , μ.

Examples of Plane strain deformations

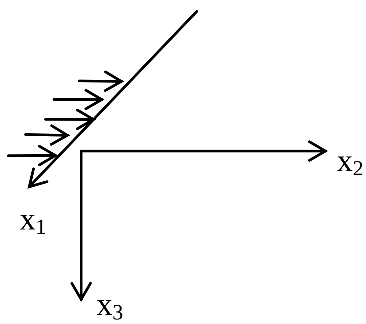
(A) The problem of stresses in an elastic semi-infinite medium subjected to a vertical line-load is a plane strain problem.



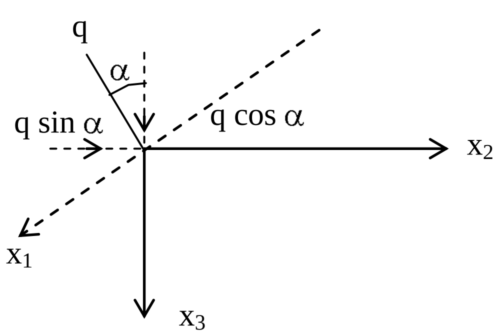
Here, the line-load extends to infinity on both sides of the origin. The displacement components are of the type

$$u_1 = 0,\ u_2 = u_2(x_2, x_3),\ u_3 = u_3(x_2, x_3)\ ,$$

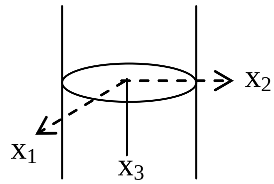
(B) The problem of determination of stresses resulting from a tangential line-load at the surface of a semi-infinite medium is a plane strain problem.



(C) The stresses and displacements in a semi-infinite elastic medium subjected to **inclined loads** can be obtained by **superposition** of the vertical and horizontal cases. If the components of the line-load are q cosα and q sinα, the stresses can be determined.



(D) The problem of deformation of an infinite cylinder by a force in the x_1x_2-plane is a plane strain problem.



In Cartesian coordinates

$$u_1 = u_1(x_1, x_2),\ u_2 = u_2(x_1, x_2),\ u_3 = 0.$$

In cylindrical coordinates

$$u_r = u(r, \theta),\ \ u_\theta = v(r, \theta)\ ,\ u_z = \ 0,$$

Principal Strains And Directions For Plane Strain Deformation

A deformation for which the strain components e_{11}, e_{22} and e_{12} are independent of x_3 and $e_{13} = e_{23} = e_{33} \equiv 0$ is called a plane strain deformation parallel to the x_1x_2-plane.

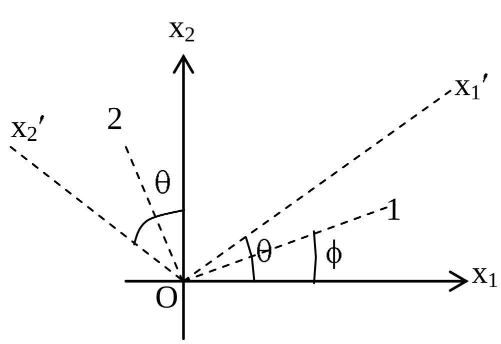
For such a deformation, the principal strain in the direction of x_3-axis is zero and the strain quadric of Cauchy

$$e_{ij}\, x_i\, x_j = \pm\, k^2 \ , \qquad \ldots(1)$$

becomes

$$e_{11}\, x_1^2 + 2\, e_{12}\, x_1x_2 + e_{22}\, x_2^2 = \pm\, k^2\ , \qquad \ldots(2)$$

which represents a cylinder in three-dimensions. Let the axes be rotated about x_3-axis through an angle θ to get new axes $Ox_1'\, x_2'\, x_3'$.



Let

$$a_{ij} = \cos(x_i', x_j) \,. \qquad \ldots(3)$$

Then

	x_1	x_2	x_3
x_1'	$\cos\theta$	$\sin\theta$	0
x_2'	$-\sin\theta$	$\cos\theta$	0
x_3'	0	0	1

…(4)

The strains e'_{pq} relative to primed system are given by the law

$$e'_{pq} = a_{pi}\, a_{qj}\, e_{ij} \,, \qquad \ldots(5)$$

for (ij) = (11), (22), (12), (21). We find

$$e'_{11} = a_{1i}\, a_{1j}\, e_{ij} = a_{11}^2\, e_{11} + a_{12}^2\, e_{22} + a_{11}\, a_{12}\, e_{12} + a_{12}\, a_{11}\, e_{12}$$

$$= \cos^2\theta .\ e_{11} + \sin^2\theta\ e_{22} + 2\sin\theta\cos\theta\ e_{12}$$

$$= e_{11}\left(\frac{1+\cos 2\theta}{2}\right) + e_{22}\left(\frac{1-\cos 2\theta}{2}\right) + e_{12}\sin 2\theta$$

$$= \frac{1}{2}(e_{11} + e_{22}) + \frac{1}{2}(e_{11} - e_{22})\cos 2\theta + e_{12}\sin 2\theta, \quad \ldots(6a)$$

Similarly

$$e'_{22} = \frac{1}{2}(e_{11} + e_{22}) - \frac{1}{2}(e_{11} - e_{22})\cos 2\theta - e_{12}\sin 2\theta, \quad \ldots(6b)$$

$$e'_{12} = -\frac{1}{2}(e_{11} - e_{22})\sin 2\theta + e_{12}\cos 2\theta \qquad \ldots(6c)$$

$$e'_{31} = e'_{32} = e'_{33} = 0 \,. \qquad \ldots(6d)$$

The principal directions in the x_1x_2-plane are given by

$$e'_{12} = 0$$

This gives

$$\frac{\sin 2\theta}{e_{12}} = \frac{\cos 2\theta}{\frac{1}{2}(e_{11} - e_{22})} = \frac{1}{\sqrt{e_{12}^2 + \frac{1}{4}(e_{11} - e_{22})^2}} \ , \qquad \ldots(7a)$$

and

$$\tan 2\theta = \frac{e_{12}}{\frac{1}{2}(e_{11} - e_{22})} = \frac{2e_{12}}{e_{11} - e_{22}} \ . \qquad \ldots(7b)$$

Let ϕ be the angle which the principal directions O_1 and O_2 make with the old axes in the x_1x_2-plane. Then

$$\tan 2\phi = \frac{2e_{12}}{e_{11} - e_{22}} \ . \qquad \ldots(8)$$

The principal strains e_1 and e_2 given by equations (6a, b) and (7a). We find (e_1

$= e_{11}^1$, $e_2 = e_{22}^1$)

$$e_1, e_2 = \frac{1}{2}(e_{11} + e_{22}) \pm \sqrt{\frac{1}{4}(e_{11} - e_{22})^2 + e_{12}^2} \ , \qquad \ldots(9)$$

the shearing strain e'_{12} will be maximum when

$$\frac{d}{d\theta} e'_{12} = 0$$

$$-(e_{11} - e_{22}) \cos 2\theta - 2\, e_{12} \sin 2\theta = 0$$

$$\frac{\cos 2\theta}{e_{12}} = \frac{\sin 2\theta}{-\frac{1}{2}(e_{11} - e_{22})} = \frac{1}{\sqrt{e_{12}^2 + \frac{1}{4}(e_{11} - e_{22})^2}} \qquad \ldots(10a)$$

This gives the direction in which the shearing strain e'_{12} is maximum and maximum value of e'_{12} is given by equations (6c) and (10a). We find

$$\left(e'_{12}\right)_{max} = \sqrt{e_{12}^2 + \frac{1}{4}(e_{11} - e_{22})^2} \quad . \qquad \ldots(10b)$$

From equations (9) and (10b), we obtain

$$\frac{e_1 - e_2}{2} = \left(e'_{12}\right)_{max}. \qquad \ldots(11)$$

This shows that maximum value of shearing strain is half of the difference of two principal strains in the x_1x_2 plane.

6.3 ANTIPLANE STRAIN DEFORMATION PARALLEL TO x_1x_2-PLANE

This deformation is characterised by

$$u_1 = u_2 \equiv 0, \;\; u_3 = u_3(x_1, x_2) .$$

The strains are

$$e_{11} = 0,\; e_{22} = e_{12} = e_{33} = 0 \;, \qquad \ldots(1)$$

$$e_{13} = \frac{1}{2}\frac{\partial u_3}{\partial x_1},\; e_{23} = \frac{1}{2}\frac{\partial u_3}{\partial x_2} \;. \qquad \ldots(2)$$

Thus, only shear strains in the x_3-direction are non-zero we can now find stresses from the Hooke’s law

$$\tau_{ij} = \lambda v.\, \delta_{ij} + 2\mu\, e_{ij} \;,$$

giving $\tau_{11} = \tau_{22} = \tau_{33} = 0,\; \tau_{12} = 0 \;,$ …(3)

and non-zero shear stresses are

$$\tau_{13} = \mu \frac{\partial u_3}{\partial x_1}, \tau_{23} = \mu \frac{\partial u_3}{\partial x_2} \;. \qquad \ldots(4)$$

The equations of equilibrium are

$$\tau_{ij,j} + F_i = 0.$$

Using the above values of stresses, we see that for i = 1, 2, we must have

$$F_1 = F_2 = 0, \qquad \ldots(5)$$

and for i = 3,

$$\tau_{31,1} + \tau_{32,2} + \tau_{33,3} + F_3 = 0$$

$$\frac{\partial \tau_{31}}{\partial x_1} + \frac{\partial \tau_{32}}{\partial x_2} + F_3 = 0 . \qquad \ldots(6)$$

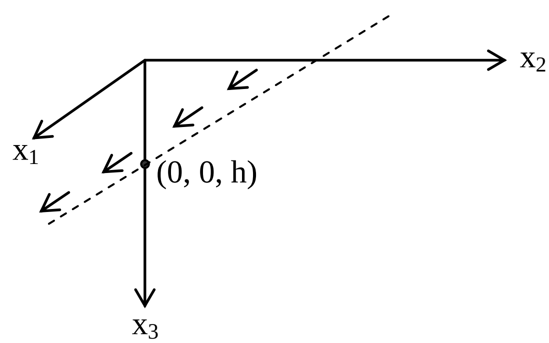
In term of u_3, this may be written as

$$\mu\left(\frac{\partial^2 u_3}{\partial x_1{}^2} + \frac{\partial^2 u_3}{\partial x_2{}^2}\right) + F_3 = 0.$$

$$\mu \nabla^2 u_3 + F_3 = 0 .$$

Example of Anti-plane Deformation

Suppose that a force is applied along the line which is parallel to x_1-axis and is situated at a depth h below the free-surface of an elastic isotropic half-space.



The resulting deformation is that of anti-plane strain deformation with

$$u_1 = u_1(x_2, x_3),\ u_2 = u_3 = 0.$$

Remark :- Two-dimensional problems in acoustics are antiplane strain problems.

6.4 PLANE STRESS

An elastic body is said to be in a state of plane stress parallel to the x_1x_2-plane if

$$\tau_{31} = \tau_{32} = \tau_{33} = 0 , \qquad \ldots(1)$$

and the remaining stress components τ_{11}, τ_{22}, τ_{12} are independent of x_3.

The equilibrium equations

$$\tau_{ij,j} + f_i = 0 ,$$

for the case of plane stress reduce to

$$\tau_{11,1} + \tau_{12,2} \;\; f_1 = 0 , \qquad \ldots(2a)$$

$$\tau_{12,1} + \tau_{22,2} + f_2 = 0 , \qquad \ldots(2b)$$

$$f_3 = 0 , \qquad \ldots(2c)$$

which are the same as for the case of plane strain deformation parallel to x_1x_2-plane. In the state of plane stress, the body force $\overline{f} = (f_1, f_2, 0)$ must be **independent of x_3** as various stress components in Cauchy's equilibrium equations in (2) are independent of x_3.

The strain components e_{ij} and stresses components τ_{ij} are connected by the

Hooke's law

$$\tau_{ij} = \lambda\, \delta_{ij}\, e_{kk} + 2\mu\, e_{ij} \quad . \qquad \ldots(3)$$

This gives

$$e_{12} = \frac{1}{2\mu}\, \tau_{12},\; e_{13} = 0,\; e_{23} = 0 ,$$

and

$$\tau_{33} = \lambda(e_{11} + e_{22} + e_{33}) + 2\mu\, e_{33}$$

$$e_{33} = -\frac{\lambda(e_{11} + e_{22})}{\lambda + 2\mu} \neq 0. \qquad \ldots(5a)$$

Hence

$$e_{kk} = e_{11} + e_{22} + e_{33} = (e_{11} + e_{22}) - \frac{\lambda(e_{11} + e_{22})}{\lambda + 2\mu}$$

$$= \frac{2\mu}{\lambda + 2\mu}(e_{11} + e_{22}). \qquad \text{…(5b)}$$

This shows that strain component e_{13} and e_{23} are zero but e_{33} is not zero.

Hence, a state of plane stress does not imply a corresponding state of plane strain.

In view of Hooke's law (3), the strain components also do not depend upon x_3.

Let

$$\bar{\lambda} = \frac{2\lambda\mu}{\lambda + 2\mu} \qquad \text{…(6)}$$

From (3), we write

$$\tau_{11} = \lambda \frac{2\mu}{\lambda + 2\mu}(e_{11} + e_{22}) + 2\mu\, e_{11} = (\bar{\lambda} + 2\mu)\, e_{11} + \bar{\lambda}\, e_{22} \qquad \text{…(7a)}$$

$$\tau_{22} = \bar{\lambda}(e_{11} + e_{22}) + 2\mu\, e_{22} = \bar{\lambda}\, e_{11} + (\bar{\lambda} + 2\mu)\, e_{22} \qquad \text{…(7b)}$$

comparing equations (7a, b) with the corresponding relations for plane strain deformation parallel to x_1x_2-plane, it is evident that solutions of plane stress problems can be obtained from the solutions of corresponding plane strain problems on replacing the true value of λ by the apparent value $\bar{\lambda} = \frac{2\lambda\mu}{\lambda + 2\mu}$.

Strain Components in terms of Stress Components

Solving equations (7a, b) for e_{11} and e_{22}, we find

$$e_{11} = \frac{2(\lambda + \mu)\tau_{11} - \lambda\tau_{22}}{2\mu(3\lambda + 2\mu)},$$

$$e_{22} = \frac{2(\lambda + \mu)\tau_{22} - \lambda\tau_{11}}{2\mu(3\lambda + 2\mu)}. \qquad \text{…(8)}$$

Substituting these values of e_{11} and e_{22} into equation (5a), we find

$$e_{33} = \frac{-\lambda(\tau_{11} + \tau_{22})}{2\mu(3\lambda + 2\mu)}. \qquad \text{…(9)}$$

Other strain components have been obtained in equation (4) already. All strain components are independent of x_3 by Hooke's law.

In this plane stress problem, two compatibility equations are identically satisfied and the remaining four are

$$e_{11,22} + e_{22,11} = 2e_{12,12}\ , \qquad \ldots(10a)$$

$$e_{33,11} = e_{33,22} = e_{33,12} = 0. \qquad \ldots(10b)$$

Since e_{33} is independent of x_3 and satisfies all conditions in (10b), so e_{33} must be of the type

$$e_{33} = c_1 + c_2\, x_2 + c_3 x_1\ , \qquad \ldots(11)$$

where c_1, c_2, c_3 are constants.

In most problems, equation (10a) is taken into consideration and requirement of equations in (10b) is ignored. This is possible, although approximately, when the dimension of the elastic body in the x_3-direction is small.

In the plane stress state, strain components e_{11}, e_{22}, e_{33} are independent of x_3 but the displacements may depend upon x_3.

Hence, plane stress problems are not truly two-dimensional.

Compatibility Equation in terms of Stresses

From equations (4), (8) and (10a), we write

$$\frac{1}{2\mu(3\lambda+2\mu)}\,[2(\lambda+\mu)\tau_{11,22} - \lambda\, \tau_{22,22} + 2(\lambda+\mu)\, \tau_{22,11} - \lambda\, \tau_{11,11}] = \frac{2}{2\mu}\tau_{12,12}$$

$$2(\lambda+\mu)(\tau_{11,22}+\tau_{22,11}) - \lambda(\tau_{22,22}\ \ \tau_{11,11}) = 2(3\,\lambda+2\mu)\, \tau_{12,12}\,. \qquad \ldots(12)$$

From equation in (2), we write

$$\tau_{11,11} + \tau_{12,12} + f_{1,1} = 0,\ \tau_{12,12} + \tau_{22,22} + f_{2,2} = 0$$

$$(\tau_{11,11} + \tau_{22,22}) + (f_{1,1} + f_{2,2}) = -2\, \tau_{12,12}\,. \qquad \ldots(13)$$

From (12) and (13) , we have

$$2(\lambda+\mu)(\tau_{11,22}+\tau_{22,11}) - \lambda(\tau_{22,22}+\tau_{11,11}) = -(3\lambda+2\mu)[\tau_{11,11}+\tau_{22,22}+f_{1,1}+f_{2,2}]$$

$$2(\lambda+\mu)(\tau_{11,22}+\tau_{22,11} + (2\lambda+2\mu)(\tau_{22,22}+\tau_{11,11}) + (3\lambda+2\mu)(f_{1,1}+f_{2,2}) = 0$$

$$2(\lambda+\mu)[(\tau_{11,22}+\tau_{22,22}) + (\tau_{11,11}+\tau_{22,11})] + (3\lambda+2\mu)(f_{1,1}+f_{2,2}) = 0$$

$$\left(\frac{\partial^2}{\partial x_1{}^2}+\frac{\partial^2}{\partial x_2{}^2}\right)(\tau_{11}+\tau_{22})+\frac{3\lambda+2\mu}{2(\lambda+\mu)}\,(f_{1,1}+f_{2,2}) = 0\,, \qquad \ldots(14)$$

which is the same as obtained from the corresponding equation for plane strain deformation parallel to x_1x_2-plane on replacing λ by $\frac{2\lambda\mu}{\lambda+2\mu}$.

Remark :- Since $\tau_{31} = \tau_{32} = 0$, so x_3-axis a principal axis of stress and the corresponding principal stress τ_3 is zero because

$$\tau_{31} = \tau_{32} = \tau_{33} = 0\,.$$

In the state of plane stress, one principal stress is zero or when one of the principal stress is zero, the state of stress is known as plane stress state.

Note :- A state of plane stress is obviously a possibility for bodies with one dimension much smaller than the other two. This type of state appears in the study of the deformation of a thin sheet plate when the plate is loaded by force applied at the boundary.

When the lengths of the generators in a cylindrical body are small in comparison with the linear dimensions of the cross-section, the body becomes a plate and the terminal sections are its faces.

The maintenance in a plate of a state of plane stress does not require the application of traction to the faces of the plate, but it required the body forces and tractions at the edge (or curved boundary) to be distributed in certain special ways.

In such a state, the stress components in the direction of the thickness of the plate are zero on both faces of the plate.

Question :- Discuss the principal stresses and principal directions of stress in a state of plane stress.

Answer :- Let an elastic body be in the state of plane stress parallel to the x_1x_2-plane. Then the stress components τ_{31}, τ_{32}, τ_{33} vanishes, i.e.,

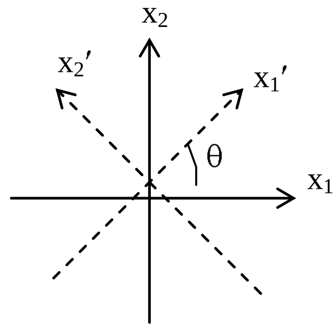
$$\tau_{31} = \tau_{32} = \tau_{33} = 0.$$

The equation of stress quadric in the state of

plane stress becomes

$$\tau_{11}\,x^2 + \tau_{22}\,y^2 + 2\tau_{12}\,xy = \pm\,k^2.$$

Let us rotate the $0x_1\,x_2\,x_3$ system about $0x_3$-axis by an amount θ .



Then

$$a_{11} = \cos\,(x_1', x_1) = \cos\theta,$$

$$a_{12} = \cos\,(x_1', x_2) = +\sin\theta\ ,$$

$$a_{13} = 0$$

$$a_{21} = \cos(x_2', x_1) = -\sin\theta,$$

$$a_{22} = \cos\,(x_2', x_2) = \cos\theta,$$

$$a_{23} = \cos\,(x_2', x_3) = 0$$

$$a_{31} = 0,\ a_{32} = 0,\ a_{33} = 1\ .$$

Let τ'_{pq} be the stresses relative to new system. Then

$$\tau'_{pq} = a_{pi}\,a_{qj}\,\tau_{ij}.$$

This gives

$$\tau'_{11} = a_{1i}\,a_{1j}\,\tau_{ij}$$

$$= \tau_{11}\cos^2\theta + \tau_{22}\sin^2\theta + 2\,\tau_{12}\cos\theta\,\sin\theta.$$

$$= \frac{1}{2}(\tau_{11} + \tau_{22}) + \frac{1}{2}(\tau_{11} - \tau_{22})\cos 2\theta + \tau_{12}\sin 2\theta$$

Similarly,

$$\tau'_{22} = \frac{1}{2}(\tau_{11} + \tau_{22}) - \frac{1}{2}(\tau_{11} - \tau_{22}) \cos 2\theta - \tau_{12} \sin 2\theta,$$

$$\tau'_{12} = -\frac{1}{2}(\tau_{11} - \tau_{22}) \sin 2\theta + \tau_{12} \cos 2\theta,$$

$$\tau'_{31} = \tau'_{32} = \tau'_{33} = 0.$$

To obtain the other two principal directions of stress, we put

$$\tau'_{12} = 0.$$

This gives $\quad \frac{1}{2}(\tau_{11} - \tau_{22}) \sin 2\theta = \tau_{12} \cos 2\theta$

$$\frac{\cos 2\theta}{\frac{1}{2}(\tau_{11} - \tau_{22})} = \frac{\sin 2\theta}{\tau_{12}} = \frac{1}{\sqrt{\frac{1}{4}(\tau_{11} - \tau_{22})^2 + \tau_{12}^2}}$$

$$\tan 2\theta = \frac{2\tau_{12}}{\tau_{11} - \tau_{22}}.$$

This determines θ and hence the directions of two principal stresses $0x'_1$ and $0x'_2$.

Let τ_1 and τ_2 be the principal stresses in the directions $0x'_1$ and $0x'_2$ respectively. Then

$$\tau_1 = \frac{1}{2}(\tau_{11} + \tau_{22}) + \frac{1}{4}\frac{(\tau_{11} - \tau_{22})^2}{\sqrt{\frac{1}{4}(\tau_{11} - \tau_{22})^2 + \tau_{12}^2}} + \frac{\tau_{22}{}^2}{\sqrt{\frac{1}{4}(\tau_{11} - \tau_{22})^2 + \tau_{12}^2}}$$

$$= \frac{1}{2}(\tau_{11} - \tau_{22}) + \sqrt{\frac{1}{4}(\tau_{11} - \tau_{22})^2 + \tau_{12}^2},$$

and

$$\tau_2 = \frac{1}{2}(\tau_{11} - \tau_{22}) - \sqrt{\frac{1}{4}(\tau_{11} - \tau_{22})^2 + \tau_{12}^2}.$$

The principal stress in the direction $0x_3$ or ox_3' is

$$\tau_3 = 0 .$$

The stress quadric with respect to principal axes becomes

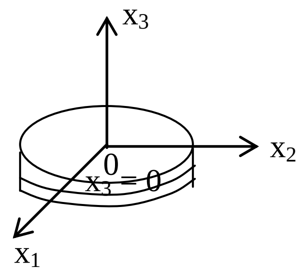
$$\tau_1\, x_1'^2 + \tau_2\, x_2'^2 = \pm k^2 ,$$

which is a cylinder whose base is a conic (where may be called stress conic); its plane contains the directions of the two principal stresses which do not vanish.

Note : (The stress of plane stress is also defined as the one in which one principal stress is zero).

6.5 GENERALIZED PLANE STRESS

Consider a thin flat plate of thickness 2h. We take the middle plane of the plate as $x_3 = 0$ plane so that the two faces of the plate are $x_3 = h$ and $x_3 = -h$. We make the following assumptions :



(a) The faces of plate are free from applied loads.

(b) The surface forces acting on the edge (curved surface) of the plate lie in planes, parallel to the middle plane ($x_3 = 0$), i.e., parallel to x_1x_2-plane and are symmetrically distributed w.r.t the middle plane $x_3 = 0$.

(c) $f_3 = 0$ and components f_1 and f_2 of the body force are symmetrically distributed w.r.t the middle plane.

Under these assumptions, the points of the middle plane will not undergo any deformation in the x_3-direction. Let

$$\bar{u}_3(x_1, x_2) = \frac{1}{2h}\int_{-h}^{h} u_3(x_1, x_2, x_3)\, dx_3 , \qquad \ldots(1)$$

denote the mean value of u_3 over the thickness of the plate. Then $\bar{u}_3(x_1, x_2)$ is independent of x_3. The symmetrical distribution of external forces w.r.t. the middle plane implies that

$$\bar{u}_3(x_1, x_2) = 0 . \qquad \ldots(2)$$

Since the faces $x_3 = \pm$ h of the plate are free from applied loads (as assumed in (a), so

$$\tau_{31}(x_1, x_2, \pm h) = \tau_{32}(x_1, x_2, \pm h) = \tau_{33}(x_1, x_2, \pm h) = 0 , \qquad \ldots(3a)$$

for all admissible values of x_1 and x_2. Hence

$$\tau_{31,1} = \tau_{32,2} = 0 , \qquad \text{at } x = \pm h . \qquad \ldots(3b)$$

Third equilibrium equation (with $f_3 = 0$) is

$$\tau_{31,1} + \tau_{32,2} + \tau_{33,3} = 0 . \qquad \ldots(4)$$

Using (2b), equation (4) reduce to

$$\tau_{33,3}(x_1, x_2, \pm h) = 0 . \qquad \ldots(5)$$

We note that τ_{33} and its derivative w.r.t. x_3 vanish on the faces of the plate. Since the thickness of plate is assumed to be very small, the stress component τ_{33} is small throughout of plate. Therefore we make assumption that

$$\tau_{33} = 0 , \qquad \ldots(6)$$

throughout the plate.

Now, we make the following definition.

Definition :- The stressed state of a thin plate for which $\tau_{33} = 0$ everywhere and τ_{31}, τ_{32} vanish on the two faces of the plate is known as generalized plane stress.

The remaining equilibrium equations for an elastic body are

$$\tau_{\alpha 1,1} + \tau_{\alpha 2,2} + \tau_{\alpha 3,3} + f_\alpha = 0 \quad \text{for } \alpha = 1, 2 .$$

Integrating w.r.t x_3 between the limits –h and +h, we obtain

$$\frac{1}{2h}\int_{-h}^{h} [\tau_{\alpha 1,1} + \tau_{\alpha 2,2} + \tau_{\alpha 3,3} + f_\alpha]\, dx_3 = 0$$

$$\bar{\tau}_{\alpha 1,1} + \bar{\tau}_{\alpha 2,2} + \bar{f}_{\alpha} = 0 \ , \qquad \ldots(7)$$

for $\alpha = 1, 2$, because,

$$\int_{-h}^{h} \tau_{\alpha 3,3}\, dx_3 = \tau_{\alpha 3}(x_1, x_2, h) - \tau_{\alpha 3}(x_1, x_2, -h)$$

$$= 0 - 0 = 0 \ , \qquad \ldots(7a)$$

as τ_{31} and τ_{32} vanish on the two faces of the plate.

Equations in (7) are the equilibrium equations for the mean values of the stresses and forces. Here $\bar{\tau}_{\alpha 1}$ etc. represents mean values.

When a plate is thin, the determination of the mean values of the components of displacement, strain and stress, taken over the thickness of the plate, may lead to knowledge nearly as useful as that of the actual values at each point. The actual values of the stresses, strains and displacements produced in the plate are determined in the case of plane stress state.

We note that the mean values of the displacements and stresses (which are independent of x_3) for the generalized plane stress problem satisfy the same set of equations that govern the plane strain problem, the only difference being is that we have to replace λ by $\dfrac{2\lambda\mu}{\lambda + 2\mu}$.

The state of generalized stress is purely two-dimensional and similar to the plane strain deformation, parallel to x_1x_2-plane.

We introduce the average field quantities $\bar{u}_i, \bar{e}_{ij}, \bar{\tau}_{ij}$ as defined in equation (1) for u_3. Then

$$\bar{u}_1 = \bar{u}_1(x_1, x_2), \quad \bar{u}_2 = \bar{u}_2(x_1, x_2), \quad \bar{u}_3 = 0 \qquad \ldots(8)$$

Since $\tau_{33} = 0$, so

$$\lambda(e_{11} + e_{22} + e_{33}) + 2\mu\, e_{33} = 0$$

$$e_{33} = \frac{-\lambda}{\lambda + 2\mu}(e_{11} + e_{22}) \qquad \ldots(8a)$$

$$e_{11} + e_{22} + e_{33} = \left(1 - \frac{\lambda}{\lambda + 2\mu}\right)(e_{11} + e_{22}) = \frac{2\mu}{\lambda + 2\mu}(e_{11} + e_{22}) \ . \quad \ldots(8b)$$

The generalized Hooke's law gives

$$\tau_{\alpha\beta} = \lambda\, \delta_{\alpha\beta}\, (e_{11} + e_{22} + e_{33}) + 2\mu\, e_{\alpha\beta} \quad \text{for } \alpha,\ \beta = 1, 2$$

$$\tau_{\alpha\beta} = \frac{2\lambda\mu}{\lambda + 2\mu} \,.\, \delta_{\alpha\beta} .\, (e_{11} + e_{22}) + 2\mu\, e_{\alpha\beta} \ .$$

Integrating over x_3 and taking mean value over the thickness, we find

$$\bar{\tau}_{\alpha\beta} = \frac{2\lambda\mu}{\lambda + 2\mu}\, \delta_{\alpha\beta}(\bar{e}_{11} + \bar{e}_{22}) + 2\mu\, \bar{e}_{\alpha\beta} \ , \qquad \ldots(9)$$

for α, $\beta = 1, 2$.

The five equations consisting of equations in (6) and (9) serve to determine the five unknown mean values $\bar{u}_1$, $\bar{u}_2$, $\bar{\tau}_{11}$, $\bar{\tau}_{22}$, $\bar{\tau}_{12}$.

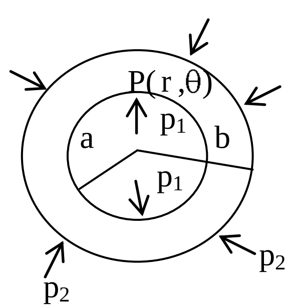
The substitution from (9) into (6) yields two equations of the Navier type

$$(\bar{\lambda} + \mu)\frac{\partial}{\partial x_\alpha}(\bar{e}_{11} + \bar{e}_{22}) + \mu\, \nabla^2\, \bar{u}_\alpha + \bar{F}_\alpha(x_1, x_2) = 0 \ , \quad \ldots(10)$$

from which the average displacements $\bar{u}_\alpha$ can be determined when the values of the $\bar{u}_\alpha$ are specified on the contour. Here $\bar{\lambda} = 2\lambda\mu/(\lambda+2\mu)$.

Example 1 :- Thick-walled Tube Under External and Internal Pressures

We consider a cross-section of a thick-walled cylindrical tube whose inner radius is a and external radius b. We shall determine the deformation of the tube due to uniform internal pressure p_1 and external pressure p_2 acting on it.



We shall use the cylindrical coordinates (r, θ, z) to solve the problem and axis of cylinder tube is taken as z-axis.

This problem is a plane strain problem and due to symmetry, the cylindrical components of displacement are of the type

$$u_r = u(r),\ u_\theta = 0,\ u_z = 0\ . \qquad \ldots(1)$$

We know that for an isotropic elastic medium, the Stoke's Navier equation of equilibrium for zero body force is

$$(\lambda + 2\mu)\ \text{grad div}\ \bar{u} - \mu\ \text{curl curl}\ \bar{u} = \bar{0}\ . \qquad \ldots(2)$$

We find

$$\text{curl}\ \bar{u} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ u_r & 0 & 0 \end{vmatrix} = \bar{0}, \qquad \ldots(3)$$

and

$$\text{div}\ \bar{u} = \frac{1}{r}\left[\frac{d}{dr}(r\,u)\right] = \frac{du}{dr} + \frac{u}{r}\ ,$$

as u is a function f r only. We have used $\frac{d}{dr}$ instead of $\frac{\partial}{\partial r}$.

From equations (2)–(4), we find

$$\text{grad div}\ \bar{u} = \bar{0}$$

$$\text{div}\ \bar{u} = \text{constt.}\ \ 2A\ (\text{say})$$

$$\frac{1}{r}\frac{d}{dr}(r\,u) = 2A$$

$$\frac{d}{dr}(r\,u) = 2Ar$$

$$ru = Ar^2 + B,\ B = \text{constt.}$$

$$U = Ar + B/r\ , \qquad \ldots(5)$$

where A and B are constants to be determined from the boundary conditions.

The strains in cylindrical coordinates (r, θ, z), using (1) and (5), are found to be

$$\left.\begin{aligned} e_{rr} &= u_{,r} = A - B/r^2 \\ e_{\theta\theta} &= \frac{u}{r} = A + B/r^2 \\ e_{r\theta} &= e_{\theta z} = e_{rz} = e_{zz} = 0 \\ \text{div } \bar{u} &= e_{rr} + e_{\theta\theta} = 2A . \end{aligned}\right\} \qquad \ldots(6)$$

The generalized Hooke's law for an isotropic material gives the expressions for stresses. We find

$$\tau_{rr} = \lambda \text{ div } \bar{u} + 2\mu\, e_{rr} = 2A\lambda + 2\mu\,(A - B/r^2) = 2(\lambda+\mu)A - 2\mu \frac{B}{r^2} ,$$

$$\tau_{\theta\theta} = \lambda \text{ div } \bar{u} + 2\mu\, e_{\theta\theta} = 2(\lambda+\mu)\, A + 2\mu\, B/r^2 ,$$

$$\tau_{zz} = \lambda \text{ div } \bar{u} + 2\mu\, e_{zz} = 2A\lambda = \sigma\,(\tau_{rr} + \tau_{\theta\theta}), \qquad \ldots(7)$$

$$\tau_{r\theta} = \tau_{\theta z} = \tau_{rz} = 0.$$

Here,

$$\sigma = \frac{\lambda}{2(\lambda + \mu)} .$$

Boundary conditions : The boundary conditions on the curved surface of the tube (or cross-section) are

$$\left.\begin{aligned} \tau_{rr} &= -p_1 \quad \text{at } r = a \\ \tau_{rr} &= -p_2 \quad \text{at } r = b \end{aligned}\right\} \qquad \ldots(8)$$

From equations (7) and (8), we write

$$-p_1 = 2(\lambda+\mu)\, A - 2\mu\, B/a^2 ,$$

$$-p_2 = 2(\lambda+\mu)\, A - 2\mu\, B/b^2 .$$

Solving these equations for A and B, we find

$$A = \frac{p_1 a^2 - p_2 b^2}{2(\lambda + \mu)(b^2 - a^2)},$$

$$B = -\frac{a^2 b^2 (p_2 - p_1)}{2\mu(b^2 - a^2)} \;. \qquad \ldots(9)$$

Putting these values of A and B in (5) and (7), we obtain the expressions for the displacement and stress which are

$$u_r = u(r) = \frac{p_1 a^2 - p_2 b^2}{2(\lambda+\mu)(b^2 - a^2)}.r - \frac{(p_2 - p_1)a^2 b^2}{2\mu(b^2 - a^2)}.\frac{1}{r} \;, \qquad \ldots(10)$$

This gives the displacement

$$\bar{u} = u\,\hat{e}_r$$

that occurs at a point distance r from the axis of the tube $(a < r < b)$.

The stresses are

$$\tau_{rr} = \frac{p_1 a^2 - p_2 b^2}{b^2 - a^2} + \frac{a^2 b^2 (p_1 - p_2)}{(b^2 - a^2)}.\frac{1}{r^2} \;, \qquad \ldots(11a)$$

$$\tau_{\theta\theta} = \frac{p_1 a^2 - p_2 b^2}{b^2 - a^2} - \frac{a^2 b^2 (p_2 - p_1)}{(b^2 - a^2)}.\frac{1}{r^2} \;, \qquad \ldots(11b)$$

$$\tau_{zz} = \sigma(\tau_{rr} + \tau_{\theta\theta}) = \frac{\lambda}{\lambda+\mu}\left(\frac{p_1 a^2 - p_2 b^2}{b^2 - a^2}\right). \qquad \ldots(11c)$$

Equation (11c) show that τ_{zz} is constant. Hence, there is a uniform extension | contraction in the direction of the axis of the tube. Moreover, cross-sections perpendicular to this axis remain plane after deformation.

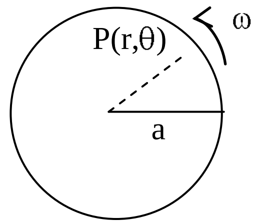
Rotating Shaft

Suppose that a solid long right circular cylinder (without an axle-hole) of radius a is rotating about its axis with uniform (constant) angular velocity ω.

We assume that the cylinder is not free to deform longitudinally.

We shall be using the cylindrical co-ordinate system (r, θ, z) to determine the displacement and stresses at any point of the cylinder.

We consider a cross-section of long right circular cylinder of radius a. This cross-section is a circle with radius a. Consider a point $P(r, \theta)$ at a distance r from the origin.



Due to symmetry, the displacement components are dependent on r only and

$$u_r = u(r),\ u_\theta = u_z = 0 \quad . \qquad \ldots(1)$$

This problem is a plane strain problem.

We know that radial and transverse components of acceleration are

$$\ddot{r} - r\,\dot{\theta}^2,\ 2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta}\ .$$

Since

$$\dot{r} = 0,\ \dot{\theta} = \omega,\ \ddot{r} = \ddot{\theta} = 0,$$

so the components of acceleration are

$$-r\,\omega^2,\ 0\ .$$

Hence equation of motion is

$$(\lambda + 2\mu)\ \text{grad div}\ \bar{u} - \mu\ \text{curl curl}\ \bar{u} = -\rho\, r\, \omega^2\ \hat{e}_r\ , \qquad \ldots(2)$$

where ρ is the density of the shaft. In view of (1),

$$\text{curl curl}\ \bar{u} = \bar{0},\ \text{div}\ \bar{u} = \frac{du}{dr} + u/r\ . \qquad \ldots(3)$$

From (2) and (3), we find

$$(\lambda + 2\mu)\ \text{grad}\left(\frac{du}{dr} + u/r\right) + \rho r\, \omega^2\, \hat{e}_r = 0$$

$$\frac{d}{dr}\left(\frac{du}{dr}+u/r\right)+\frac{\rho\omega^2}{\lambda+2\mu}\ r = 0\ .$$

Integrating, we obtain

$$\frac{du}{dr}+u/r+\frac{\rho\omega^2}{\lambda+2\mu}.\frac{r^2}{2}=2A$$

$$\frac{1}{r}\frac{d}{dr}(u.r)+\frac{\rho\omega^2}{\lambda+2\mu}.\frac{r^2}{2}=2A$$

$$\frac{d}{dr}(u.r)+\frac{\rho\omega^2}{\lambda+2\mu}.\frac{r^3}{2}=2Ar$$

$$u.\ r+\frac{\rho\omega^2}{\lambda+2\mu}.\frac{r^4}{8} = Ar^2 + B$$

$$u(r) = A\ r + \frac{B}{r}-\frac{\rho\omega^2}{\lambda+2\mu}.\frac{r^3}{8}\ , \qquad \ldots(4)$$

where A and B are constants to be determined from boundary conditions.

Since cylinder is a solid cylinder, we must take

$$B = 0,$$

since, otherwise, $|u|\to\infty$ as $r\to 0$. So (4) reduces to

$$u(r) = Ar-\frac{\rho\omega^2}{\lambda+2\mu}\left(\frac{r^3}{8}\right)\ . \qquad \ldots(5)$$

We know that the generalized Hooke's law in term of cylindrical coordinates gives

$$\tau_{rr} = \lambda\ \text{div}\ \bar{u} + 2\mu\ e_{rr} = \lambda\ \text{div}\ \bar{u} + 2\mu\ \frac{du}{dr}$$

$$= \lambda\left[\frac{du}{dr}+\frac{u}{r}\right]+2\mu\frac{du}{dr}$$

$$= \lambda\left[2A - \frac{\rho\omega^2}{\lambda+2\mu}.\frac{r^2}{2}\right] + 2\mu\left[A - \frac{\rho\omega^2}{\lambda+2\mu}.\frac{3r^2}{8}\right]$$

$$= 2(\lambda+\mu)\ A - \frac{\rho\omega^2}{\lambda+2\mu}(2\lambda+3\mu)\frac{r^2}{4} \quad . \qquad \ldots(6)$$

The surface r = a of the shaft is traction free, so the boundary condition is

$$\tau_{rr} = 0 \text{ at } r = a \qquad \ldots(7)$$

From equations (7) and (8); we find

$$A = \frac{\rho\omega^2(2\lambda+3\mu)}{(\lambda+\mu)(\lambda+2\mu)}.\frac{a^2}{8} \qquad \ldots(8)$$

Hence, at any point P(r, θ) of the shaft, the displacement and stress τ_{rr} due to rotation with angular velocity ω is

$$\bar{u} = u(r)\ .\ \hat{e}_r = \frac{\rho\omega^2 r}{8(\lambda+2\mu)}\left[\frac{2\lambda+3\mu}{\lambda+\mu}a^2 - r^2\right]\hat{e}_r\ , \qquad \ldots(9)$$

$$\tau_{rr} = \frac{\rho\omega^2}{4}\left(\frac{2\lambda+3\mu}{\lambda+2\mu}\right)\ [a^2 - r^2]\ . \qquad \ldots(10)$$

The other non-zero stresses are

$$\tau_{\theta\theta} = \lambda \text{ div } \bar{u} + 2\mu e_{\theta\theta} = \lambda \text{div } \bar{u} + 2\mu.\left(\frac{u}{r}\right)$$

$$= \frac{\rho\omega^2}{4(\lambda+2\mu)}[(2\lambda+3\mu)\ a^2 - (2\lambda+\mu)\ r^2]\ , \qquad \ldots(11a)$$

$$\tau_{zz} = \sigma(\tau_{rr} + \tau_{\theta\theta}) = \lambda \text{ div } \bar{u} = \lambda\left[2A - \frac{\rho\omega^2}{\lambda+2\mu}.\frac{r^2}{2}\right]$$

$$= \frac{\lambda\rho\omega^2}{2(\lambda+2\mu)}\left[\frac{2\lambda+3\mu}{2(\lambda+\mu)}a^2 - r^2\right]. \qquad \ldots(11b)$$

6.6 AIRY'S STRES FUNCTION FOR PLANE STRAIN PROBLEMS

In plane elastostatic problems, it is convenient to use the standard notation x, y, z instead of x_1, x_2, x_3 for Cartesian system.

The **plane strain** Cauchy's equilibrium equations on the xy-plane are

$$\tau_{xx,x} + \tau_{xy,y} + f_x = 0\,, \qquad \ldots(1)$$

$$\tau_{xy,x} + \tau_{yy,y} + f_y = 0\,. \qquad \ldots(2)$$

Assume that the external body force is conservative, so that

$$\underline{f} = -\underline{\nabla}\, V,$$

where V is the force potential. This gives

$$f_x = -V_{,x} \text{ and } f_y = -V_{,y}.$$

Using this, equations (1) and (2) can be put in the form

$$(\tau_{xx} - V)_{,x} + \tau_{xy,y} = 0 \qquad \ldots(3)$$

$$\tau_{xy,x} + (\tau_{yy} - V)_{,y} = 0\,. \qquad \ldots(4)$$

Equations (3) and (4) can be satisfied identically through the introduction of a stress function $\Phi = \Phi(x, y)$ such that

$$\tau_{xx} = \frac{\partial^2\phi}{\partial y^2} + V,$$

$$\tau_{yy} = \frac{\partial^2\phi}{\partial x^2} + V,$$

$$\tau_{xy} = \frac{-\partial^2\phi}{\partial x\partial y}\,. \qquad \ldots(5)$$

The function Φ is know as Airy's stress function, after the name of a British astronomer G. B. Airy. The function ϕ is called a stress function as ϕ generates stresses.

The Beltrani–Michell compatibility equation for plane strain deformation (in term of stresses) is

$$\nabla^2(\tau_{xx} + \tau_{yy}) + \frac{2(\lambda+\mu)}{\lambda+2\mu}\{\text{div}\ \vec{f}\} = 0\ ,$$

which now in the case of conservative body force and stress function Φ becomes, using (5),

$$\nabla^2\ \{\nabla^2\phi + 2V\} + \frac{2(\lambda+\mu)}{\lambda+2\mu}\ \{-\nabla^2\ V\} = 0$$

$$\nabla^2\ \nabla^2\phi + \left(\frac{2\mu}{\lambda+2\mu}\right)\nabla^2\ V = 0\ , \qquad \ldots(6)$$

where $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}.$

Equation (6) shows that the stress-function Φ is a biharmonic function whenever V is harmonic.

If the body force is absent/vanish, then for plane strain problem, the stress function Φ satisfies the biharmonic equation

$$\nabla^2\ \nabla^2\ \phi = 0 \qquad \ldots(7)$$

i.e. $$\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0. \qquad \ldots(7a)$$

The formula's for displacements in terms of stress function ϕ can be obtained by integrating the stress-strain relations for plane strain, which are

$$\tau_{11} = \phi_{,22} = (\lambda + 2\mu)e_{11} + \lambda\ e_{22}\ , \qquad \ldots(8)$$

$$\tau_{22} = \phi_{,11} = \lambda\ e_{11} + (\lambda+2\mu)\ e_{22}\ , \qquad \ldots(9)$$

$$\tau_{12} = -\phi_{,12} = 2\mu\, e_{12} = \mu(u_{1,2} + u_{2,1})\,. \qquad \ldots(10)$$

Solving equations (8) and (9), for e_{11} and e_{22}, we get

$$2\mu\, u_{1,1} = 2\mu\, e_{11} = -\phi_{,11} + \frac{\lambda + 2\mu}{2(\lambda+\mu)}\, \nabla^2\phi\,, \qquad \ldots(11)$$

$$2\mu\, u_{2,2} = 2\mu\, e_{22} = -\phi_{,22} + \frac{\lambda + 2\mu}{2(\lambda+\mu)}\, \nabla^2\phi\,. \qquad \ldots(12)$$

The integration of equations (11) and (12) yields

$$2\mu\, u_1 = -\phi_{,1} + \frac{\lambda + 2\mu}{2(\lambda+\mu)} \int \nabla^2\phi\, dx + f(y)\,, \qquad \ldots(13)$$

$$2\mu\, u_2 = -\phi_{,2} + \frac{\lambda + 2\mu}{2(\lambda+\mu)} \int \nabla^2\phi\, dy + g(x)\,, \qquad \ldots(14)$$

where f and g are arbitrary functions. Substituting the values of u_1 and u_2 from (13) and (14) into equation (10),, we find that

$$f'(y) + g'(x) = 0$$

$$f'(y) = -g'(x) = \text{constt} = \alpha(\text{say})$$

$$f(y) = \alpha y + \beta\,,$$

$$g(x) = -\alpha x + \gamma\,,$$

where α, β, γ are constants.

The form of f and g indicates that they represent a rigid body displacement and can thus be ignored in the analysis of deformation.

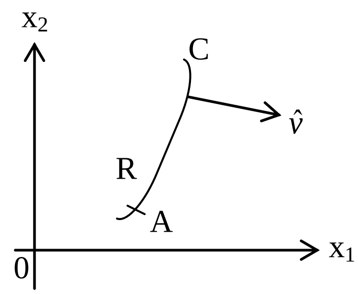
Thus, whenever ϕ becomes knows; the displacements, strains and stresses can be obtained for the plane strain problem.

Biharmonic Boundary-Value Problems

Let the body occupies the region bounded by the curve C. Then the boundary conditions are of the form

$$\tau_{\alpha\beta}\, v_\beta = T_\alpha(s)\ , \qquad \ldots(1)$$

where $T_\alpha(s)$ are known functions of the are parameter s on C, the are length s being measured along C from a fixed point, say A.



The direction cosines of a tangent are $\left(\frac{dx_1}{ds}, \frac{dx_2}{ds}\right)$, so the d. c'.s of a normal to the curve is

$$\left(\frac{dx_2}{ds}, -\frac{dx_1}{ds}\right).$$

Choosing

$$v_1 = \frac{dx_2}{ds}, \quad v_2 = -\frac{dx_1}{ds},$$

equation (1) can be written as

$$\left.\begin{aligned} \tau_{11}\frac{dx_2}{ds} - \tau_{12}\frac{dx_1}{ds} &= T_1(s) \\ \tau_{21}\frac{dx_2}{ds} - \tau_{22}\frac{dx_1}{ds} &= T_2(s)\,. \end{aligned}\right\} \qquad \ldots(2)$$

The Airy stress function ϕ generates the stresses as given below

$$\tau_{11} = \frac{\partial^2\phi}{\partial x_2^2},\ \tau_{22} = \frac{\partial^2\phi}{\partial x_1^2},\ \tau_{12} = -\frac{\partial^2\phi}{\partial x_1 \partial x_2}\ . \qquad \ldots(3)$$

From (2) and (3), we write

$$\phi_{,22}\ \frac{dx_2}{ds} + \phi_{,12}\frac{dx_1}{ds} = T_1(s)$$

$$\Rightarrow\ \frac{\partial}{\partial x_2}\left\{\frac{\partial\phi}{\partial x_2}\right\}\frac{dx_2}{ds} + \frac{\partial}{\partial x_1}\left\{\frac{\partial\phi}{\partial x_2}\right\}\frac{dx_1}{ds} = T_1(s)\,,$$

and

$$-\,\phi_{,12}\ \frac{dx_2}{ds} - \phi_{,11}\frac{dx_1}{ds} = T_2(s)$$

$$\Rightarrow\quad \frac{\partial}{\partial x_2}\left\{\frac{\partial\phi}{\partial x_1}\right\}\frac{dx_2}{ds} + \frac{\partial}{\partial x_1}\left\{\frac{\partial\phi}{\partial x_1}\right\}\frac{dx_1}{ds} = -T_2(s)\,.$$

Using chain rule, we write

$$\frac{d}{ds}\left(\frac{\partial\phi}{\partial x_2}\right) = T_1(s),$$

$$\frac{d}{ds}\left(\frac{\partial\phi}{\partial x_1}\right) = -\ T_2(s)\ .$$

Integrating these equations along C, we get

$$\frac{\partial\phi}{\partial x_1} = -\int T_2(s)\ ds = f_1(s)\ ,$$

…(4a)

$$\frac{\partial\phi}{\partial x_2} = \int T_1(s)\ ds = f_2(s)\ . \qquad \text{…(4b)}$$

Hence, the stress boundary problem of elasticity is related to the boundary value problem of the type

$$\nabla^2\,\nabla^2\,\phi = 0 \quad \text{in } R\ ,$$

$$\phi_{,\alpha} = f_\alpha(s) \quad \text{on } C\,, \qquad \ldots(5)$$

where $f_\alpha(s)$ are known functions.

The boundary value problem (5) is know as the fundamental biharmonic boundary-value problem.

This boundary-value problem can be phrased in the following form.

Normal derivative of $\phi = \dfrac{\partial\phi}{\partial\nu} = \underline{\nabla}\,\phi.\ \hat{\nu}$

$$= \phi_{,\alpha}\, u_\alpha$$

$$= f_1(s)\,\frac{dx_2}{ds} - f_2(s)\frac{dx_1}{ds}$$

$$\equiv g(s),\ \text{say}\,, \qquad \ldots(6)$$

on C,

Since $\quad d\phi = \phi_{,\alpha}\, dx_\alpha$

$$= f_\alpha\, dx_\alpha\,,$$

so $\quad \phi = \int f_\alpha\, dx_\alpha$

$$= \int\left(f_\alpha\,\frac{dx_\alpha}{ds}\right)ds$$

$$= f(s)\ \text{say}. \qquad \ldots(7)$$

on C.

Thus, the knowledge of the $\phi_{,\alpha}(s)$ on C leads to compute the value of $\phi(s)$ and its normal derivative $\dfrac{\partial\phi}{\partial\nu}$ on C .

Conversely, if ϕ and $\dfrac{\partial\phi}{\partial\nu}$ are know on C, we can compute $\phi_{,\alpha}(s)$.

Consequently, the boundary value problem in (5) can be written in an equivalent form

$$\left.\begin{aligned}&\nabla^2\nabla^2\phi = 0 \text{ in } R\\ &\phi = f(s) \text{ and } \frac{d\phi}{d\nu} = g(s) \text{ on } C\,.\end{aligned}\right\} \qquad \ldots(8)$$

The boundary value problem in (8) is more convenient in some problems.

6.7 STRESS FUNCTION FOR PLANE STRESS PROBLEM

For plane stress case, the Airy stress function ϕ is defined in the same way as for the plane strain problem. The generalized Hooke's law gives

$$e_{11} = \frac{1}{E}(\tau_{11} - \sigma\,\tau_{22}),$$

$$e_{22} = \frac{1}{E}\,(\tau_{22} - \sigma\,\tau_{11}),$$

$$e_{12} = \frac{1+\sigma}{E}\tau_{12}.$$

$$e_{11} = \frac{1}{E}[\phi_{,22} - \sigma\,\phi_{,11} + (1-\sigma)V],$$

$$e_{22} = \frac{1}{E}[\phi_{,11} - \sigma\,\phi_{,22} + (1-\sigma)V],$$

$$E_{12} = -\frac{1+\sigma}{2}\phi_{,12}\,.$$

Substitution into the Cauchy's compatibility equation, we find

$$\nabla^2\nabla^2\,\phi + (1-\sigma)\,\nabla^2\,V = 0.$$

For zero body forces, the stress function ϕ in plane stress problems, satisfies the biharmonic equation

$$\nabla^2\,\nabla^2\,\phi\; = 0.$$

Exercise :- Show that, when body force is absent, the stress function ϕ is a biharmonic function for both elasto-static problems-plane strain problem and plane stress state of the body

Airy Stress Function in Polar Coordinate

(for both cases of plane stress and plane strain)

For zero body force, components f_r, f_θ; the equilibrium equations in terms of 2–D polar coordinates (r, θ)

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,\tau_{rr}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{r\theta} - \frac{\tau_{\theta\theta}}{r} = 0,$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\,\tau_{\theta r}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta\theta} = 0,$$

are identically satisfied if the stresses are derived from a function $\phi = \phi(r, \theta)$

$$\tau_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2},$$

$$\tau_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2},$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right).$$

The function $\phi(r, \theta)$ is the Airy stress function.

In this case, the compatibility equation shows that $\phi(r, \theta)$ satisfies the biharmonic equation

$$\nabla^2\,\nabla^2\,\phi = 0,$$

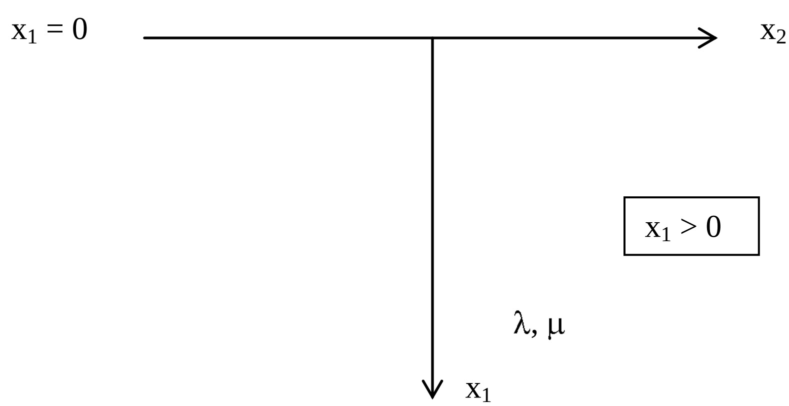
$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right) = 0. \quad \ldots(*)$$

which is know as the compatibility equation to be satisfied by the Airy stress function $\phi(r, \theta)$.

If one can find a solution of (*) that also satisfies the given boundary conditions, then the problem is solved, since by uniqueness theorem (called Kirchoff's uniqueness theorem), such a solution is unique.

6.8. DEFORMATION OF A SEMI-INFINITE ELASTIC ISOTROPIC SOLID WITH DISPLACEMENTS OR STRESSES PRESCRIBED ON THE PLANE BOUNDARY

We consider a semi-infinite elastic medium with x_1-axis pointing into the medium so that the medium occupies the region $x_1 > 0$ and $x_1 = 0$ is the plane boundary.



($x_3 = 0$ cross-section of the medium)

We consider the plane strain deformation parallel to x_1x_2-plane. Then, the displacements are of the type

$$u_1 = u_1(x_1, x_2)\ ,\ \ u_2 = u_2(x_1, x_2)\ ,\ \ u_3 = 0\ . \tag{1}$$

The stresses are generated by the Airy stress function $\phi = \phi(x_1, x_2)$ such that

$$\tau_{11} = \frac{\partial^2\phi}{\partial x_2^2}\ ,\quad \tau_{22} = \frac{\partial^2\phi}{\partial x_1^2}\ ,\quad \tau_{12} = -\frac{\partial^2\phi}{\partial x_1 \partial x_2}\ , \tag{2}$$

where ϕ satisfies the biharmonic equation

$$\frac{\partial^4\phi}{\partial x_1^4} + 2\frac{\partial^4\phi}{\partial x_1^2\partial x_2^2} + \frac{\partial^4\phi}{\partial x_2^4} = 0\ . \tag{3}$$

For convenience, we write

$$(x_1, x_2, x_3) \equiv (x, y, z), \ (u_1, u_2, u_3) = (u, v, w). \qquad (4)$$

We use the Fourier transform method to solve the biharmonic equation (4). We use $\bar{f}(x, k)$ to denote the Fourier transform of f(x,y). Then

$$\bar{f}(x, k) = F[f] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x, y) \exp\{-iky\}\, dy. \quad (5)$$

If f(x, y) satisfies the Dirichlet conditions then at the points where it is continuous, we have the inverse transformation

$$f(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(x, k) \exp(iky)\, dk, \qquad (6)$$

for $-\infty < y < \infty$. Also if

$$\frac{\partial^n f}{\partial y^n} \to 0, \qquad \text{when } y \to \pm\infty$$

for the case n = 0, 1,2,…., r –1, then

$$F\left[\frac{\partial^r f}{\partial y^r}\right] = (-ik)^r \ \bar{f}(x, k). \qquad (7)$$

Taking the Fourier transform of equation (3) w.r.t. the variable y, we obtain

$$\frac{d^4\Phi}{dx^4} - 2k^2 \frac{d^4\Phi}{dx^2} + k^4 \Phi = 0, \qquad (8)$$

where $\Phi(x, k)$ is the Fourier transform of $\phi(x,y)$. Equation (8) is an ODE of fourth order. Its solution is

$$\Phi(x, k) = (A + Bx) \exp(-|k|\, x) + (C + Dx) \exp(|k|\, x), \qquad (9)$$

where A, B, C, D are constants and they may depend upon k also.

Since we require that (9) is bounded as $x \to \infty$, we conclude that

$$C = D = 0\,, \tag{10}$$

so that (9) becomes

$$\Phi(x, k) = (A + Bx)\exp(-|k|\,x)\,. \tag{11}$$

Inverting (11) with the help of (6), we write

$$\phi(x, y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[\,(A+Bx)\exp(-\,|k|x)]\exp\,(iky)\,dk\,. \tag{12}$$

From equations (2) and (12), the stresses are found to be

$$\tau_{11} = \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} k^2[(A + Bx)\exp(-|k|\,x)]\exp\,(iky)\,dk\,, \tag{13}$$

$$\tau_{12} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} |k|\,[|k|\,A + B(\,|k|\,x-2)]\exp(-\,|k|\,x)\exp\,(iky)\,dk\,, \tag{14}$$

$$\tau_{12} = -\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} ik\,[\,|k|A - B(1-\,|k|\,x)\,]\exp\,(-|k|\,x)\exp(iky)\,dk\,. \tag{15}$$

We know that the displacements u(x,y) and v(x, y) are given by

$$2\mu u = -\,\frac{\partial\phi}{\partial x} + \frac{1}{2\alpha}\int\nabla^2\phi\,dx\,, \tag{16}$$

$$2\mu v = -\,\frac{\partial\phi}{\partial x} + \frac{1}{2\alpha}\int\nabla^2\phi\,dy\,, \tag{17}$$

after neglecting the rigid body displacements, and

$$\alpha = \frac{\lambda+\mu}{\lambda+2\mu}\,. \tag{18}$$

we find

$$\nabla^2\phi = \tau_{11} + \tau_{22}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-2B)\,|k|\,\exp(-\,|k|x)\,\exp(iky)\,dk\,. \qquad (19)$$

Hence, we find

$$2\mu u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\,+\,|k|A + B(-1 + \frac{1}{\alpha} + |k|x\,)\,\exp(-|k|x)\,\exp(iky)\,dk\,, \qquad (20)$$

$$2\mu v\,(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [(-ik)A + B(-ikx - \frac{i\,|\,k\,|}{k\alpha})]\;\exp(-\,|k|x)\,\exp(iky)\,dk\,. \qquad (21)$$

Case I. When stresses are prescribed on x = 0.

Let the boundary conditions be

$$\tau_{11}(0,y) = h(y), \qquad (22)$$

$$\tau_{12}(0,\,y) = g(y), \qquad (23)$$

where $h(y)$ and $g(y)$ are known functions of y. Then

$$\bar{\tau}_{11} = \bar{g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y)\,\exp(-iky)\,dy, \qquad (24)$$

$$\bar{\tau}_{12} = \bar{h}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(y)\,\exp(-iky)\,dy, \qquad (25)$$

so that

$$\tau_{11}(0,y) = h(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{h}(k)\,\exp(iky)\,dk\,, \qquad (26)$$

$$\tau_{12}(0,y) = g(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{g}(k)\,\exp(iky)\,dk\,. \qquad (27)$$

Putting $x = 0$ in equations (13) and (15), and comparing with respective equations (26) and (27), we obtain

$$-k^2 [\, A] = \bar{h}(k)\,, \tag{28}$$

$$ik\,[|k|\, A - B\,] = \bar{g}(k)\,. \tag{29}$$

Solving these equations for A and B, we obtain

$$A = -\frac{[\bar{h}(k)]}{k^2}\,, \tag{30}$$

$$B = -\left(|k|\,\frac{\bar{h}(k)}{k^2} + \frac{\bar{g}(k)}{ik}\right). \tag{31}$$

Putting the values of coefficients A and B from equations (30) and (31) into (13) – (15) and (20) and (21), we obtain the integral expressions for displacements and stresses.

$$2\mu\, u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\left(\frac{1}{ik} - \frac{1}{i\alpha k} - \frac{|k|\,x}{ik}\right)\bar{g}(k) + \left(-\frac{1}{\alpha\,|k|}\,x\right)\bar{h}(k)\right],$$
$$\exp(-|k|\,x)\exp(iky)dk \tag{32}$$

$$2\mu v\,(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\left(\frac{-1}{\alpha\,|k|} + x\right)\bar{g}(k) + \left(\frac{i}{k} + \frac{ikx}{|k|} - \frac{i}{k\alpha}\right)\bar{h}(k)\right].$$
$$\exp(-|k|\,x)\exp(iky)dk \tag{33}$$

$$\tau_{11}(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [(-ikx)\;\bar{g}(k) + (1 + x\,|k|)\;\bar{h}(k)]\;\exp(-|k|x)\;\exp(iky)\;dk\;, \tag{34}$$

$$\tau_{22}(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\left(ixk - \frac{2\,|k|}{k}\right)\bar{g}(k) + (1 - x\,|k|)\bar{h}(k)\right],$$
$$\exp(-|k|\,x)\exp(iky)dk \tag{35}$$

$$\tau_{12}(x,y) \quad = \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[(1 - x\,|\,k\,|)\,\bar{g}(k) + (-ixk)\,\bar{h}(k) \right] \exp(-\,|\,k\,|\,x) \exp(iky)\,dk \, . \tag{36}$$

Now we consider two particular situations in which specific surface stresses are known.

Particular Cases

(a) Normal line-load :- In this particular case, a normal line-load, P, per unit length acts on the z-axis, then

$$h(y) = -P\delta(y), \tag{37}$$

$$g(y) = 0 \, . \tag{38}$$

Consequently,

$$\bar{h}(k) = \frac{-P}{\sqrt{2\pi}} \, . \tag{39}$$

$$\bar{g}(k) = 0 \, . \tag{40}$$

Putting the values of $\bar{h}(k)$ and $\bar{g}(k)$ from equations (39) and (40) into equations (34) to (36), we find the following integral expressions for stresses at any point of isotropic elastic half-space due to a normal line-load acting on the z-axis.

$$\tau_{11}(x,y) = \frac{-P}{2\pi} \int_{-\infty}^{\infty} (1 + x\,|k|\,) \exp(-|k|x) \exp\,(iky)\, dk \, , \tag{41}$$

$$\tau_{22}(x,y) = \frac{-P}{2\pi} \int_{-\infty}^{\infty} (1 - x\,|k|\,) \exp(-|k|\, x) \exp(iky)\, dk \, , \tag{42}$$

$$\tau_{12}(x,y) = \frac{-P}{2\pi} \int_{-\infty}^{\infty} (-ixk) \exp(-\,|k|\, x) \exp\,(iky)\, dk \, . \tag{43}$$

We shall evaluate the integrals (41) to (43). We find

$$\tau_{11}(x,y) = \frac{-P}{2\pi}\left[\frac{2x}{(y^2+x^2)} + \frac{2x(x^2-y^2)}{(x^2+y^2)^2}\right]$$

$$= \frac{-2P}{\pi}\left[\frac{x^3}{(x^2+y^2)^2}\right], \tag{44}$$

$$\tau_{22}(x,y) = \frac{-P}{2\pi}\left[\frac{2x}{y^2+x^2} - \frac{2x(x^2-y^2)}{(x^2+y^2)^2}\right]$$

$$= \frac{-2P}{\pi}\left[\frac{xy^2}{(x^2+y^2)^2}\right], \tag{45}$$

$$\tau_{12}(x, y) = \frac{-P}{2\pi}\left[-ix\left\{\frac{4ixy}{(x^2+y^2)^2}\right\}\right]$$

$$= \frac{-2P}{\pi}\left[\frac{x^2y}{(x^2+y^2)^2}\right], \tag{46}$$

using the following standard integrals.

(1) $$\int_{-\infty}^{\infty} e^{+ik(y)}\, dk = 2\pi\, \delta(-y)$$

(2) $$\int_{-\infty}^{\infty} (\,|k|\,)^{-1} e^{-|k|x} e^{iky}\, dk = -\log(y^2 + x^2)$$

(3) $$\int_{-\infty}^{\infty} k^{-1} e^{-|k|x} e^{iky}\, dk = 2i \tan^{-1}\left(\frac{y}{x}\right)$$

(4) $$\int_{-\infty}^{\infty} e^{-|k|x} e^{iky}\, dk = \frac{2x}{y^2+x^2}$$

(5) $$\int_{-\infty}^{\infty} \frac{k}{|k|}\, e^{-|k|x}\, e^{iky}\, dk = \frac{-2i(-y)}{y^2+x^2} = \frac{2iy}{x^2+y^2}$$

(6) $$\int_{-\infty}^{\infty} k\, e^{-|k|x}\, e^{iky}\, dk = \frac{4iyx}{(y^2+x^2)^2}$$

(7) $$\int_{-\infty}^{\infty} |k|\, e^{-|k|x}\, e^{iky}\, dk = \frac{2(x^2-y^2)}{(x^2+y^2)^2}$$

(8) $$\int_{-\infty}^{\infty} k^2\, e^{-|k|x}\, e^{iky}\, dk = \frac{4x(x^2-3y^2)}{(x^2+y^2)^3}$$

(9) $$\int_{-\infty}^{\infty} k|k|\, e^{-|k|x}\, e^{iky}\, dk = \frac{4iy(3x^2-y^2)}{(x^2+y^2)^3}$$

The corresponding displacements can be found from equations (32), (33), (39) and (40). We find

$$2\mu u = \frac{-P}{2\pi}\int_{-\infty}^{\infty}\left(-x-\frac{1}{\alpha|k|}\right)\exp(-|k|x)\exp(iky)dk$$

$$= \frac{P}{\pi}\left[\frac{x^2}{y^2+x^2}-\frac{\log(x^2+y^2)}{2\alpha}\right], \tag{47}$$

$$2\mu v = \frac{-P}{2\pi}\int_{-\infty}^{\infty}\left(\frac{i}{k}+\frac{ixk}{|k|}+\frac{1}{i\alpha k}\right)\exp(-|k|x)\exp(iky)dk$$

$$= \frac{P}{\pi}\left[\frac{xy}{x^2+y^2}+\left(\frac{\alpha-1}{\alpha}\right)\tan^{-1}\left(\frac{y}{x}\right)\right]. \tag{48}$$

(b) Normal pressure : Suppose that a uniform normal pressure p_0 acts over the strip $-a \le y \le a$ on the surface $x = 0$ in the positive x-direction. The corresponding boundary conditions give

$$h(y) = \begin{cases} -p_0 & |y| \le a \\ 0 & |y| > a \end{cases}, \tag{49}$$

$$g(y) = 0 . \tag{50}$$

We find

$$\bar{h}(k) = -2p_0 \left(\frac{\sin ka}{k}\right), \quad \bar{g}(k) = 0 . \tag{51}$$

Proceeding as in the previous case, we find the following integral expressions for the stresses and displacements at any point of an isotropic elastic half-space due to normal pressure.

$$\tau_{11}(x, y) = -\frac{2p_0}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (1 + x\,|k|)\left(\frac{\sin ka}{k}\right) \exp(-|k|x) \exp(iky)\, dk , \tag{52}$$

$$\tau_{22}(x,y) = \frac{-2p_0}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (1 - x\,|k|)\left(\frac{\sin ka}{k}\right) \exp(-|k|x) \exp(iky) , \tag{53}$$

$$\tau_{12}(x,y) = \frac{-2p_0}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-ixk)\left(\frac{\sin ka}{k}\right) \exp(-|k|x) \exp(iky)\, dk , \tag{54}$$

$$2\mu u(x,y) = \frac{-2p_0}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(-x - \frac{1}{\alpha\,|k|}\right)\left(\frac{\sin ka}{k}\right) \exp(-|k|x) \exp(iky)\, dk , \tag{55}$$

$$2\mu v(x,y) = \frac{-2p_0}{\sqrt{2\pi}} \int_{-\infty}^{\infty} i\left(\frac{1}{k} + \frac{xk}{|k|} - \frac{1}{\alpha k}\right)\left(\frac{\sin ka}{k}\right) \exp(-|k|x) \exp(iky)\, dk . \tag{56}$$

The earlier case of the normal line-load, P per unit length, becomes the particular case of the above uniform normal strip-loading case, by taking

$$P_0 = \frac{P}{2a} , \tag{57}$$

and using the relation

$$\lim_{a \to 0} \left(\frac{\sin ka}{ka} \right) = 1 \,. \tag{58}$$

Case 2. When surface displacements are prescribed on the boundary x = 0.

Let the boundary conditions be

$$u(0, y) = h_1(y) \,, \tag{59}$$

$$v(0, y) = g_1(y) \,. \tag{60}$$

Then, we write, as before,

$$u(0, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{h}_1(k) \exp(iky)\, dk \,, \tag{61}$$

$$v(0, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{g}_1(k) \exp(iky)\, dk \,. \tag{62}$$

Proceeding as earlier, we shall get the result.

6.9 GENERAL SOLUTION OF THE BIHARMONIC EQUATION

The biharmonic equation in two-dimension is

$$\nabla^2 \nabla^2 \phi \,(x_1, x_2) = 0 \quad \text{in } R \,. \qquad \ldots(1)$$

Here R is a region of x_1x_2-plane.

Let $$\nabla^2 \phi = P_1(x_1, x_2) \,. \qquad \ldots(2)$$

Then

$$\nabla^2 P_1 = 0, \qquad \ldots(3)$$

showing that $P_1(x_1, x_2)$ is a harmonic function. Let $P_2(x_1, x_2)$ be the conjugate harmonic function. Then, the function

$$F(z) = P_1 + i\, P_2 \,, \qquad \ldots(4)$$

is an analytic function of the complex variable

$$z = x_1 + ix_2$$

satisfying the Cauchy-Riemann equations

$$P_{1,1} = P_{2,2},$$

$$P_{2,1} = -P_{1,2}. \qquad \ldots(5)$$

Let $$G(z) = \frac{1}{4}\int F(z)\,dz = p_1 + ip_2. \qquad \ldots(6)$$

Then G(z) is also an analytic function such that

$$G'(z) = \frac{1}{4}\ F(z)\ , \qquad \ldots(7)$$

and by virtue of CR-equations for G(z),

$$p_{1,1} = p_{2,2}\,,$$

$$p_{1,2} = -p_{2,1}\ , \qquad \ldots(8)$$

we find $$\frac{\partial p_1}{\partial x_1} + \frac{i\partial p_2}{\partial x_1} = \frac{1}{4}\,(P_1 + iP_2)\ . \qquad \ldots(9)$$

This gives $$p_{1,1} = p_{2,2} = \frac{1}{4}\,P_1,$$

$$P_{1,2} = -p_{2,1} = \frac{-1}{4}\ P_2\,. \qquad \ldots(10)$$

Now $$\nabla^2(p_1\,.\,x_1) = p_1\nabla_2\,x_1 + x_1\,\nabla^2\,p_1 + 2\,\nabla p_1\,.\,\nabla x_1$$

$$= 0 + 0 + 2\left[\frac{\partial p_1}{\partial x_1}\frac{\partial x_1}{\partial x_1} + \frac{\partial p_1}{\partial x_2}\cdot\frac{\partial x_1}{\partial x_2}\right]$$

$$= 2[p_{1,1} + 0]$$

$$= 2p_{1,1} \qquad \ldots(11)$$

Similarly, $$\nabla^2(p_2.\,x_2) = 2\,p_{2,2}\,. \qquad \ldots(12)$$

As p_1 and p_2 are harmonic functions therefore,

$$\nabla^2(\phi - p_1\,x_1 - p_2x_2) = \nabla^2\phi - 2(p_{1,1} + p_{2,2})$$

$$= P_1 - 2\left[\frac{1}{4}P_1 + \frac{1}{4}P_1\right].$$

This implies

$$\nabla^2 (\phi - p_1x_1 - p_2x_2) = 0 \text{ in R} \qquad \ldots(13)$$

Let $\phi - p_1x_1 - p_2x_2 = q_1 (x_1, x_2)$, say . …(14)

Then $q_1(x_1, x_2)$ is a harmonic function in R. Let $q_2(x_1, x_2)$ be a conjugate function of $q_1(x_1, x_2)$ and let

$$H(z) = q_1 + iq_2. \qquad \ldots(15)$$

Then H(z) is an analytic function of z.

From (14), we write

$$\phi(x_1, x_2) = p_1x_1 + p_2x_2 + q_1$$

$$\boldsymbol{\phi(x_1, x_2) = \mathrm{Re}\, [\, \bar{z}\, G(z) + H(z)].} \qquad \ldots(16)$$

Here, $\bar{z} = x_1 - i\, x_2$ and Re denotes the real part of the bracketed expression. The representation (16) of the biharmonic function $\phi(x_1, x_2)$ in terms of two analytic functions G(z) and H(z) was first obtained by GOURSAT.

Deduction : From (16), we write

$$\phi = p_1x_1 + p_2x_2 + q_1$$

$$\phi_{(z,\ \bar{z})} = \frac{1}{2} [\ \bar{z}\, G(z) + z\,\overline{G(z)} + H(z) + \overline{H(z)}] \ , \qquad \ldots(17)$$

since $H(z) + \overline{H(z)} = 2q_1$ and $G(z) = p_1 + ip_2$, $\overline{G(z)} = p_1 - ip_2$.

Some Terminology Involving Conjugate of Complex Functions

Suppose that t is a real variable and

$$f(t) = U(t) + iV(t) . \qquad \ldots(1)$$

is a complex valued function of the real variable t, where U(t) and V(t) are functions of t with real coefficients. The conjugate $\bar{f}(t)$ of f(t) is defined

$$\bar{f}(t) = U(t) - iV(t) \qquad \ldots(2)$$

If we replace the real variable t by the complex variable z(= x + iy), then f(z) and $\bar{f}$(z) are defined to be

$$\left.\begin{aligned} f(z) &= U(z) + iV(z) \\ \bar{f}(z) &= U(z) - iV(z). \end{aligned}\right\} \qquad \ldots(3)$$

Similarly f($\bar{z}$) and $\bar{f}$($\bar{z}$), functions of the complex variable $\bar{z} = \overline{x + iy}$ = x –iy, are defined to be

$$\left.\begin{aligned} f(\bar{z}) &= U(\bar{z}) + i\,V(\bar{z}) \\ \bar{f}(\bar{z}) &= U(\bar{z}) - i\,V(\bar{z}). \end{aligned}\right\} \qquad \ldots(4)$$

Now $\overline{f(z)}$ = conjugate of f(z)

= conjugate of {U(z) + iV(z)}

= U($\bar{z}$) –i V($\bar{z}$) (on changing i $\to$ –i and z to $\bar{z}$)

= $\bar{f}$ ($\bar{z}$), using (4)

(i) Thus $\overline{f(z)} = \bar{f}(\bar{z})$. …(5)

(ii) $\frac{d}{dz}[\overline{f(\bar{z})}] = \frac{d}{dz}$[U(z) – iV(z)], using (4a)

= U′(z) –i V′(z)

= U′($\bar{z}$) + i V′($\bar{z}$)

= $\overline{[U(\bar{z}) + iV(\bar{z})]'}$

= $\overline{f'(\bar{z})}$. …(6)

Similarly

$$\frac{d}{d\bar{z}}[\overline{f(z)}] = U'(\bar{z}) - iV'(\bar{z}) = [\overline{f'(z)}]. \qquad \ldots(7)$$

Stresses and Displacements in terms of Analytic Functions G(z) and H(z).

The stresses in terms of Airy stress function $\phi = \phi(x_1, x_2)$ are given by

$$\tau_{11} = \phi_{,22},\ \tau_{22} = \phi_{,11},\ \tau_{12} = -\phi_{,12}. \qquad \ldots(1)$$

This gives

$$\tau_{11} + i\,\tau_{12} = \phi_{,22} - i\,\phi_{,12}$$

$$= -i\frac{\partial}{\partial x_2}[\phi_{,1} + i\,\phi_{,2}], \qquad \ldots(2)$$

$$\tau_{22} - i\,\tau_{12} = \phi_{,11} + i\,\phi_{,12}$$

$$= \frac{\partial}{\partial x_1}[\phi_{,1} + i\,\phi_{,2}]. \qquad \ldots(3)$$

We have

$$z = x_1 + i\,x_2,\quad \bar{z} = x_1 - i\,x_2$$

$$x_1 = \frac{1}{2}(z + \bar{z}),\ x_2 = \frac{1}{2i}(z - \bar{z}). \qquad \ldots(4)$$

By chain rule

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}, .$$

$$\frac{\partial}{\partial x_2} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right). \qquad \ldots(5)$$

Now $$\frac{\partial\phi}{\partial x_1} + i\frac{\partial\phi}{\partial x_2} = \left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)\phi$$

$$= \left[\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} - \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)\right]\phi$$

$$= 2\,\frac{\partial\phi}{\partial \bar{z}}. \qquad \ldots(6)$$

Since

$$\phi = \frac{1}{2}[\bar{z}\ G(z) + z\ \overline{G(z)} + H(z) + \overline{H(z)}] \qquad \ldots(7)$$

where G and H are analytical functions, we find

$$2\frac{\partial\phi}{\partial\bar{z}} = G(z) + z\frac{\partial}{\partial\bar{z}}\{\overline{G(z)}\} + 0 + \frac{\partial}{\partial\bar{z}}\{\overline{H(z)}\}$$

$$= G(z) + z\,\overline{G'(z)} + \overline{H'(z)}$$

$$= G(z) + z\ \overline{G'(z)} + \overline{K(z)}\ , \qquad \ldots(8)$$

where

$$K(z) = H'(z)\ . \qquad \ldots(9)$$

From equations (2), (3), (5), (6) and (8), we find

$$\tau_{11} + i\,\tau_{12} = \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial\bar{z}}\right)[G(z) + z\overline{G'(z)} + \overline{K(z)}]$$

$$= \{G'(z) + 1\ \overline{G'(z)}\} - \{z.\overline{G''(z)}\ + \overline{K'(z)}\}$$

$$= G'(z) + \overline{G'(z)} - z.\overline{G''(z)}\ - \overline{K'(z)}, \qquad \ldots(10)$$

because

$$\frac{\partial}{\partial\bar{z}}\overline{G'(z)} = \overline{\frac{\partial}{\partial z}(G'(z)} = \overline{G''(z)}\ . \qquad \ldots(11)$$

Also

$$\tau_{22} - i\,\tau_{12} = \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial\bar{z}}\right)[G(z) + z.\overline{G'(z)} + \overline{K(z)}]$$

$$= G'(z) + \overline{G'(z)} + z.\overline{G''(z)} + \overline{K'(z)}\,. \qquad \ldots(12)$$

Adding and subtracting (10) from (10a), we obtain

$$\tau_{11} + \tau_{22} = 2[G'(z) + \overline{G'(z)}\,]$$

$$= 2.2 \text{ Re } \{G'(z)\}$$

$$= 4 \text{ Re}[G'(z)], \qquad \ldots(13)$$

$$\tau_{22} - \tau_{11} - 2i\,\tau_{12} = 2\,[\overline{zG''(z)} + \overline{K'(z)}]$$

On taking conjugate both sides, we obtain

$$\tau_{22} - \tau_{11} + 2i\,\tau_{12} = 2\,[z\,G''(z) + K'(z)]. \qquad \ldots(14)$$

Equations (13) and (14) provide expressions for stresses for plane strain deformation in terms of analytic functions G(z) and H(z).

Expressions for Displacements :- We know that, for plane strain deformation parallel to x_1x_2-plane,

$$\tau_{11} = \phi_{,22} = (\lambda+2\mu)\,u_{1,1} + \lambda\,u_{2,2}\,, \qquad \ldots(15)$$

$$\tau_{22} = \phi_{,11} = \lambda u_{1,1} + (\lambda + 2\mu)\,u_{2,2}\,, \qquad \ldots(16)$$

$$\tau_{12} = -\phi_{,12} = \mu\,(u_{1,2} + u_{2,1})\,. \qquad \ldots(17)$$

Solving equations (15) and (16) for $u_{1,1}$ and $u_{2,2}$ in terms of $\phi_{,11}$ and $\phi_{,22}$, we find

$$2\mu\,u_{1,1} = \phi_{,11} + \frac{\lambda+2\mu}{2(\lambda+\mu)}\,\nabla^2\phi, \qquad \ldots(18)$$

$$2\mu\,u_{2,2} = -\phi_{,22} + \frac{\lambda+2\mu}{2(\lambda+\mu)}\,\nabla^2\phi\,. \qquad \ldots(19)$$

We know that

$$\nabla^2\phi = P_1 = 4\,p_{1,1} = 4\,p_{2,2}\,. \qquad \ldots(20)$$

Putting the values of $\nabla^2\,\phi$ from equations (20) into equations (18) and (17), we obtain

$$2\mu\, u_{1,1} = -\,\phi_{,11} + \frac{2(\lambda+2\mu)}{(\lambda+\mu)}\, p_{1,1}, \qquad \ldots(21)$$

$$2\mu\, u_{2,2} = -\phi_{,22} + \frac{2(\lambda+2\mu)}{\lambda+\mu}\, p_{2,2}. \qquad \ldots(22)$$

The integration of these equations yields,

$$2\mu\, u_1 = -\phi_{,1} + \frac{2(\lambda+2\mu)}{\lambda+\mu}\, p_1 + f(x_2), \qquad \ldots(23)$$

$$2\mu\, u_2 = -\phi_{,2} + \frac{2(\lambda+2\mu)}{\lambda+\mu}\, p_2 + g(x_1). \qquad \ldots(24)$$

where $f(x_2)$ and $g(x_1)$ are, as yet, arbitrary functions. Equation (17) serves to determine f and g.

Since $p_{1,2} = -p_{2,1}$, we easily obtain from equations (17), (23) and (24) that

$$f'(x_2) + g'(x_1) = 0$$

$$f'(x_2) = -g'(x_1) = \text{constt} = \alpha \text{ (say)}.$$

Hence $\qquad f(x_2) = \alpha\, x_2 + \beta,$

$$g(x_1) = -\alpha x_1 + \gamma\,. \qquad \ldots(25)$$

where α, β and γ are constants.

From equations (23), (24) and (25), we note that f and g represent a rigid body displacements and therefore can be neglected in the analysis of deformation.

Setting $f = g = 0$ in (23) and (24), we write

$$2\mu\, u_1 = -\phi_{,1} + \frac{2(\lambda+2\mu)}{\lambda+\mu} p_1, \qquad \ldots(26)$$

$$2\mu\, u_2 = -\phi_{,2} + \frac{2(\lambda+2\mu)}{\lambda+\mu}\, p_2. \qquad \ldots(27)$$

This implies, in compact form,

$$2\mu\,(u_1 + iu_2) = -(\phi_{,1} + i\,\phi_{,2}) + \frac{2(\lambda+2\mu)}{\lambda+\mu}\,(p_1 + i\,p_2)$$

$$= -[G(z) + z\,\overline{G'(z)} + \overline{H'(z)}] + \frac{2(\lambda+2\mu)}{\lambda+\mu}G(z)$$

$$= \left[\frac{2(\lambda+2\mu)}{\lambda+\mu} - 1\right]G(z) - z\overline{G'(z)} - \overline{H'(z)}$$

$$= k_0\,G(z) - z\,\overline{G'(z)} - \overline{H'(z)}, \qquad \ldots(28)$$

where

$$k_0 = \frac{\lambda+3\mu}{\lambda+\mu} = 3 - 4\sigma. \qquad \ldots(29)$$

The quantity k_0 is called the Muskhelishvile's constant.

The formulas given by (13), (14) and (28) are called Kolosov-Muskhelishvilli formulas. This result corresponds to the state of plane strain.

Remark :- In the generalized plane-stress problem, λ must be replaced by $\bar{\lambda} = \frac{2\lambda\mu}{\lambda+2\mu}$ and if k_0 is the corresponding value of in (29). We find

$$k_0 = \frac{\bar{\lambda}+3\mu}{\bar{\lambda}+\mu} = \frac{5\lambda+6\mu}{3\lambda+2\mu} = \frac{3-\sigma}{1+\sigma}.$$

We note that both k and k_0 are greater than 1.

First and Second Boundary Values Problem of Plane Elasticity

We first consider the first boundary-value problem in which the stress component $\tau_{\alpha\beta}$ must be such that

$$\tau_{\alpha\beta}\,\nu_\beta = T_\alpha(s),\ \ \alpha, \beta = 1, 2\ , \qquad \ldots(1)$$

where the stress vector $T_\alpha(s)$ is specified on the boundary.

In terms of Airy stress function $\phi = \phi(x_1, x_2)$, the condition (1) is equivalent to

$$\phi_{,1}(s) = -\int T_2(s)\,ds\ , \qquad \ldots(2)$$

$$\phi_{,2}(s) = \int T_1(s)\, ds\ ,$$

on C. Now, we write

$$\phi_{,1} + i\,\phi_{,2} = i \int [T_1(s) + iT_2(s)]\, ds \quad \text{on C.} \qquad \ldots(3)$$

we know that

$$\phi_{,1} + i\,\phi_{,2} = G(z) + z\,\overline{G'(z)} + \overline{H'(z)}\ , \qquad \ldots(4)$$

where G and H are analytic functions. Thus, the boundary condition in terms of G and H is

$$G(z) + z\,\overline{G'(z)} + \overline{H'(z)} = i \int [T_1(s) + i\,T_2(s)]\, ds \text{ on C.} \qquad \ldots(5)$$

The determination of the corresponding boundary conditions in the second boundary-value problem is as follows.

In this type of boundary value problem, boundary conditions are

$$u_\alpha = g_\alpha(s), \text{ on C,} \qquad \ldots(6)$$

where the functions $g_\alpha(s)$ are know functions. Equation (6) yields

$$2\mu(u_1 + i\,u_2) = 2\mu\,[g_1(s) + ig_2(s)], \text{ on C,}$$

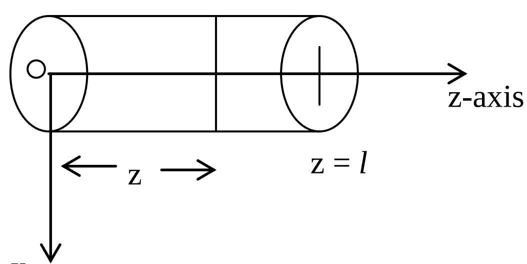
This implies

$$\Rightarrow \qquad k_0\, G(z) - z\,\overline{G'(z)} - \overline{H'(z)} = 2\mu\,[g_1(s) + i\,g_2(s)], \text{ on C .} \qquad \ldots(7)$$

Chapter-7
Torsion of Bars

7.1 TORSION OF A CIRCULAR SHAFT

Let us consider an elastic right circular beam of length *l*. We choose the z-axis along the axis of the beam so that its ends lie in the planes z = 0 and z = l, respectively. The end z = 0 is fixed in the xy-plane and a couple of vector moment $\vec{M} = M\hat{e}_3$ about the z-axis is applied at the end z = *l* . The lateral surface of the circular beam is stress-free and body forces are neglected.



(Torsion of a circular beam)

The problem is to compute the displacements, strains and stresses developed in the beam because of the twist (or torsion) it experiences due to the applied couple.

Because of the symmetry of the cross-section of the beam by planes normal to the z-axis, these sections will remain planes even after deformation.

That is, if (u, v, w) are the displacements, then

$$\mathbf{w = 0,} \qquad \text{….(1)}$$

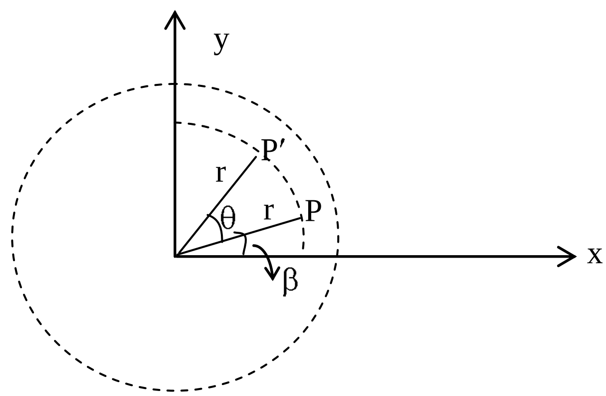
However, these plane sections will get rotated about the z-axis through some angle θ. The rotation θ will depend upon the distance of the cross-section from the fixed end z = 0 of the beam. For small deformations, we assume that

$$\theta = \alpha\, z \qquad \text{....(2)}$$

where α is a constant.

The constant α represents the twist per unit length (for z = 1).

Now, we consider a cross-section of the right circular beam. Let P(x, y) be a point on it before deformations and P′ (x′, y′) be the same material point after deformation.



(Rotation of a section of the circular shaft)

Then

$$u = x' - x$$
$$= r\,[\,\cos(\theta+\beta) - r\cos\beta]$$
$$= r\,[\cos\theta\cos\beta - \sin\theta\sin\beta] - r\cos\beta$$
$$= r\cos\beta\,(\cos\theta - 1) - y\sin\theta$$
$$= x\,(\cos\theta - 1) - y\sin\theta\,. \qquad \text{....(3)}$$

Since θ is small, so

$$\cos\theta \approx 1\,,\ \ \sin\theta \approx \theta\,.$$

Therefore

$$\mathbf{u = -y\theta \ = -\alpha yz\,.} \qquad \textbf{....(4)}$$

Similarly

$$v = y' - y$$
$$= r\sin(\beta + \theta) - r\sin\beta$$
$$\mathbf{= x\sin\theta + y\,(\cos\theta - 1)\,.} \qquad \textbf{....(5)}$$

For small deformations, θ is small, so

$$\mathbf{v = x\,\theta = \alpha\,xz\,.} \qquad \textbf{....(6)}$$

Thus, the displacement components at any point (x, y, z) of the beam due to twisting are

$$\mathbf{u = -\alpha\, yz, \qquad v = \alpha\, xz, \qquad w = 0,} \qquad \text{….(7)}$$

where α is the twist per unit length. Therefore, the displacement vector $\bar{u}$ is

$$\bar{u} = -\alpha z\ (y\hat{i} - x\hat{j})\ . \qquad \text{….(8)}$$

Then

$$\bar{u}\,.\,\bar{r} = -\alpha z\,(y\hat{i} - x\hat{j})\,.\,(x\hat{i} + y\hat{j}) = 0\ , \qquad \text{….(9)}$$

and

$$|\bar{u}| = \alpha z\sqrt{x^2 + y^2} = \alpha zr\ , \qquad \text{...(10)}$$

in polar coordinates (r, θ).

That is, the displacement vector is in the tangential direction and is of magnitude αrz.

The corresponding strains are

$$e_{xx} = 0,\ e_{yy} = 0,\ e_{zz} = 0,$$

$$e_{xy} = 0,\ e_{yz} = \frac{1}{2}\alpha x,\ e_{zx} = -\frac{1}{2}\alpha y\ . \qquad \text{...(11)}$$

The stress-strain relations

$$\tau_{ij} = \lambda\ \delta_{ij}\ e_{kk} + 2\mu e_{ij},$$

yield

$$\tau_{xx} = \tau_{yy} = \tau_{zz} = \tau_{xy} = 0\ ,$$

$$\tau_{yz} = \mu\alpha x\ , \quad \tau_{xz} = -\ \mu\alpha y\ . \qquad \text{….(12)}$$

This system of stresses clearly satisfies the equations of equilibrium (for zero body)

$$\tau_{ij\ ,\ j} = 0\ \ .$$

Also the Beltrami-Michell compatibility conditions for zero body force

$$\nabla^2\tau_{ij} + \frac{1}{1+\sigma}\tau_{kk,\ ij} = 0\ \ , \qquad \text{...(13)}$$

are obviously satisfied.

Since the lateral surface is stress-free, so the boundary conditions to be satisfied on the lateral surface are

$$\tau_{ij}\, \nu_{j} = 0\ , \qquad \text{for } i = 1, 2, 3\ . \qquad ...(14)$$

Since the normal is perpendicular to z-axis, so

$$\nu_3 = 0\ , \qquad ...(15)$$

on the lateral surface of the beam. The first two conditions are identically satisfied. The third condition becomes

$$\tau_{xz}\, \nu_x + \tau_{yz}\nu_y = 0\ . \qquad ...(16)$$

Let 'a' be the radius of the cross-section. Let (x, y) be a point on the boundary C of the cross-section. Then

$$x^2 + y^2 = a^2$$

and

$$\nu_x = \cos(\hat{\nu}, x) = x/a\ ,$$

$$\nu_y = \cos(\hat{\nu}, y) = y/a\ .$$

Therefore, on the boundary C,

$$\tau_{xz}\, \nu_x + \tau_{yz}\nu_y = -\mu\,\alpha y.(x/a) + \mu\alpha x.(y/a) = 0. \qquad ..(17)$$

That is, the lateral surface of the circular beam is stress-free.

On the base z = l , let

$$\bar{F} = (F_x, F_y, F_z)\ ,$$

be the resultant force. Then

$$F_x = \iint_R \tau_{zx}\, dx\, dy$$

$$= -\mu\alpha \iint_R y\ dx\, dy$$

$$\mathbf{= 0,} \qquad \mathbf{..(18)}$$

since the y coordinate of the C. G. is zero. Also,

$$F_y = \iint_R \tau_{zy}\, dx\, dy = \alpha x \iint_R x\ dx\, dy = 0\ , \qquad ..(19)$$

$$Fz = \iint_R \tau_{zz}\, dx\, dy = 0\ , \qquad ...(20)$$

because the x coordinate of the C. G. is zero as the C. G. lies on the z-axis.

Thus, on the base z = *l*, the resultant force $\bar{F}$ is zero.

Let $\overline{M} = (M_x, M_y, M_z)$ be the resultant couple on the base z = *l* . Then

$$\overline{M} = \iint_R (x\hat{e}_1 + y\hat{e}_2 + l\hat{e}_3) \times (\tau_{xz}\hat{e}_1 + \tau_{zy}\hat{e}_2 + \tau_{zz}\hat{e}_3) dxdy. \qquad ..(21)$$

This gives

$$M_x = \iint_R (y\tau_{zz} - l\tau_{zy}) dxdy = \mu\alpha l \iint_R yx\, dxdy = 0\ , \qquad ..(22)$$

$$M_y = \iint_R (l\tau_{xz} - x\tau_{zz}) dxdy = -\mu\alpha l \iint_R y\, dxdy = 0, \qquad ...(23)$$

$$M_z = \iint_R (x\tau_{zy} - y\tau_{zx}) dxdy$$

$$= \mu x \iint_R (x^2 + y^2\ ) dxdy$$

$= \mu\alpha x$ [moment of inertia of the cross-section z = *l* about the z-axis]

$$= \mu\alpha \frac{\pi a^4}{2}. \qquad ...(24)$$

As $\overline{M} = M\,\hat{e}_3$, so we write

$$M = M_z = \frac{\pi\mu\alpha a^4}{2}\ , \qquad ...(25)$$

where a is the radius of the circular cross-section at z =ℓ.

This gives

$$\alpha = \frac{2M}{\mu\pi a^4}\ , \qquad ...(26)$$

which determines the constant α in term of moment of the applied couple M, radius of the cross-section 'a' and rigidity μ of the medium of the beam.

The constant α is the twist per unit length. With α given by (26), the displacements, strains and stresses at any point of the beam due to applied twist became completely known by equations (7), (11), (12).

Stress -Vector

The stress vector at any point P(x, y) in any cross-section (z = constant) is given by

$$\overset{z}{\underset{\sim}{T}} = \tau_{zx}\,\hat{i} + \tau_{zy}\,\hat{j} + \tau_{zz}\,\hat{k} \,. \qquad ..(27)$$

Using (12), we find

$$\overset{z}{\underset{\sim}{T}} = \tau_{zx}\,\hat{i} + \tau_{zy}\,\hat{j}$$

$$= \mu\alpha\,(-y\,\hat{i} + x\,\hat{j})\,, \qquad ...(28)$$

which lies in the cross-section itself, i.e., the stress-vector is tangential.

Moreover, we note that the stress-vector $\overset{z}{\underset{\sim}{T}}$ is perpendicular to the radius vector $\underset{\sim}{r} = x\,\hat{i} + y\,\hat{j}$ as

$$\overset{z}{\underset{\sim}{T}} \cdot \underset{\sim}{r} = 0.$$

The magnitude τ of the shier vector $\overset{z}{\underset{\sim}{T}}$ is given by

$$\tau = \sqrt{\tau^2_{zx} + \tau^2_{zy}} = \mu\alpha\sqrt{x^2 + y^2} = \mu\alpha r, \qquad ...(29)$$

which is maximum when r = a , and

$$\tau_{max} = \mu\alpha a = \frac{2M}{\pi a^3}\,. \qquad ...(30)$$

Note:- The torsinal rigidity of the beam, denoted by D_0, is defined by

$$D_0 = \frac{M}{\alpha} = \mu.\frac{\pi a^4}{2}\,. \qquad ...(31)$$

The constant D_0 provides a measure of the rigidity of the beam. It depends on the modulus of rigidity μ and the shape of the cross-section of the beam only. The constant D_0 (which is equal to $\frac{M}{\alpha}$) represents the moment of the couple required to produce a unit angle of twist per unit of length.

It is also called the torsinal stiffness of the beam.

Example: Consider a circular shaft of length *l*, radius a, and shear modulus μ, twisted by a couple M. Show that the greatest angle of twist θ and the maximum shear stress

$$\tau = \sqrt{\tau^2_{zx} + \tau^2_{zy}}$$

are given by

$$\theta_{max} = \frac{2Ml}{\pi.\mu a^4}, \qquad \tau_{max} = \frac{2M}{\pi a^3}$$

Solution: We know that

$$\theta = \alpha z, \qquad M = \frac{\mu\alpha\pi a^4}{2}, \qquad T = \mu\alpha r. \qquad \text{....(1)}$$

We find

$$\alpha = \frac{2M}{\mu\pi a^4} \qquad \text{....(2)}$$

Now

$$\theta_{max} = \alpha.(z)_{max} = \alpha l = \frac{2M}{\mu\pi a^4},$$

and

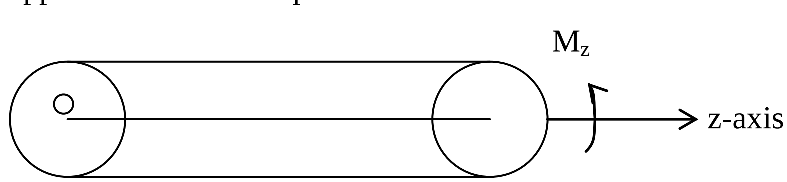
$$\tau_{max} = \mu\alpha\,(r)_{max} = \mu\alpha a = \frac{2M}{\pi a^3}.$$

7.2. TORSION OF BARS OF ARBITRARY CROSS-SECTION

Consider an elastic beam of length *l* of uniform but arbitrary cross-section. The lateral surface is stress free and body forces are absent. Suppose that a couple of vector moment $\vec{M}$ about the axis of the beam is applied at one end and the other end is fixed.

The problem is to compute the displacements, strains and stresses developed in the beam.

We choose z-axis along the axis of the beam. Let the end z = 0 be fixed and the end $z = \ell$ be applied the vector couple of moment $\vec{M}$.



(Torsion of a beam of arbitrary cross-section)

Let (u, v, w) be the components of the displacement. In the case of a circular cross-section, a plane cross-section remains plane even after the deformation. However, when the cross-section is arbitrary, a plane cross-section will not remain plane after deformation, it gets warped (curved surface).

This phenomenon is known as warping.

We assume that each section is warped in the same way, i.e., warping is independent of z . We write

$$\mathbf{w = \alpha\, \phi\, (x, y)}\ , \qquad \text{...(1)}$$

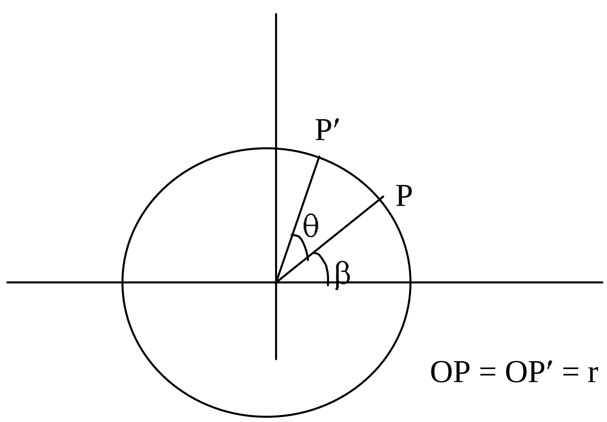
which is the same for all sections. Here α is the angle of twist per unit length of the beam.

The function $\phi = \phi$ (x, y) is called the Saint-Venant's warping function or torsion function.

Due to twisting, a plane section get rotated about the axis of the beam. The angle θ of rotation depends upon the distance of the section from the fixed end. For small deformations, we assume that

$$\theta = \alpha z\ . \qquad ..(2)$$

Let P (x, y) be any point on a cross section before deformations P′ (x′, y′) be the same material point after deformation.



Then

$$\begin{aligned} u = x' - x &= r\cos(\theta + \beta) - r\cos\beta \\ &= r\cos\theta\cos\beta - r\sin\theta\sin\beta - r\cos\beta \\ &= x(\cos\theta - 1) - y\sin\theta \\ &= -y\,\theta\ . \qquad ..(3) \end{aligned}$$

Similarly

$$v = x\,\theta\ . \qquad ..(4)$$

Hence, for small deformations, the displacement components are given by the relations

$$\mathbf{u = -\alpha yz\ ,}$$

$$\mathbf{v = \alpha xz\ ,}$$

$$\mathbf{w = \alpha\phi\ (x, y)\ .} \qquad \mathbf{..(5)}$$

The strain components are

$$e_{xx} = \frac{\partial u}{\partial x} = 0,$$

$$e_{yy} = \frac{\partial v}{\partial y} = 0,$$

$$e_{zz} = \frac{\partial w}{\partial z} = 0\ , \qquad ..(6)$$

and

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\left(-\alpha z + \alpha z\right) = 0,$$

$$e_{yz} = \frac{1}{2}\alpha\left[x + \frac{\partial \phi}{\partial y}\right],$$

$$e_{zx} = \frac{1}{2}\alpha\left[-y + \frac{\partial \phi}{\partial x}\right]\ . \qquad ..(7)$$

The stresses are to be found by using the Hooke's law

$$\tau_{ij} = \lambda\ \delta_{ij}\ e_{kk} + 2\ \mu\ e_{ij}. \qquad ..(8)$$

We find

$$\tau_{xx} = \tau_{yy} = \tau_{zz} = \tau_{xy} = 0, \qquad \ldots(9)$$

$$\tau_{yz} = \mu\ \alpha\left(x + \frac{\partial \phi}{\partial y}\right), \qquad ..(10)$$

$$\tau_{zx} = \mu\ \alpha\left(-y + \frac{\partial \phi}{\partial x}\right)\ . \qquad ...(11)$$

These stresses must satisfy the following equilibrium equations for zero body force

$$\tau_{ij,j} = 0 \ , \qquad ..(12)$$

for i = 1, 2, 3 in R. First two equations are identically satisfied. The third equation gives

$$\tau_{zx,x} + \tau_{zy,y} = 0 \qquad \text{in R}$$

or $$\mu\alpha \frac{\partial}{\partial x}\left[-y + \frac{\partial \phi}{\partial x}\right] + \mu\alpha \frac{\partial}{\partial y}\left[x + \frac{\partial \phi}{\partial y}\right] = 0$$

or $$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \qquad ...(13)$$

in the region R of the cross-section.

This shows that the torsion function ϕ is a harmonic function.

Let C be the boundary curve of the region R representing the cross-section of the beam. Since the lateral surface is stress free, so the boundary conditions to be satisfied are

$$\tau_{ij}\, v_j = 0 \ , \qquad \text{on C} \qquad ...(14)$$

for i = 1, 2, 3. The first two conditions are identically satisfied as $v_3 = 0$. The third condition is

$$\tau_{zx}\, v_x + \tau_{zy}\, v_y = 0 \qquad \text{on C.}$$

or

$$\left(-y + \frac{\partial \phi}{\partial x}\right) v_x + \left(x + \frac{\partial \phi}{\partial y}\right) v_y = 0 \qquad \text{on C}$$

or

$$\frac{\partial \phi}{\partial x} v_x + \frac{\partial \phi}{\partial y} v_y = y\, v_x - x\, v_y \qquad \text{on C}$$

or

$$\frac{d\phi}{dv} = y\, v_x - x\, v_y \qquad \text{on C.} \qquad ...(15)$$

since

$$\frac{d\phi}{dv} = \text{normal derivative of } \phi$$

$$= \nabla\phi.\hat{v}$$

$$= \frac{\partial\phi}{\partial x}v_x + \frac{\partial\phi}{\partial y}v_y . \qquad ...(16)$$

Thus, the torsion function/ warping functions ϕ must be a solution of the two dimensional following Neumann boundary value problem.

$$\nabla^2\phi = 0 \quad , \qquad \text{in R} \ ,$$

$$\frac{d\phi}{dv} = \text{normal derivations of } \phi = (y.\, v_x - x.\, v_y) \qquad \text{on C} . \qquad ...(17)$$

So, we solve the torsion problem as a Neummann problem (17).

On the base z = l , let $\bar{F}$ be the resultant force. Then

$$F_x = \iint_R \tau_{zx} dx\, dy = \mu\alpha \iint_R \left(\frac{\partial\phi}{\partial x} - y\right) dx\, dy$$

$$= \mu\alpha \iint_R \left[\frac{\partial}{\partial x}\left\{x\left(\frac{\partial\phi}{\partial x} - y\right)\right\} + \frac{\partial}{\partial y}\left\{x\left(\frac{\partial\phi}{\partial y} + x\right)\right\}\right] dx\, dy$$

$$= \mu\alpha \int_C \left[-x\left(\frac{\partial\phi}{\partial y} + x\right)dx + x\left(\frac{\partial\phi}{\partial x} - y\right)dy\right], \qquad ...(18)$$

using Green's theorem

$$\int_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy \ , \qquad ...(19)$$

which converts surface integral into a line integral.
In case of two-dimensional curve C, directions cosines of the tangent are $< \frac{dx}{ds}, \frac{dy}{ds} >$, and therefore, d. c.'s of the normal are $< +\frac{dy}{ds}, -\frac{dx}{ds} >$, i. e.,

$$v_x = dy/ds, \qquad v_y = -dx/ds. \qquad ...(20)$$

From equations (18) and (20), we write

$$F_x = \mu\alpha \int_C x\left[\left(-x-\frac{\partial\phi}{\partial y}\right)\frac{dx}{ds}+\left(\frac{\partial\phi}{\partial x}-y\right)\frac{dy}{ds}\right]ds$$

$$= \mu\alpha \int_C x\left[\frac{\partial\phi}{\partial x}.\nu_x+\frac{\partial\phi}{\partial y}\nu_y+(x\nu_y-y\nu_x)\right]ds$$

$$= \mu\alpha \int_C x\left[\frac{\partial\phi}{\partial \nu}+(x\nu_y-y\nu_x)\right]ds$$

$$\mathbf{= 0,} \qquad \textbf{...(21)}$$

since, on the boundary C, the integrand is identically zero.

Similarly

$$F_y = F_z = 0\ .$$

Thus, the resultant force $\overline{F}$ on the cross-section R of the beam vanishes.

On the base z = *l*, let $\overline{M}$ be the resultant couple. Then

$$M_x = \iint_R (y\tau_{zz} - z\tau_{zy})\,dx\,dy$$

$$= -l \iint_R \tau_{zy}dxdy \quad ; \qquad (\because z = l)$$

$$= -l\, F_y$$

$$\mathbf{= 0\ .} \qquad \textbf{...(23)}$$

Similarly

$$M_y = 0. \qquad ...(24)$$

Now

$$M_z = \iint_R (x\tau_{zy} - y\tau_{zx})\,dx\,dy = \mu\alpha \iint_R \left[x^2+y^2+x\frac{\partial\phi}{\partial y}-y\frac{\partial\phi}{\partial x}\right]dx\,dy\ ,$$

This gives

$$M = \mu\alpha \iint_R \left[x^2+y^2+x\frac{\partial\phi}{\partial y}+y\frac{\partial\phi}{\partial z}\right]dx\,dy\,, \qquad ...(25)$$

as it is given that M is the moments of the torsion couple about z-axis.

We write

$$\mathbf{M = \alpha\, D} \qquad \textbf{...(26)}$$

where M is applied moment, α is the twist per unit length and D is the torsinal rigidity given by

$$D = \mu \iint_R \left[x^2 + y^2 + x\frac{\partial \phi}{\partial y} - y\frac{\partial \phi}{\partial x} \right] dx\, dy\ , \qquad ...(27)$$

which depends upon μ (i. e., the material of the beam) and the shape of the cross-section R (i. e., the geometry of the beam).

For given M, α can be determined from equation (26).

However, when α is given, then we can calculate the required moment M, from equation (27), to produce the twist α per unit length.

After finding ϕ (by solving the Neumann boundary value problem) the torsion function ϕ becomes know and consequently the torsinal rigidity D becomes known.

7.3. DIRICHLET BOUNDARY VALUE PROBLEM

Let ψ (x, y) be the conjugate harmonic function of the harmonic function ϕ(x, y). Then

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}, \qquad ..(28)$$

and the function $\phi + i\psi$ is an analytic function of the complex variable x + iy.

Now

$$\frac{d\phi}{d\nu} = \frac{\partial \phi}{\partial x}\nu_x + \frac{\partial \phi}{\partial y}\nu_y$$

$$= \frac{\partial \phi}{\partial x}\frac{dy}{ds} - \frac{\partial \phi}{\partial y}\frac{dx}{ds}$$

$$= \frac{\partial\psi}{\partial y}\frac{dy}{ds} + \frac{\partial\psi}{\partial x}\frac{dx}{ds},$$

$$= \frac{d\psi}{ds} . \qquad \text{....(29)}$$

On the boundary C, the boundary condition (15) now becomes

$$\frac{d\psi}{ds} = y\nu_x - x\nu_y \quad \text{on C} ,$$

or

$$\frac{d\psi}{ds} = y\frac{dy}{ds} + x\frac{dx}{ds} \quad \text{on C} ,$$

or

$$\psi = \frac{1}{2}\left(x^2 + y^2\right) + \text{constt}, \quad \text{on C} . \qquad \text{.....(30)}$$

Thus, the determination of the function $\psi = \psi$ (x, y) is the problem of solving the Dirichlet problem

$$\frac{\partial^2 \boldsymbol{\psi}}{\partial \mathbf{x}^2} + \frac{\partial^2 \boldsymbol{\psi}}{\partial \mathbf{y}^2} = \mathbf{0} \quad \textbf{in R}$$

$$\boldsymbol{\psi} = \frac{\mathbf{1}}{\mathbf{2}}\left(\mathbf{x}^2 + \mathbf{y}^2\right) + \textbf{constt.} \quad \textbf{on C} \qquad \textbf{...(31)}$$

whose solution is unique.

Once ψ becomes known, the torsion function $\phi = \phi$ (x, y) can be obtained from relations in (28).

We generally, take, constt. = 0 in (31).

The Dirichlet problem of potential theory can be solved by standard techniques.

7.4. STRESS- FUNCTION

Stress-Function: The stress function, denoted by Ψ, is defined as

$$\Psi\,(x,y) = \psi\,(x, y) - \frac{1}{2}\left(x^2 + y^2\right) \qquad \text{...(32)}$$

where ψ (x, y) is the solution of the Dirichlet problem (31).

We know that ψ (x, y) is also the conjugate function of the harmonic torsion function ϕ (x, y).

The stress $\Psi(x,y)$ was introduced by Prandtl.

From equation (32), we find

$$\frac{\partial \Psi}{\partial x} = \frac{\partial \psi}{\partial x} - x \quad ,$$

or

$$\frac{\partial \psi}{\partial x} = \frac{\partial \Psi}{\partial x} + x \quad , \qquad ..(33)$$

and

$$\frac{\partial \Psi}{\partial y} = \frac{\partial \psi}{\partial y} - y$$

or

$$\frac{\partial \psi}{\partial y} = \frac{\partial \Psi}{\partial y} + y \quad . \qquad ..(34)$$

Further, we find

$$\nabla^2 \ \Psi = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2}$$

$$= \left(\frac{\partial^2 \psi}{\partial x^2} - 1 \right) + \left(\frac{\partial^2 \psi}{\partial y^2} - 1 \right)$$

$$\mathbf{= -2 \qquad in\ R.} \qquad \mathbf{..(35)}$$

From equations (31) and (32), we find

$$\Psi \ = \text{constt on } C \ . \qquad ..(36)$$

The differential equation in (35) is called Poisson's equation.

Thus, the stress function $\Psi(x, y)$ is a solution of the boundary value problem consists of equations (35) and (36).

The shear stresses τ_{zx} and τ_{zy}, given in (11), can also be expressed in terms of stress function Ψ. We find

$$\tau_{zx} = \mu\alpha \left(\frac{\partial \phi}{\partial x} - y \right)$$

$$= \mu\alpha \left(\frac{\partial \psi}{\partial y} - y \right)$$

$$= \mu\alpha \, \frac{\partial \Psi}{\partial y}, \qquad ..(37)$$

$$\tau_{zy} = \mu\alpha \left(\frac{\partial \phi}{\partial y} + x \right)$$

$$= \mu\alpha \left(-\frac{\partial \psi}{\partial x} + x \right)$$

$$= -\mu\alpha \, \frac{\partial \Psi}{\partial x} \, . \qquad ...(38)$$

The torsional rigidity D in term of stress function $\Psi(x, y)$ can be found.

We know that

$$D = \frac{1}{\alpha} \iint_R (x\tau_{zy} - y\tau_{zx}) \, dx\,dy$$

$$= \frac{1}{\alpha} \iint_R \left[x\left\{ -\mu\alpha \frac{\partial \Psi}{\partial x} \right\} - y\left\{ \mu\alpha \frac{\partial \Psi}{\partial y} \right\} \right] dx\,dy$$

$$= -\mu \iint_R \left[x\frac{\partial \Psi}{\partial x} + y\frac{\partial \Psi}{\partial y} \right] dx\,dy$$

$$= -\mu \iint_R \left[\frac{\partial}{\partial x}(x\Psi) + \frac{\partial}{\partial y}(y\Psi) \right] dx\,dy + 2\mu \iint_R \Psi \, dx\,dy$$

$$= -\mu \int_C \Psi \, (x\nu_x + y\nu_y) \, ds + 2\mu \iint_R \Psi \, dx\,dy \, , \qquad ..(39)$$

using the Green theorem for plane.

On the boundary, C,

Ψ = constt.

We choose

$$\Psi = 0 \text{ on } C \, . \qquad ..(40)$$

Then

$$D = 2\mu \iint_R \Psi dxdy, \qquad ..(41)$$

which is the expression for torsinal rigidity D in terms of stress function Ψ

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7.5. LINES OF SHEARING SHEER

Consider the family of curves in the xy-plane given by

$$\Psi = \text{constt.} \qquad ...(42)$$

Then

$$\frac{\partial\Psi}{\partial x} + \frac{\partial\Psi}{\partial y}\frac{dy}{dx} = 0.$$

This gives

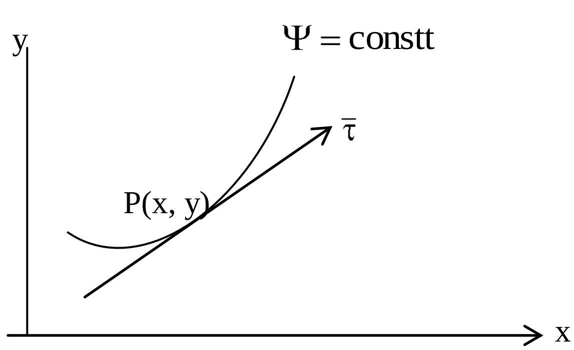
$$\frac{dy}{dx} = -\frac{\Psi x}{\Psi y} \;.$$

$$= \frac{\mu\alpha\Psi x}{\mu\alpha\Psi y}$$

$$= \frac{\tau_{zy}}{\tau_{zx}} \;. \qquad ..(43)$$

This relation shows that at each of the curve, Ψ = constt, the stress vector

$$\bar{\tau} = \tau_{zx}\,\hat{i} + \tau_{zy}\,\hat{j} \qquad \textbf{..(44)}$$



is defined along the tangent to the curve Ψ = constt. at that point.

The curves

$$\Psi = \text{constt}$$

are called lines of shearing stress.

7.6. SPECIAL CASES OF BEAMS : TORSION OF AN ELLIPTIC BEAM

Let the boundary C of the cross-section be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \; . \qquad ..(A1)$$

We assume that the solution of the Dirichlet boundary values problem (31) is of the type

$$\psi\,(x, y) = c^2\,(x^2 - y^2) + k^2 \, , \qquad ..(A2)$$

for constants c^2 and k^2.

It is obvious that

$$\nabla^2\,\psi = 0 \; . \qquad ..(A3)$$

At each point (x, y) on the boundary C, we must have

$$c^2\,(x^2 - y^2) + k^2 = \frac{1}{2}\,(x^2 + y^2)$$

or

$$x^2(\tfrac{1}{2} - c^2) + y^2(\tfrac{1}{2} + c^2) = k^2 \qquad ..(A4)$$

which becomes the ellipse (A1) if

$$c^2 < \tfrac{1}{2}$$

and

$$a = \frac{k}{\sqrt{\tfrac{1}{2}-c^2}}\,, \qquad b = \frac{k}{\sqrt{\tfrac{1}{2}+c^2}}\,,$$

or

$$a^2 = \frac{k^2}{\tfrac{1}{2}-c^2}\,, \qquad b^2 = \frac{k^2}{\tfrac{1}{2}+c^2}$$

or

$$c^2 = \tfrac{1}{2}\left(\frac{a^2 - b^2}{a^2 + b^2}\right), \qquad k^2 = \frac{a^2b^2}{a^2 + b^2}\,. \qquad ..(A5)$$

Therefore, solution of the Dirichlet problem (31) for this particular type of elliptic beam is

$$\psi\,(x, y) = \frac{a^2 - b^2}{2\,(a^2 + b^2)}(x^2 - y^2) + \frac{a^2b^2}{a^2 + b^2}\,. \qquad ..(A6)$$

The torsion function ϕ is given by the formula

$$\phi\,(x,y) = \int\left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy\right)$$

$$= \int\left[\frac{\partial\psi}{\partial y}dx - \frac{\partial\psi}{\partial x}dy\right]$$

$$= -\frac{a^2 - b^2}{a^2 + b^2}\int (y\,dx + x\,dy)$$

$$= -\frac{a^2 - b^2}{a^2 + b^2}\,xy\,. \qquad ..(A7)$$

The stress function $\Psi(x,y)$ is given by the formula

$$\Psi = \psi\,(x, y) - \tfrac{1}{2}(x^2 + y^2)$$

$$= \frac{-a^2b^2}{a^2 + b^2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right), \qquad ..(A8)$$

using (A6).

To calculate the torsional rigidity D_e and twist α .

The non-zero shear stresses are

$$\tau_{zx} = \mu\alpha\left(\frac{\partial\psi}{\partial y} - y\right)$$

$$= \mu\alpha\left[-\frac{a^2 - b^2}{a^2 + b^2}y - y\right]$$

$$= \frac{-2\mu\alpha a^2 y}{a^2 + b^2}, \qquad ..(A9)$$

and

$$\tau_{zy} = \mu\alpha \left(-\frac{\partial\psi}{\partial x} + x\right)$$

$$= \mu\alpha \left[-\frac{a^2 - b^2}{a^2 + b^2} x + x\right]$$

$$= \frac{2\mu\alpha b^2 x}{a^2 + b^2}. \qquad ..(A10)$$

Let M be the torsion moment of the couple about z-axis. Then

$$M = \iint_R (x\tau_{zy} - y\tau_{zx})\, dxdy$$

$$= \frac{2\mu\alpha}{a^2 + b^2}\left[b^2 \iint_R x^2 dxdy + a^2 \iint_R y^2 dxdy\right]$$

This implies

$$M = \frac{2\mu\alpha}{a^2 + b^2}\left[b^2 I_y + a^2 I_x\right], \qquad \ldots(A11)$$

where I_x and I_y are the moments of inertia of the elliptic cross-section about x– and y–axis, respectively.

We know that

$$I_x = \frac{\pi ab^3}{4}, \qquad I_y = \frac{\pi a^3 b}{4} \qquad \ldots(A12)$$

Putting these values in (A11), we find

$$M = \frac{\pi\mu\alpha a^3 b^3}{a^2 + b^2}. \qquad ..(A13)$$

The torsional rigidity D_e is given by the formula

$$M = D_e\alpha. \qquad \ldots(A14)$$

Hence, we find

$$D_e = \frac{\pi\mu a^3 b^3}{a^2 + b^2}, \qquad ..(A15)$$

$$\alpha = \frac{M(a^2 + b^2)}{\pi\mu a^3 b^3}. \qquad \ldots(A16)$$

Equations (A9) , (A10) and (A16) imply

$$\tau_{zx} = \frac{-2My}{\pi ab^3}, \qquad 1 \qquad \ldots(A17)$$

$$\tau_{zy} = \frac{-2Mx}{\pi a^3 b}. \qquad \ldots(A18)$$

Lines of shearing stress: The line of shearing stress are given by the formula

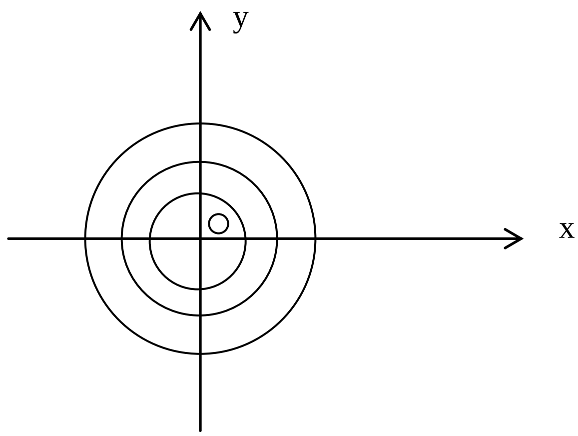
$$\Psi = \text{constt.} \qquad \ldots(A19)$$

Equations (A8) and (A19) imply

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \text{constt} \qquad \ldots(A20)$$

as the lines the shearing stress.

This is a family of concentric ellipses, similar to the given ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



Displacements. The displacements at any point of the given elliptic beam are now given by the formulas

$$u = -\alpha\, yz, \qquad v = \alpha\, xz, \qquad w = \alpha\, \phi = -\left(\frac{a^2 - b^2}{a^2 + b^2}\right)\alpha\, xy \qquad \ldots(A21)$$

with the twist per unit length, α, as given by equation (A16).

Maximum shear stress: The shear stress τ is given by

$$\tau = \sqrt{\tau^2{}_{zx} + \tau^2{}_{zy}} \quad .$$

$$= \frac{2\mu\alpha}{a^2 + b^2}\sqrt{a^4 y^2 + b^4 x^2}\,. \qquad \ldots(A22)$$

We know that the maximum shear stress occurs on the boundary C

$$C \ : \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

So, on the boundary C , the shear stress becomes

$$\tau = \frac{2\mu\alpha}{a^2+b^2}\sqrt{a^4b^2\left(1-\frac{x^2}{a^2}\right)+b^4x^2}$$

$$= \frac{2\mu\alpha ab}{a^2+b^2}\sqrt{a^2-x^2+b^2x^2/a^2}$$

$$= \frac{2\mu\alpha ab}{a^2+b^2}\sqrt{a^2-x^2(1-b^2/a^2)}$$

$$(\tau)_C = \frac{2\mu\alpha ab}{a^2+b^2}\sqrt{a^2-e^2x^2}\ , \qquad \ldots(A23)$$

where

$$e^2 = 1 - b^2/a^2 \ , \qquad \ldots(A24)$$

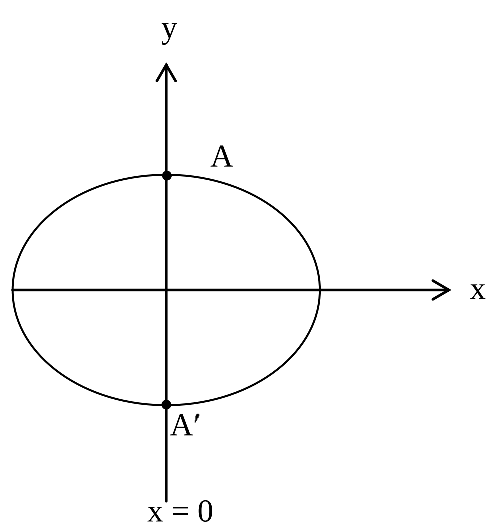
is the eccentricity of the ellipse.

We note that the maximum value of shearing stress (A23) occurs when x = 0.

Thus,

$$\tau_{max} = \frac{2\mu\alpha a^2 b}{a^2+b^2} = \frac{2M}{\pi ab^2} \quad \text{at the points } (0, \pm b)\,. \qquad \ldots(A25)$$

Thus, the maximum shear stress occurs at the extremities of the minor axis



of the ellipse and maximum shear stress is given by (A25).

Warping curves: The warping curves are given by the relation

$$w = \text{constt.}$$

or

$$xy = \text{constt.}, \qquad \ldots(A26)$$

using (A21).

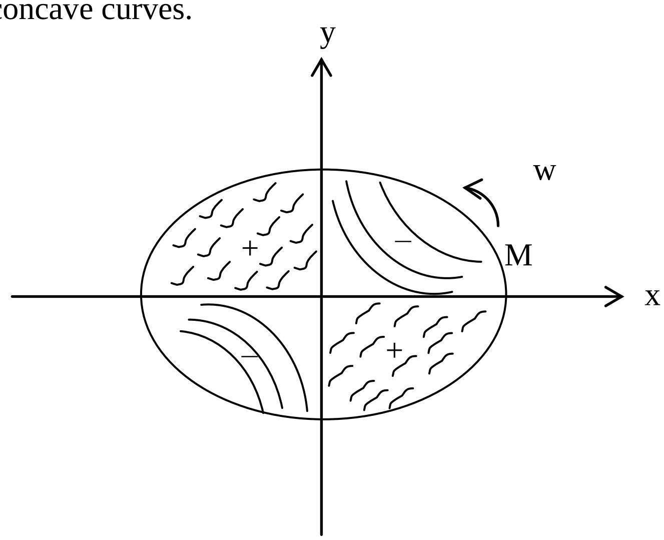
These are rectangular hyperbolas. We know that

$$w = -\frac{a^2 - b^2}{a^2 + b^2}\alpha\, xy\,.$$

In the first quadrant, x and y are both positive so w < 0.

In the third quadrant x < 0, y < 0, so w < 0.

Therefore, in I and III quadrants, the curves becomes concave. In figure, –ve sign represents concave curves.



In the II and IV quadrants, the curves become convex which are represented by + sign in the figure.

Remark 1. Let A be the area of the ellipse and I be the moment of inertia about z-axis. Then

$$A = \pi\, ab,$$

$$I = I_x + I_y = \frac{\pi ab}{4}(a^2 + b^2),$$

$$D_e = \frac{\mu A^4}{4\pi^2 I} \; . \qquad \ldots(A27)$$

Remark 2. The results for a circular bar can be derived as a particular case of the above–a bar with elliptic cross section, on taking b = a. We find

$$\phi\,(x, y) = 0\,,$$

$$w\,(x, y) = 0\,,$$

$$\Psi\,(x, y) = 0\,.$$

7.7. TORSION OF BEAMS WITH TRIANGULAR CROSS-SECTION

Consider a cylinder of length l whose cross section is a triangular prism. Let z-axis lie along the central line of the of the beam and one end of the beam lying in the plane z = 0 is fixed at the origin and the other end lie in the plane z = l. A couple of moment M is applied at the centriod (0, 0,l) of end.

We shall determine the resulting deformation.

Let

$$\phi + i\psi = i\,c\,(x + iy)^3 + ik, \qquad ..(B1)$$

where c and k are constants. We find

$$\phi = c\,(-3x^2y + y^3), \qquad ..(B2)$$

$$\psi = c\,(x^3 - 3x\,y^2) + k\;. \qquad ..(B3)$$

We shall be solving the Dirichlet problem in ψ:

$$\text{(i)}\quad \nabla^2\,\psi = 0 \qquad \text{in } R, \qquad ..(B4)$$

$$\text{(ii)}\quad \psi = \tfrac{1}{2}(x^2 + y^2) \qquad \text{on } C, \qquad ..(B5)$$

where C is the boundary of a triangular cross section occupying the region R.

We note that ψ, given by (B3), is a harmonic function in the two-dimensional region R.

On the boundary C, we must have

$$c\,(x^3 - 3x\,y^2) + k = \tfrac{1}{2}(x^2 + y^2) \qquad ..(B6)$$

By altering the value of constant c and k, we obtain various cross-sections of the beam. In particularly, if we set

$$C = -\frac{1}{6a} \qquad \text{and} \qquad k = \frac{2}{3}a^2, \qquad ..(B7)$$

in equation (B6), we have

$$-\frac{1}{6a}(x^3 - 3x\,y^2) + \frac{2}{3}a^2 = \tfrac{1}{2}(x^2 + y^2)$$

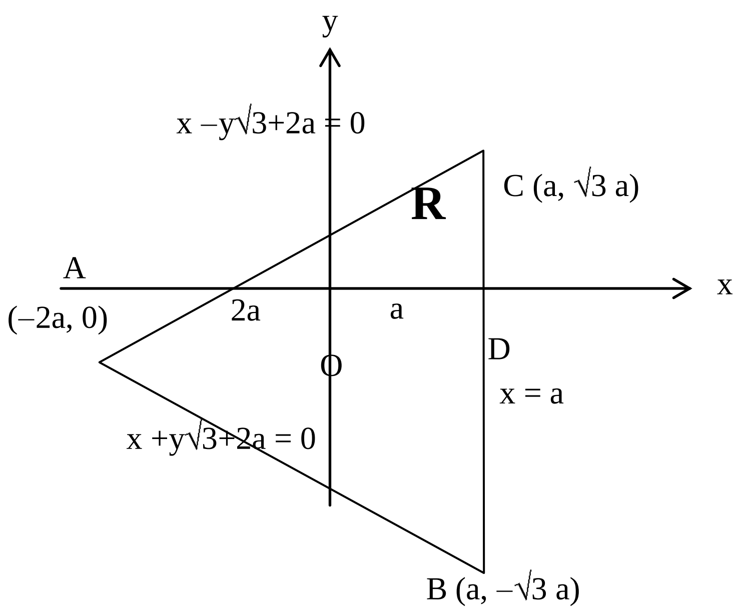
or

$$x^3 - 3x\,y^2 + 3a\,x^2 + 3a\,y^2 - 4\,a^3 = 0$$

or

$$(x - a)\,(x - y\sqrt{3} + 2a)\,(x + y\sqrt{3} + 2a) = 0\,. \qquad ..(B8)$$

This shows that the boundary C consists of an equilateral triangle ABC



formed by the st. lines

$$x = a\,,$$

$$x + \sqrt{3}y + 2a = 0,$$

$$x - \sqrt{3}\,y + 2a = 0\,. \qquad ..(B9)$$

The altitude of this cross-section is 3a and each length of the triangle is 2√3 a. The centroid of this cross-section is origin , lying on the central line of the beam.

The unique solution of the Dirichlet problem (B4, 5) is

$$\psi = -\frac{1}{6a}(x^3 - 3x\,y^2) + \frac{2}{3}a^2\,. \qquad \ldots(B10)$$

The torsion function becomes

$$\phi = -\frac{1}{6a}(y^3 - 3x^2y) . \qquad \ldots(B11)$$

The stress function Ψ is given by

$$\Psi = \psi(x, y) - \tfrac{1}{2}(x^2 + y^2)$$

$$= -\frac{1}{6a}[x^3 - 3x y^2 + 3x^2a + 3y^2 - 4a^3] . \qquad ..(B12)$$

The non-zero shear stresses are

$$\tau_{zx} = \mu\alpha \left(\frac{\partial\psi}{\partial y} - y\right)$$

$$= \frac{\mu\alpha}{a}(x - a)\, y , \qquad \ldots(B13)$$

$$\tau_{zy} = \mu\alpha \left(-\frac{\partial\psi}{\partial x} + x\right)$$

$$= \frac{\mu\alpha}{2a}(x^2 + 2ax - y^2) . \qquad \ldots(B14)$$

The displacement at any points of the triangular beam are given by

$$u = -\alpha\, yz,$$

$$v = \alpha\, xz,$$

$$w = \alpha\phi = \frac{\alpha}{6a}(3x^2y - y^3) \qquad \ldots(B15)$$

where α is the twist per unit length.

We know that the maximum value of the shearing sheers τ occurs at the boundary C.

On the boundary x = a, we find

$$\tau_{zx} = 0, \qquad \tau = \tau_{zy} = \frac{\mu\alpha}{2a}(3a^2 - y^2), \qquad \ldots(B16)$$

which is maximum where y = 0. Thus

$$\tau_{max} = \frac{3}{2}\mu\alpha a , \qquad \ldots(B17)$$

at the point D(a,0), which is the middle point of BC.

Also, $\tau = 0$ at the points (corner pts B & C) where $y = \pm\sqrt{3}a$.

At the point A($-2a$, 0), τ is zero. Thus, the stress τ is zero at the corner points A, B, C. We note that at the centroid O(0, 0), the shear stress is also zero.

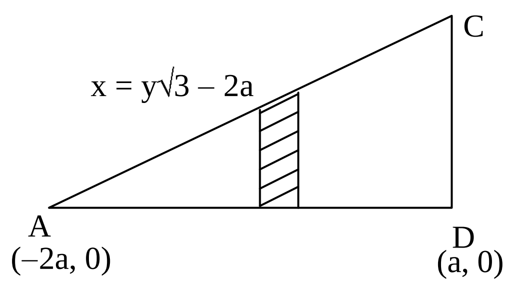
Similarly, we may check that the shearing stress τ is maximum at the middle points of the sides AC and AB, and the corresponding maximum shear stress is each equal to $\frac{3}{2}\mu\alpha a$.

Torsional rigidity: We know that M is the moment of the applied couple along z-axis. Therefore

$$M = \iint_R \left(x\tau_{zy} - y\tau_{zx} \right) dx\,dy$$

$$= \frac{\mu\alpha}{2a} \iint_R \left(x^3 + 2ax^2 - xy^2 - 2y^2x + 2y^2a \right) dx\,dy$$

$$= 2.\ \frac{\mu\alpha}{2a} \int_{x=-2a}^{a} \int_{y=0}^{y=\frac{x+2a}{\sqrt{3}}} \left(x^3 + 2ax^2 - xy^2 - 2y^2x + 2ay^2 \right) dy\,dx\,. \quad \ldots\text{(B18)}$$



This gives

$$M = \frac{9\sqrt{3}}{5}\mu\,\alpha a^4, \qquad \ldots\text{(B19)}$$

Consequently, we obtain

$$D = \frac{9\sqrt{3}}{5}\mu\ \ a^4, \qquad \ldots\text{(B20)}$$

$$\alpha = \frac{5M}{9\sqrt{3}\mu a^4}, \qquad \ldots\text{(B21)}$$

with a = $\frac{1}{3}$rd of the altitude of the equilateral triangle, each side being equal to $2\sqrt{3}a$.

Equation (B21) determines the constant α when the moment M and the cross-section are known. Equation (B13), (B14), and (B21) yield

$$\tau_{zx} = \frac{5M}{9\sqrt{3}a^5} y(x - a), \qquad \ldots(B22)$$

$$\tau_{zy} = \frac{5M}{18\sqrt{3}a^5} (x^2 + 2\,ax - y^2)\,. \qquad \ldots(B23)$$

Equations (B17) and (B21) imply

$$\tau_{max} = \frac{15M}{18\sqrt{3}a^4}\,, \qquad \ldots(B24)$$

at the point D (a, 0).

Theorem. Show that the points at which the shearing stress is maximum lie on the boundary C of the cross-section of the beam.

Proof. To prove this theorem, we use the following theorem from analysis.

"Let a function f(x, y) be continuous and has continuous partial derivatives of the first and second orders and not identically equal to a constant and satisfy the in equality $\nabla^2 f \geq 0$ in the region R. Then f (x, y) attains its maximum value on the boundary C of the region R".

We knows that the shear stress τ is given by

$$\tau^2 = \mu^2\,\alpha^2\,(\Psi_x^2 + \Psi_y^2\ )\qquad , \qquad \ldots.(1)$$

where Ψ is the stress function. Now

$$\frac{\partial}{\partial x}\tau^2 = \mu^2\alpha^2\,\left[2\Psi_x\Psi_{xx} + 2\Psi_y\Psi_{yx}\right]$$

$$= 2\,\mu^2\alpha^2[\Psi_x\Psi_{xx+}\Psi_y\Psi_{yy}]\,, \qquad \ldots.(2)$$

and

$$\frac{\partial^2}{\partial x^2}\tau^2 = 2\mu^2\alpha^2\left[\Psi_{zz}^2 + \Psi_x\Psi_{xxx} + \Psi_{yx}^2 + \Psi_y\Psi_{yxx}\right]. \quad \ldots(3)$$

Similarly

$$\frac{\partial^2}{\partial y^2}\tau^2 = 2\mu^2\alpha^2\left[\Psi_{yy}^2 + \Psi_y\Psi_{yyy} + \Psi_{xy}^2 + \Psi_x\Psi_{xyy}\right]. \quad \ldots(4)$$

Adding (3) and (4),

$$\nabla^2\tau^2 = 2\mu^2\alpha^2\left[\left(\Psi_{xx}^2 + \Psi_{yy}^2\right) + \Psi_x \frac{\partial}{\partial x}\left(\Psi_{xx} + \Psi_{yy}\right) + \Psi_y \frac{\partial}{\partial y}\left(\Psi_{xx} + \Psi_{yy}\right) + 2\Psi_{xy}^2 \right]$$

$$= 2\mu^2\propto^2 \left(\Psi_{xx}^2 + \Psi_{yy}^2 + 2\Psi_{xy}^2\right) \qquad ...(5)$$

$$\geq 0 \qquad \text{in } R.$$

Therefore, by above result, τ^2 (and hence stress τ) attains the maximum value on the boundary C of the region R.

Question. Let D_0 be the torsional rigidity of a circular cylinder, D_e that of an elliptic cylinder, and D_t that of a beam whose cross-section is an equilateral triangle. Show that for cross-section of equal areas

$$D_e = kD_0, \qquad D_t = \frac{2\pi\sqrt{3}}{15} D_0, \text{ where } k = \frac{2ab}{a^2 + b^2} \leq 1,$$

and a, b are the semi-axis of the elliptical section.

Solution. We know that for a circular cylinder of radius r,

$$D_0 = \frac{\pi}{2}\mu r^4 . \qquad ...(1)$$

We know that for an elliptic cylinder $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

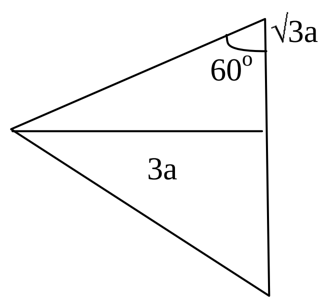
$$D_e = \frac{\pi\mu a^3 b^3}{a^2 + b^2} . \qquad ...(2)$$

We know that for an equilateral triangle (with each side of length $2\sqrt{3}x$)

$$D_t = \frac{9\sqrt{3}}{5}\mu\, x^4 . \qquad ...(3)$$

Since the areas of all the cross-section are equal, so

$$\pi r^2 = \pi\, ab = 3\sqrt{3}\, x^2 . \qquad ...(4)$$



Now

$$\frac{D_e}{D_0} = \frac{\pi\mu a^3 b^3}{a^2 + b^2} \times \frac{2}{\pi\mu r^2}$$

$$= \frac{2a^3b^3}{r^4(a^2+b^2)}$$

$$= \frac{2a^3b^3}{a^2b^2(a^2+b^2)}$$

$$= \frac{2ab}{a^2+b^2} \quad ,$$

$$= k . \qquad \text{...(5)}$$

Also

$$\frac{D_t}{D_0} = \frac{a\sqrt{3}\mu x^4}{5} \times \frac{2}{\pi \mu r^4}$$

$$= \frac{18\sqrt{3}\,x^4}{5\pi r^4}$$

$$= \frac{18\sqrt{3}x^4}{5\pi(27\,x^4/\pi)}$$

$$= \frac{18\sqrt{3}\pi}{5\times 27} = \frac{2\pi\sqrt{3}}{15} \qquad \text{..(6)}$$

Hence, the result.

Chapter-8
Variational Methods

8.1. INTRODUCTION

We shall be using the minimum principles in deriving the equilibrium and compatibility equations of elasticity.

8.2. DEFLECTION OF AN ELASTIC STRING

Let a stretched string, with the end points fixed at (0, 0) and (ℓ, 0), be deflected by a distributed transverse load f(x) per unit length of the string. We suppose that the transverse deflection y(x) is small and the change in the stretch force T produced by deflection is neglible.

These are the usual assumption used in deriving the equation for y(x) from considerations of static equilibrium.

We shall deduce this equation from the Principle of Minimum Potential Energy.

We know that the potential energy V is defined by the formula

$$V = U - \int_0^{\ell} f(x)\, y\, dx, \qquad \text{...(1)}$$

where the strain energy U is equal to the product of the tensile force T by the total stretch e of the string. That is

$$U = T\,e, \qquad \text{...(2)}$$

where

$$e = \int_0^{\ell} (ds - dx)$$

$$= \int_0^{\ell} \left[\sqrt{1 + (y')^2} - 1\right] dx. \qquad \text{...(3)}$$

Since, we are dealing with the linear theory, so

$$(y')^2 << 1, \qquad ...(4)$$

and equation (3) can be written as

$$e = \frac{1}{2}\int_0^{\ell} (y')^2 dx. \qquad ...(5)$$

From equations (1), (2) and (5), finally we write

$$V = \int_0^{\ell}\left[\frac{1}{2}T(y')^2 - f(x)\,y\right]dx. \qquad ...(6)$$

The appropriate Euler's equation of the functional (6) is (left as an exercise)

$$T\frac{d^2y}{dx^2} + f(x) = 0. \qquad ...(7)$$

This equation is the familiar / well known equation for the transverse deflection of the string under the load *f*(x).

8.3. DEFLECTION OF THE CENTRAL LINE OF A BEAM

Let the axis of a beam of constant cross-section coincides with the x-axis. Suppose that the beam is bent by a transverse load

$$p = f(x), \qquad ...(1)$$

estimated per unit length of the beam.

As per theory of deformation of beams, we suppose that the shearing stresses are negligible in comparison with the tensile stress

$$\tau_{xx} = \frac{My}{I}, \qquad ....(2)$$

where M is the magnitude of the moment about the x-axis and I is the moment of inertia of the cross-section about x-axis.

The strain e_{xx} is then given by

$$e_{xx} = \frac{\tau_{xx}}{E}$$

$$= \frac{My}{IE}, \qquad \text{…(4)}$$

where E is the Young's modulus.

The strain-energy function W is given by

$$W = \frac{1}{2}\tau_{xx}\, e_{xx}$$

$$= \frac{M^2 y^2}{2EI^2}. \qquad \text{...(5)}$$

The strain energy per unit length of the beam is found by integrating W over the cross-section of the beam, and we get

$$\int_R W d\sigma = \frac{M^2}{2EI^2} \int_R y^2 d\sigma$$

$$= \frac{M^2}{2EI}. \qquad \text{...(6)}$$

The well known Bernoulli- Euler law is

$$M = -EI\frac{d^2y}{dx^2}. \qquad \text{….(7)}$$

The total strain energy U obtained by integrating the expression (6) over the length of the beam, and using (7), we find

$$U = \frac{1}{2}\int_0^{\ell} E\, I\left(\frac{d^2y}{dx^2}\right)dx. \qquad \text{….(8)}$$

We suppose that the ends of the beam are clamped, hinged, or free, so that the supporting forces do not work and contribute nothing to potential energy V.

If we neglect the weight of the beam, the only external load is p = *f*(x), then the formula

$$V = U - \int_0^{\ell} f(x)\, y\, dx, \qquad \text{…(9)}$$

for the potential energy gives

$$V = \int_0^{\ell} \left[\tfrac{1}{2} EI \left(\frac{d^2y}{dx^2} \right) - f(x)\, y \right] dx . \qquad ..(10)$$

The Euler's equation of the above function V is (left as an exercise)

$$\frac{d^2}{dx^2}\left(EI \frac{d^2y}{dx^2} \right) - f(x) = 0. \qquad ...(11)$$

8.4. DEFLECTION OF AN ELASTIC MEMBRANE

Let the membrane, with fixed edges, occupy some region in the xy-plane. We suppose that the membrane is stretched so that the tension T is uniform and that T is so great that it is not appreciably changed when the membrane is deflected by a distributed normal load of intensity $f(x, y)$.

We first compute the strain energy U. The total stretch e of the surface

$$\mathbf{z = u(x, y)}, \qquad \mathbf{...(1)}$$

is

$$e = \iint_R \left(d\sigma - dx\,dy \right)$$

$$= \iint_R \left(\sqrt{u_x^2 + u_y^2 + 1} - 1 \right) dx\, dy, \qquad ....(2)$$

where

$$d\sigma = \left(\sqrt{u_x^2 + u_y^2 + 1} \right) dx\,dy \qquad ....(3)$$

is the element of area of the membrane in the deformed state.

As usual, it is assumed that the displacement u and its first derivatives are small. Then, we can write (2) as

$$e = \frac{1}{2} \iint_R \left(u_x^2 + u_y^2 \right) dx\,dy . \qquad ....(4)$$

Hence, the strain energy U is given by

U = T e

$$= \frac{T}{2} \iint_R \left(u_x^2 + u_y^2\right) dx\, dy. \qquad \text{....(5)}$$

We know that the potential energy is given by the formula

$$V = U - \iint_R f(x, y)\, u\, dx\, dy\ . \qquad \text{...(6)}$$

Equations (5) and (6) give the potential energy as

$$V = \iint_R \left\{ \frac{T}{2} \left(u_x^2 + u_y^2\right) - f(x, y) \right\} u\ dx\, dy. \qquad \text{....(7)}$$

The equilibrium state is characterized by the condition

$$\delta\ V = 0. \qquad \text{....(8)}$$

This gives (left as an exercise)

$$T\ \nabla^2\ u + f(x, y) = 0. \qquad \text{...(9)}$$

8.5. TORSION OF CYLINDERS

We consider the Saint –Venant torsion problem for a cylinder of arbitrary cross-section. We shall use the Principle of Minimum Complementary Energy to deduce the appropriate Compatibility equation.

We know that the displacement components in the cross-section are

$$u\ (x, y, z) = -\alpha zy, \qquad v\ (x, y, z) = \alpha\ zx, \qquad \text{...(1)}$$

where α is the twist per unit length of the cylinder.

We assume with Saint-Venant principle that nonvanishing stresses are τ_{zx} and τ_{zy}.

The formula for complementary energy is

$$V^* = U - \int_{\sum u} T_i u_i \quad d\sigma, \qquad \ldots(2)$$

where the surface integral is evaluated over the ends of the cylinder, and strain energy U is given by

$$U = \int_{\tau} W \ d\tau, \qquad \ldots.(3)$$

where

$$W = \frac{1}{2} \tau_{ij} e_{ij}$$

$$= (\tau_{zx} e_{zx} + \tau_{zy} e_{zy}), \qquad \ldots.(4)$$

From shear strain relations, we have

$$\tau_{zx} = 2 \mu e_{zx},$$

$$\tau_{zy} = 2 \mu e_{zy}. \qquad \ldots.(5)$$

From equations (4) and (5), we find

$$W = \frac{1}{2\mu} (\tau^2_{zx} + \tau^2_{zy}), \qquad \ldots.(6)$$

and hence

$$U = \frac{1}{2\mu} \int_{\tau} \left(\tau^2_{zx} + \tau^2_{zy} \right) d\tau. \qquad \ldots(7)$$

Now, we shall compute the surface integral in (2).

i). For the end z = 0, we have

$$u + v = 0,$$

so

$$\int_R T_i . u_i \ d\sigma = 0. \qquad \ldots(8)$$

ii). On the end $z = \ell$, we have

$$\int_R T_i . u_i \ d\sigma = \alpha \iint_R \left(-\ell y \tau_{zx} + \ell x \tau_{zy} \right) dx\, dy. \qquad \ldots.(9)$$

Thus, we find

$$V^* = \frac{\ell}{2\mu}\iint_R \left(\tau^2{}_{zx} + \tau^2{}_{zy}\right) dx\,dy - \alpha\ell \iint_R \left(x\,\tau_{zy} - y\,\tau_{zx}\right) dx\,dy. \quad ...(10)$$

In this case, the admissible stresses satisfy the equilibrium equation (left as an exercise)

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} = 0, \qquad \text{in R} \;, \qquad ...(11)$$

and the boundary condition

$$\tau_{zx} \cos\,(x, \nu) + \tau_{zy} \cos\,(y, \nu) = 0, \qquad \text{on C.} \qquad ...(12)$$

Equilibrium equation (11) will clearly be satisfied identically if we introduce the stress function.

$$\Psi = \Psi(x, y)$$

such that

$$\tau_{zx} = \mu\alpha \frac{\partial \Psi}{\partial y},$$

$$\tau_{zy} = -\mu\alpha \frac{\partial \Psi}{\partial x}. \qquad ...(13)$$

The boundary condition (12) then requires that

$$\mu\alpha \left(\frac{\partial \Psi}{\partial y}\frac{dy}{ds} + \frac{\partial \Psi}{\partial x}\frac{dx}{ds} \right) = 0,$$

or

$$\frac{d\Psi}{ds} = 0,$$

or

$$\mathbf{\Psi = constant, \qquad on\ C.} \qquad \mathbf{...(14)}$$

On substituting expressions for stresses from (13) into equation (10), we get

$$V^* = \left(\frac{\mu\alpha^2\ell}{2}\right)\iint_R \left[(\Psi_x)^2 + (\Psi_y)^2 + 2(x\Psi_x + y\Psi_y)\right] dx\,dy. \qquad ...(15)$$

The corresponding Euler equation (exercise) is

$$\nabla^2\ \Psi = -2, \quad \text{in } R, \qquad ...(16)$$

which is precisely the equation for the Prandtl stress function.

Remark: The formula (15) for V^* can be written in a simple form which we shall find useful in subsequent considerations.

We note that

$$\iint_R (x\Psi_x + y\Psi_y)\,dx\,dy = \iint_R \left[\frac{\partial}{\partial x}(x\Psi) + \frac{\partial}{\partial y}(y\Psi)\right] dx\,dy - 2\iint_R \Psi\,dx\,dy. \quad ..(17)$$

But

$$\iint_R \left[\frac{\partial}{\partial x}(x\Psi) + \frac{\partial}{\partial y}(y\Psi)\right] dx\,dy = \int_C \Psi\left[x\cos(x,\nu) + y\cos(y,\nu)\right] ds. \qquad ...(18)$$

So that we obtain

$$V^* = \left(\frac{\mu\alpha^2\ell}{2}\right)\iint \left[(\nabla\Psi)^2 - u\Psi\right] dx\,dy + 2\int_C \Psi\left[x\cos(x,\nu) + y\cos(y,\nu)\right] ds, \quad ..(19)$$

where

$$(\nabla\Psi)^2 = (\Psi_x)^2 + (\Psi_y)^2. \qquad ...(20)$$

If the region R is simply connected, we can take

$$\Psi = 0, \qquad \text{on } C, \qquad ...(21)$$

and for the determination of Ψ, we have the functional

$$V^* = \left(\frac{\mu\alpha^2\ell}{2}\right)\iint_R \left[(\nabla\Psi)^2 - 4\Psi\right] dx\,dy, \qquad ...(22)$$

This functional (22) is to be minimized on the set of all functions of class C^2 vanishing on the boundary C of the simply connected region R.

8.6. VARIATIONAL PROBLEM RELATED TO THE BIHARMONIC EQUATION

Consider the variational problem

$$I(u) = \iint_R \left[(\nabla^2 u)^2 - 2fu \right] dx\, dy = \min, \qquad \text{....(1)}$$

where the admissible functions u(x, y) satisfy on the boundary C of the region R the conditions

$$\mathbf{u = \phi(s)\,,} \qquad \textbf{...(2)}$$

$$\frac{\partial u}{\partial \eta} = h(s)\,, \qquad \text{...(3)}$$

Suppose that the set { u(x, y)} of all admissible functions includes the minimizing function u(x, y). We represent an arbitrary function $\bar{u}(x, y)$ of this set in the form

$$\bar{u}(x, y) = u(x, y) + \in \eta(x, y), \qquad \text{...(4)}$$

where $\in$ is a small real parameter.

We know that the necessary condition that u minimize the integral (1) is

$$\delta I = \left[\frac{d}{d\in} I(u + \in \eta) \right]_{\in = 0} = 0. \qquad \text{...(5)}$$

Using (1), we write

$$I(u + \in\eta) = \iint_R \left[\{\nabla^2 (u + \in \eta)\}^2 - 2f(u + \in \eta) \right] dx\, dy. \qquad \text{...(6)}$$

Therefore equations (5) and (6) give

$$\iint_R \left(\nabla^2 u \nabla^2 \eta - f\,\eta\right) dx\,dy = 0\,. \qquad \ldots(7)$$

We note that

$$\nabla^2 u\, \nabla^2 \eta, = (\nabla^2 u)\left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}\right)$$

$$= \begin{array}{l} \left[\frac{\partial}{\partial x}\left(\nabla^2 u \frac{\partial \eta}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nabla^2 u \frac{\partial \eta}{\partial y}\right)\right] - \left[\frac{\partial}{\partial x}\left(\eta \frac{\partial}{\partial x}\nabla^2 u\right) + \frac{\partial}{\partial y}\left(\eta \frac{\partial}{\partial y}\nabla^2 u\right)\right] \\ + \left[\frac{\partial^2}{\partial x^2}\left(\nabla^2 u\right) + \frac{\partial^2}{\partial y^2}\left(\nabla^2 u\right)\right]\eta\,. \end{array}$$

$$\ldots(8)$$

Applying Gauss Diwergence & Stoke’s theorems we get

$$\iint_R \left[\frac{\partial}{\partial x}\left(\nabla^2 u \frac{\partial \eta}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nabla^2 u \frac{\partial \eta}{\partial y}\right)\right] dx\,dy = \int_C \nabla^2 u \frac{\partial \eta}{\partial \nu} ds\,, \qquad \ldots(9)$$

$$\iint_R \left[\frac{\partial}{\partial x}\left(\eta \frac{\partial}{\partial x}\nabla^2 u\right) + \frac{\partial}{\partial y}\left(\eta \frac{\partial}{\partial y}\nabla^2 u\right)\right] dx\,dy = \int_C \eta \frac{\partial}{\partial \nu}\left(\nabla^2 u\right) ds\,. \qquad \ldots(10)$$

For equations (7) to (10), we obtain

$$\delta I = \iint_R \left(\nabla^2 \nabla^2 u - f\right)\eta\, dx\,dy + \int_C \nabla^2 u \frac{\partial \eta}{\partial \nu} ds - \int_C \eta \frac{\partial}{\partial \nu}\left(\nabla^2 u\right) ds = 0 \qquad \ldots(11)$$

From equations (2) to (4), we have

$$\eta = 0 \text{ and } \frac{\partial \eta}{\partial \nu} = 0 \qquad \text{on } C. \qquad \ldots(12)$$

Using result (12), equation (11) gives

$$\nabla^4 u = f(x, y) \quad \text{in the region R.} \qquad ...(13)$$

This is the same differential equation which arises in the study of the transverse deflection of thin elastic plates.

We have assumed in the foregoing that the admissible functions in the set {u (x, y)} satisfy the boundary conditions (2) and (3).

If we consider a larger set S of all functions u belonging to class C^4, then (11) must hold for every u in this set. But the set S includes functions that satisfy the boundary conditions (2) and (3), and thus we must have,

$$\iint_R \left(\nabla^4 u - f\right) \eta \, dx\, dy = 0. \qquad ...(14)$$

Since η is arbitrary, it follows that the minimizing function u(x, y) again satisfy (13) and we conclude from (11) that

$$\int_C \nabla^2 u \frac{\partial \eta}{\partial \nu} ds - \int_C \eta \frac{\partial}{\partial \nu}\left(\nabla^2 u\right) ds = 0, \qquad ...(15)$$

for every η of class C^4.

Now if we consider first all η such that

$$\eta = 0 \quad \text{on C and} \quad \frac{\partial \eta}{\partial \nu} \neq o \qquad \text{on C,} \qquad ...(16)$$

it follows from (15) that

$$\nabla^2 u = 0, \qquad \text{on C.} \qquad ...(17)$$

On the other hand, if we consider only those η such that

$$\eta \neq 0, \quad \frac{\partial \eta}{\partial \nu} = 0, \qquad \text{on C,} \qquad ...(18)$$

We get the condition

$$\frac{\partial}{\partial \nu}\left(\nabla^2 u\right) = o, \qquad \text{on C.} \qquad ...(19)$$

Hence if the functional in (1) is minimized on the set S of all u of class C^4, the minimizing function will be found among those functions of S which satisfy the conditions (17) and (19) on the boundary of the region.

Remark:- For this variational problem (1), we shall obtain the same differential equation (13) when the minimizing function u = u (x, y), instead of the system given in (2) and (3), satisfies the following boundary condition.

$$\nabla^2 u = 0,$$

$$\frac{\partial}{\partial \nu}\left(\nabla^2 u\right) = 0, \quad \text{on } C.$$

8.7. RITZ METHOD :– ONE DIMENSIONAL CASE

Consider the variational problem

$$I(y) = \int_{x_0}^{x_1} F(x, y, y')\, dx, \qquad \ldots(1)$$

in which all admissible function y = y (x) are such that

$$y(x_0) = y_1, \quad y(x_1) = y_1. \qquad \ldots(2)$$

We know that such a function y is a solution of the Euler's equation

$$F_y - \frac{d}{dx} F_{y'} = 0. \qquad \ldots(3)$$

A direct method to obtain the desired function was proposed by W. Ritz in 1911.

In this method, we construct a sequence of functions which converge to desired solution of the Euler's equation (3).

Outlines of the Ritz Method:

Let $y = y^*(x)$ be the exact solution of the given variational problem. Let $I(y^*) = m$ be the minimum value of the functional in (1).

In this method, one tries to find a sequence $\{\bar{y}_n(x)\}$ of admissible functions such that

$$\lim_{n\to\infty} I(\bar{y}_n(x)) = m, \qquad ...(4)$$

so that

$$\lim_{n\to\infty} \bar{y}_n(x) = y^*(x), \qquad ...(5)$$

is the required function.

Ritz proposed to construct a function $\bar{y}(x)$ by choosing a family of functions

$$y(x) = \phi(x, a_1, a_2 \ldots\ldots\ldots\ldots, a_k) \qquad ....(6)$$

depending on k real parameters a_i, where ϕ is such that for all values of the a_i, the end conditions given in (2) are satisfied.

Then, on putting the value of y from (6) in (1), one obtains

$$I(a_1, a_2, a_3, \ldots\ldots\ldots\ldots, a_k).$$

This functional can be minimized by determining the values of the parameters a_i from the following system of equations:

$$\frac{\partial I(a_1, a_2, \ldots\ldots\ldots a_k)}{\partial a_r} = 0 \qquad \text{for } r = 1, 2, \ldots\ldots k. \qquad ...(7)$$

Let this system has solution $\bar{a}_1, \bar{a}_2, \ldots\ldots, \bar{a}_k$. Then, the minimizing function, say $\bar{y}(x)$, is

$$\bar{y}(x) = \phi(x, \bar{a}_1, \bar{a}_2, \ldots\ldots\ldots, \bar{a}_k). \qquad ...(8)$$

It is expected that $\bar{y}(x)$ will be a fair approximation to the minimizing function $y^*(x)$ when the number of parameters in (3) is made sufficiently large.

Now, we construct a sequence $\{\bar{y}_n(x)\}$ of functions $\bar{y}_n(x)$ such that

$$\lim_{n\to\infty} I(\bar{y}_n) = m. \qquad ...(9)$$

Consider a sequence of families of functions of the type (6), namely,

$$\left.\begin{array}{l} y_1(x) = \phi_1(x, a_1) \\ y_2(x) = \phi_2(x, a_1, a_2) \\ y_3(x) = \phi_3(x, a_1, a_2, a_3) \\ -- ------ -----, \\ y_n(x) = \phi_n(x, a_1, a_2, \ldots\ldots\ldots, a_n) \end{array}\right\}, \qquad ...(10)$$

in which the family $y_k(x) = \phi_k(x, a_1, \ldots\ldots\ldots, a_k)$ includes in it all functions in the families with subscript less than k.

The parameters a_i in each function y_k can be determined so as to minimize the integral $I(y_k)$. We denote the values of the parameters thus obtained by $\bar{a}_i$, so that the minimizing functions are

$$\bar{y}_n(x) = \phi_n(x, \bar{a}_1, \bar{a}_2, \ldots\ldots\ldots, \bar{a}_n), \qquad \ldots(11)$$

for $n = 1, 2, \ldots$ Since each family $y_k(x)$ includes the families $y_{k-1}(x)$ for special values of parameters a_i, the successive minima $I(\bar{y}_k)$ are non-increasing, therefore,

$$I(\bar{y}_1) \geq I(\bar{y}_2) \geq I(\bar{y}_3) \geq \ldots\ldots\ldots \geq I(\bar{y}_{n-1}) \geq I(\bar{y}_n) \geq \ldots\ldots\ldots \qquad \ldots(12)$$

Since the sequence $\{I(\bar{y}_n)\}$ of real numbers is bounded below by m and is non-increasing (m being the exact minima), therefore, it is a convergent sequence.

In order to ensure the convergence of this sequence to $I(y^*)$, one must impose some conditions on the choice of functions ϕ_i in (10). We take the set of functions in (10) to be relatively complete.

Then for each $\in \to 0$, there exists (by definition of relatively complete) in the family (10), a function

$$y^*_n(x) = y_n(x, a^*_1, a^*_2, \ldots\ldots\ldots a^*_n)$$

such that

$$\left| y^*_n - y^* \right| < \in, \qquad \ldots(13)$$

and

$$\left| y^{*\prime}_n - y^{*\prime} \right| < \in, \qquad \ldots(14)$$

for all $x \in (x_0, x_1)$. But, it is known that $F(x, y, y')$ is a continuous function of its arguments, therefore,

$$\left| F(x, y^*_n, y^{*\prime}_n) - f(x, y^*, y^{*\prime}) \right| < \in, \qquad \ldots(15)$$

for all x in (x_0, x_1). Consequently

$$I(y^*_n) - I(y^*) = \int_{x_0}^{x_1} \left[F(x, y^*_n, y^{*\prime}_n) - F(x, y^*, y^{*\prime}) \right] dx$$

$$= \int_{x_0}^{x_1} \left| F(x, y^*_n, y^{*}_n{}') - F(x, y^*, y^*{}') \right| dx$$

$$< \in' \text{, say.}$$

This gives

$$I(y^*_n) < I(y^*) + \in' . \qquad ...(16)$$

As y^*_n is a function of the set (10) and I ($\bar{y}_n$) is a minimum of I (y) on the family y_n, therefore,

$$I(y^*_n) \geq I(\bar{y}_n) . \qquad ...(17)$$

As y^* is the exact solution of the problem and $\bar{y}_n$ is an approximation of the same, therefore,

$$I(y^*) \leq (\bar{y}_n). \qquad ...(18)$$

Combining the in equalities (16) to (18), we find

$$I(y^*) \leq I(\bar{y}_n) \leq I(\bar{y}_n) < I(y^*) + \in, \qquad ...(19)$$

but $\in'$ can be made as small as we wish, therefore, we get

$$\underset{n\to\infty}{\text{Lt}}\ I(\bar{y}_n) = \underset{n\to\infty}{\text{Lt}}\ I(y^*_n) = I(y^*). \qquad ...(20)$$

This completes the proof.

Definition. Let y (x) be an admissible function satisfying the end conditions

$$y(x_0) = y_0,\ y(x_1) = y_1.$$

If , for each $\in > 0$, their exists in the family (6) a function

$$y^*_n(x) = y_n(x, a^*_1, a^*_2, a^*_3, \ldots, a^*_n)$$

such that

$$\left| y^*_n - y^* \right| < \in \qquad \text{and} \qquad \left| y^*_n{}' - y^*{}' \right| < \in$$

for all x in (x_0, x_1), then the family of functions in (6) is said to be **relatively complete.**

Remark:– Among useful, relatively complete sets of functions in the interval $(0, \ell)$ are

(i) trigonometrically polynomials : $\sum_{k=1}^{n} a_k \sin\left(\frac{k\pi x}{\ell}\right)$,

(ii) algebraic polynomials : $\sum_{k=1}^{n} a_k x^k (\ell - x)$.

Question:– Show that the system of equations

$$\frac{\partial}{\partial a_j} I(y_n) = 0 \text{ (for } j = 1, 2, 3, \ldots\ldots., n)$$

for the coefficients in the approximate solution

$$y_n(x) = \sum_{k=1}^{n} a_k \phi_k(x)$$

of the variational problem

$$I(y) = \int_0^{\ell} \left[p y'^2 + q y^2 + 2 f y\right] dx = \min,$$

$$y(0) = y(\ell) = 0,$$

by the Ritz method is

$$\int_0^{\ell} \left(p y'_n \phi'_j + q y_n \phi_j + f \phi_j\right) dx = 0, \; j = 1, 2, \ldots\ldots.., n.$$

Solution:– We have

$$I(y_n) = \int_0^{\ell} \left[p y_n'^2 + q y_n^2 + 2 f y_n\right] dx. \qquad \ldots(1)$$

Therefore, we find

$$\frac{\partial}{\partial a_j} I(y_n) = \int_0^{\ell} \left[2p\, y_n' . \frac{\partial}{\partial a_j}.(y_n') + 2\, q\, y_n \frac{\partial}{\partial a_j}.y_n + 2f \frac{\partial}{\partial a_j} y_n\right] dx$$

$$= 2\int_0^{\ell} \left(p y_n' \phi'_j + q y_n \phi_j + f \phi_j\right) dx. \qquad \ldots(2)$$

Hence, the system of equations

$$\frac{\partial}{\partial a_j} I(y_n) = 0 \qquad \text{...(3)}$$

becomes, using (2),

$$\int_0^{\ell} \left[p\, y_n' \phi_j' + q\, y_n \phi_j + f \phi_j\right] dx = 0 \quad , \qquad \text{…(4)}$$

for j = 1, 2, ……..,n.

This completes the solution.

8.8. RITZ METHOD :– TWO-DIMENSIONAL CASE

Consider the functional in the form

$$I(u) = \iint_R F(x, y, u, u_x, u_y)\, dx\, dy\,. \qquad \text{…(1)}$$

We suppose that the admissible functions in the variational problem,

$$I(u) = \text{minimum}, \qquad \text{…(2)}$$

satisfy the condition

$$u = \varphi(s)$$

on the boundary C of the region R.

Let u^* (x, y) be an exact solution of the variational problem (obtained by solving the corresponding Euler's equation) and let

$$I(u^*) = m\ , \qquad \text{...(3)}$$

be the minimum value of the functional (1).

We now introduce a sequence $\{u_n(x, y)\}$ of families of admissible functions

$$u_n(x, y) = \phi_n(x, y, a_1, a_2, \ldots\ldots., a_n), \qquad \text{...(4)}$$

with parameters a_i, and suppose that each family u_i (x, y) includes in it families with subscripts less than i.

We further assume that the set (4) is relatively complete.
Then, for each $\in > 0$, there exists a function

$$u^*_n(x, y) = \phi_n(x, y, a^*_1, a^*_2, \ldots\ldots\ldots, a^*_n)$$

belonging to the set (4) such that

$$\left| u^*_n - u^* \right| < \in, \qquad \left| \frac{\partial u^*_n}{\partial x} - \frac{\partial u^*}{\partial x} \right| < \in, \ \left| \frac{\partial u^*_n}{\partial y} - \frac{\partial u^*}{\partial y} \right| < \in \ , \qquad \ldots(5)$$

for all $(x, y) \in R$. With the help of (4), we form $I(u_n)$ and determine the parameters a_i so that $I(u_n)$ is a minimum.

Let $\bar{a}_i$ be the values of the a_i obtained by solving the system of equations (known as Ritz's equations)

$$\frac{\partial I(u_n)}{\partial a_j} = 0, \qquad \ldots(6)$$

for j =1, 2, …..,n. We write

$$\bar{u}_n(x, y) = \varphi_n(x, y, \bar{a}_1, \bar{a}_2, \ldots\ldots\ldots \ldots\ldots \bar{a}_n) . \qquad \ldots(7)$$

The sequence $\{I(\bar{u}_n)\}$ of real numbers then converges to $I(u^*) = m$, where $u^*(x, y)$ is the function that minimizes (1).

The remaining proof is similar to the proof for one dimensional case.

Illustration. Find an approximate solution to the problem of extremising the functional

$$I(z) = \iint_D \left(z_x^2 + z_y^2 - 2z\right) dx\, dy , \qquad \ldots(1)$$

where the region R is a sequence, $-a \le x \le a$, $-a \le y \le a$ and $z = 0$ on the boundary of the sequence D.

Solution: We shall seek an approximate solution in the form

$$z_1 = z = \alpha_1(x^2 - a^2)(y^2 - a^2) , \qquad \ldots(2)$$

in which α_1 is a constant to be determined.

It is clear that this function z_1 satisfies the boundary condition. Putting the value of z from (2) into (1), we find

$$I(z_1) = \iint_D \left[4\alpha_1^2 x^2 \left(y^2 - a^2\right)^2 + 4\alpha_1^2 y^2 \left(x^2 - a^2\right)^2 - 2\alpha_1 \left(x^2 - a^2\right)\left(y^2 - a^2\right)\right] dx\, dy$$

$$= 4\alpha_1^2$$

$$\int_{-a}^{a} x^2 dx \int_{-a}^{a} (y^2 - a^2)^2 dy + 4\alpha_1^2 \int_{-a}^{a} y^2 dy. \int_{-a}^{a} (x^2 - a^2)^2 dx - 2\alpha_1 \int_{-a}^{a} (x^2 - a^2) dx \int_{-a}^{a} (y^2 - a^2) dy$$

$$= 4\alpha_1^2 \left[\frac{x^3}{3}\right]_{-a}^{a} \left[\frac{y^5}{5} - \frac{2a^3 y^3}{3} + a^4 y\right]_{-a}^{a}$$

$$+ 4\alpha_1^2 \left[\frac{y^3}{3}\right]_{-a}^{a} \left[\frac{x^5}{5} - 2a^2 \frac{x^3}{3} + a^4 x\right]_{-a}^{a}$$

$$- 2\alpha_1 \left[\frac{x^3}{3} - a^2 x\right]_{-a}^{a} \left[\frac{y^3}{3} - a^2 y\right]_{-a}^{a}$$

$$= 8\alpha_1^2 \frac{1}{3} (a^3 + a^3) \left(\frac{a^5}{5} - \frac{2a^5}{3} + a^5\right)(2) - 2\alpha_1 \left(\frac{a^3}{3} - a^3\right).4$$

$$= \frac{32}{3}\alpha_1^2 a^3 \left(\frac{8a^5}{15}\right) - 8\alpha_1^2 \left(\frac{2a^3}{3}\right)^2 = \frac{32 \times 8}{3 \times 15}\alpha_1^2 a^8 - \frac{32}{9}\alpha_1 a^6. \qquad ...(3)$$

For an extremum value, we have

$$\frac{dI(\alpha_1)}{d\alpha_1} = 0 \qquad ...(4)$$

This gives

$$\alpha_1 = \frac{5}{16a^2}. \qquad ...(5)$$

Thus, an approximate solution is

$$z = \frac{5}{16a^2}(x^2 - a^2)(y^2 - a^2). \qquad \text{...(6)}$$

Question. Show that the system of equations $\frac{\partial}{\partial a_j} I(u_n) = 0$, $(j = 1, 2, \ldots\ldots, n)$ for the determination of coefficients in the minimizing function

$$u_n = \sum_{i=1}^{n} a_i \phi_i (x, y)$$

for the problem

$$I(u) = \iint_R \left(u_x^2 + u_y^2 + 2fu \right) dx\, dy = \text{min}, \qquad u = 0 \quad \text{on } C$$

is

$$\iint_R \left[\frac{\partial u_n}{\partial x}\frac{\partial \phi_j}{\partial x} + \frac{\partial u_n}{\partial y}\frac{\partial \phi_i}{\partial y} + f\,\phi_j \right] dx\, dy = 0\,, \qquad j = 1, 2, 3, \ldots\ldots., n.$$

Solution:– The minimizing function is

$$u_n = \sum a_i \phi_i \qquad \ldots\ldots\ldots(1)$$

We form

$$I(u_n) = \iint_R \left(u_{n,x}^2 + u_{n,y}^2 + 2fu_n \right) dx\, dy\,. \qquad \ldots(2)$$

Therefore, we find

$$\frac{\partial}{\partial a_j} I(u_n) = \iint_R \left[2u_{n,x}\frac{\partial}{\partial a_j}(u_{n,x}) + 2u_{x,y}\frac{\partial}{\partial a_j}(u_{n,y}) + 2f\frac{\partial}{\partial a_j}u_n \right] dx\, dy \qquad \ldots(3)$$

We have

$$u_{n,x} = \sum_{i=1}^{n} a_i \frac{\partial \phi_i}{\partial x}\,, \qquad \ldots(4)$$

and

$$\frac{\partial}{\partial a_j}(u_{n,x}) = \frac{\partial \phi_j}{\partial x}\,. \qquad \ldots(5)$$

Similarly,

$$\frac{\partial}{\partial a_j}(u_{n,y}) = \frac{\partial \phi_j}{\partial y}\,, \qquad \ldots(6)$$

$$\frac{\partial}{\partial a_j} u_n = \phi_j \ . \qquad \ldots(7)$$

Using (5) to (7) in relation (3), we write

$$\frac{\partial}{\partial a_j} I(u_n) = 2 \iint_R \left[\frac{\partial u_n}{\partial x} \frac{\partial \phi_j}{\partial x} + \frac{\partial u_n}{\partial y} \frac{\partial \phi_j}{\partial y} + f \phi_j \right] dx\, dy. \qquad \ldots(8)$$

Therefore, the system of equations

$$\frac{\partial}{\partial a_j} I(u_m) = 0 \ , \qquad \ldots(9)$$

becomes

$$\iint_R \left[\frac{\partial u_n}{\partial x} \frac{\partial \phi_j}{\partial x} + \frac{\partial u_n}{\partial y} \frac{\partial \phi_j}{\partial y} + f \phi_j \right] dx\, dy = 0, \qquad ..(10)$$

for $j = 1, 2, \ldots..,n$.

Question:– Show that the system of equations $\frac{\partial I(u_n)}{\partial a_j} = 0$, $(j = 1, 2, \ldots\ldots.,n)$,

for determining the coefficients in the approximate solution

$$u_x = \sum_{i=1}^{n} a_i \phi_i$$

for the problem

$$I(u) = \iint_R \left[(\nabla^2 u)^2 - fu \right] dx\, dy = \min,$$

$$u = 0, \qquad \frac{\partial u}{\partial \eta} = 0 \qquad \text{on } C \text{ , (C being the boundary of R) ,}$$

is

$$\iint_R \left[\sum_{i=1}^{\eta} a_i \nabla^2 \phi_i \nabla^2 \phi_j - f\phi_j \right] dx\, dy = 0$$

for $j = 1, 2, \ldots..,n$.

Solution:– We form $I(u_n)$. We obtain

$$I(u_n) = \iint_R \left[(\nabla^2 u_n)^2 - 2fu_n \right] dx\, dy. \qquad ...(1)$$

We find

$$\frac{\partial}{\partial a_j} I\,(u_n) = 2 \iint_R \left[\nabla^2 u_n \cdot \frac{\partial}{\partial a_J} \left(\nabla^2 u_n \right) - f \frac{\partial}{\partial a_J} u_n \right] dx\,dy$$

$$= 2 \iint_R \left(\nabla^2 u_n . \nabla^2 \phi_j - f\,\phi_j \right) dx\,dy$$

$$= 2 \iint_R \left(\sum_{i=1}^{\eta} \nabla^2 \phi_i \nabla^2 \phi_j - f\phi_j \right) dx\,dy. \qquad ...(2)$$

Therefore, the system of equations

$$\frac{\partial}{\partial a_j} I\,(u_n) = 0 \;, \qquad ...(3)$$

becomes

$$\iint_R \left(\sum_{i=1}^{\eta} \nabla^2 \phi_i \nabla^2 \phi_j - f\phi_j \right) dx\,dy = 0 \;, \qquad ...(4)$$

for j = 1, 2,……..,n. This completes the solution.

8.9. GALERKIN METHOD

In 1915, Galerkin proposed a method of finding an approximate solution of the boundary value problems in mathematical physics. This method shall have wider scope than the method of Ritz.

Method **: Let it be required to solve a linear differential equation**

$$\mathbf{L\,(u) = 0 \qquad in\ R,} \qquad \mathbf{…(1)}$$

subject to some homogeneous boundary conditions, L being a linear differential operator.

It is assumed, for simplicity that the domain R is two-dimensional.

We seek an approximate solution of the problem of the type

$$u_n\,(x, y) = \sum_{i=1}^{n} a_i \phi_i\,(x, y) \qquad …(2)$$

where the ϕ_i are suitable coordinate functions and a_i are constant.

We suppose that the functions ϕ_i satisfy the same boundary conditions as the exact solution u(x, y). We further suppose that the set $\{\phi_i)$ is complete in the sense that every piecewise continues function *f*(x, y), say, can be approximated in R by the sum $\sum_{i=1}^{n} c_i\phi_i$ in such a way that

$$\delta_N = \iint_R \left(f - \sum_{i=1}^{N} c_i\phi_i \right) dx\,dy \qquad \ldots(3)$$

can be made as small as we wish.

Ordinarily, u_n given in (2) will not satisfy (1). Let

$$L\,(u_n) = \in_n (x, y), \qquad \text{where } \in_x(x, y) \neq 0\,, \quad \text{in R}\,. \quad \ldots(4)$$

If maximum of $\in_n$ (x, y) is small, we can consider u_n (x, y) given is (2) as a satisfactory approximation to the exact solution u(x, y).

Thus, to get a good approximation, we have to choose the constants a_i so as to minimize the error function $\in_n$(x, y).

A reasonable minimization technique is suggested by the following:

If one represents u(x, y) by the serious u(x, y) = $\sum_{i=1}^{\infty} a_i\phi_i$, with suitable properties and consider $u_n = \sum_{i=1}^{\eta} c_i\phi_i$ as the nth partial sum, then, the orthogality condition,

$$\iint_R L\,(u_n)\,\phi_i\,(x, y)\,dx\,dy = 0, \qquad \ldots(5)$$

as $n \to \infty$ is equivalent to the statement

$$\mathbf{L\,(u) = 0,} \qquad \mathbf{\ldots(6)}$$

by virtue of (1).

This led Galerkin to impose on the error function $\in_n$ a set of orthogality conditions **(now called Galerkin conditions)**

$$\iint_R L\,(u_n)\,\phi_i\,(x, y)\,dx\,dy = 0, \qquad \ldots(7)$$

for i =1, 2, ……….,n. This yields the set of equations

$$\iint_R L\left(\sum_{j=1}^{\eta} a_j\phi_j\right)\phi_i \; dxdy = 0, \qquad \ldots(8)$$

for i =1, 2,………,n.

This set of n equations determine the constants a_i in the approximate solution (2).

Remark 1. When the differential equation and the boundary conditions are self-adjoint and the corresponding functional I (u) in the problem

$$I(u) = \min, \qquad \ldots(9)$$

is positive definite, then the system of Galerkin equation in (8) is equivalent to the Ritz system

$$\frac{\partial}{\partial a_j} I(u_n) = 0. \qquad \ldots(10)$$

Remark 2. It is important to the note that in Galerkin's formulation, there is no reference to any connection of equation (1) with a variational problem. Indeeds, the Galerkin method can be applied to a wider class of problems phrased in terms of integrals and other types of functional equations.

Question. Solve the variational problem

$$\int_0^1 \left[y'^2 - y^2 - 2xy\right] dx = \min,$$

$$y(0) = y(1) = 0, \qquad \ldots(1)$$

by the Galerkin method.

Solution:– Here

$$F = y^{1^2} - y^2 - 2xy,$$

and Euler's equation is

$$y'' + y - x = 0, \qquad \text{in } 0 < x < 1,$$

$$L[y] = \frac{d^2y}{dx^2} + y - x = 0. \qquad \ldots(2)$$

We consider an approximate solution of the problem of the form

$$y_n = (1 - x)\,[a_1 x + a_2 x^2 + \ldots\ldots\ldots + a_n x^n]\,, \quad \text{...(3)}$$

which satisfy the boundary conditions.

The first approximation is

$$y_1 = a_1 x(1 - x)$$

$$= a_1 (x - x^2). \qquad \ldots(4)$$

Here,

$$\phi_1 = x - x^2\,. \qquad \ldots(5)$$

We find

$$L[y_1] = a_1(-2) + a_1(x - x^2) + (-x) = a_1(x - x^2 - 2) - x, \quad \ldots(6)$$

and the coefficient a_1 is determined from the Galerkin's equation

$$\int_0^1 L(u_1)\,\phi_1\,dx = 0. \qquad \ldots(7)$$

This yields

$$\int_0^1 \left[a_1 (x - x^2 - 2) - x\right].(x - x^2)\,dx = 0$$

$$\int_0^1 \left[a_1 (x - x^2 - 2)(x - x^2) - (x^2 - x^3)\right] dx = 0$$

$$\int_0^1 \left[a_1 (-2x + 3x^2 - 2x^3 + x^4) - (x^2 - x^3)\right] dx = 0\,.$$

This gives

$$a_1 = \frac{-5}{18}\,. \qquad \ldots.(8)$$

Thus, an approximate solution of the given variational problem, using Galerkin method, is

$$y \equiv y_1(x) = \frac{-5}{18}\,(x - x^2)\,.$$

8.10 APPLICATION OF GALERKIN METHOD TO THE PROBLEM OF TORSION OF BEAMS

Consider a cylindrical bar subjected to no body forces and free external forces on its lateral surface. One end of the bar is fixed in the plane z= 0 while the other end is in the plane z = ℓ (say). The bar is twisted by a couple of magnitude M whose moment is directed along the axis of the bar (i.e., z-axis). Prandtl introduced a function $\Psi(x, y)$, **known as Prandtl stress function,** such that

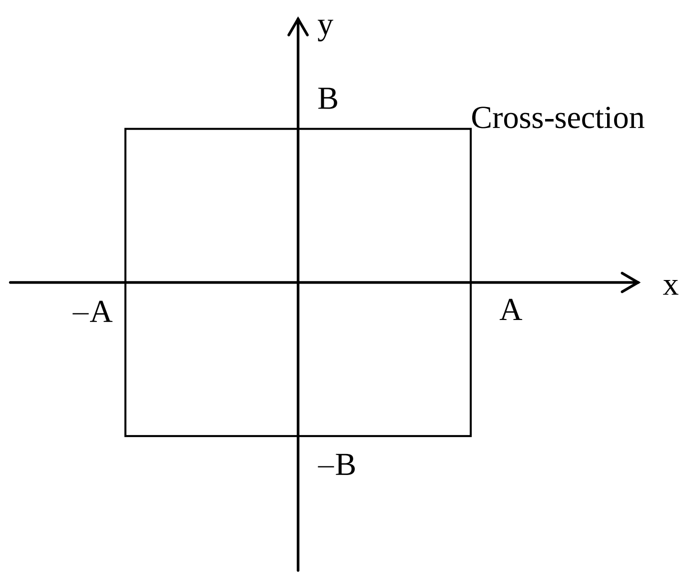
$$\tau_{zx} = \mu\,\alpha\,\frac{\partial\Psi}{\partial y}, \qquad \tau_{zy} = -\mu\,\alpha\,\frac{\partial\Psi}{\partial x}, \qquad \text{….(1)}$$

and stress function Ψ is determined from the system

$$\nabla^2\,\Psi = -2 \quad \text{in R}, \qquad \text{….(2)}$$

$$\boldsymbol{\Psi = 0 \quad \text{on C}}, \qquad \textbf{….(3)}$$

where R is the region of the cross-section of the bar and C its boundary.
Let R be the rectangle $|x| \le A$, $|y| \le B$.



Now we have to solve the system consisting of equations (2) and (3) by using the Galerkin method. We write (2) as

$$\mathbf{L\,(\Psi) = 0}, \qquad \textbf{….(4)}$$

where

$$L \equiv \nabla^2 + 2. \qquad \text{...(5)}$$

We take an approximate solution in the form

$\Psi_n(x, y) = (x^2 - A^2)(y^2 - B^2)(a_1 + a_2 x^2 + a_3 y^2 + \ldots\ldots.. a_n x^{2k} y^{2k})$. …(6)

This approximate solution satisfies the boundary conditions in (3). Here a_1, $a_2, \ldots.., a_n$ are constants to be determined by using Galerkin method.

The first approximation is

$$\Psi_1 = a_1\phi_1 = a_1 (x^2 - A^2)(y^2 - B^2)\,, \qquad \text{...(7)}$$

with

$$\phi_1 = (x^2 - A^2)(y^2 - B^2)\,. \qquad \text{...(8)}$$

The coefficient a_1 is determined from the Galerkin equation

$$\int_{-B}^{B}\int_{-A}^{A} L(\Psi_1)\,\phi_1\, dxdy = 0\,. \qquad \text{...(9)}$$

This implies

$$\int_{-B}^{B}\int_{-A}^{A} (\nabla^2\Psi_1 + 2)\,\phi_1\, dxdy = 0$$

$$\int_{-B}^{B}\int_{-A}^{A} [2a_1(y^2 - B^2) + 2a_1(x^2 - A^2) + 2](x^2 - A^2)(y^2 - B^2)\, dx\, dy = 0$$

$$2a_1 \int_{-A}^{A} (x^2 - A^2)\, dx \int_{-B}^{B} (y^2 - B^2)^2\, dy + 2a_1 \int_{-A}^{A} (x^2 - A^2)\, dx \int_{-B}^{B} (y^2 - B^2)^2\, dy$$

$$+ 2\int_{-A}^{A} (x^2 - A^2)\, dx \int_{-B}^{B} (y^2 - B^2)^2\, dy = 0$$

Integration yields

Hence

$$a_1 = \frac{5}{4}\left(\frac{1}{A^2 + B^2}\right). \qquad \text{..(10)}$$

Therefore, an approximate solution, by Galerkin method, to the given boundary value problem is

$$\Psi_1 = \frac{5}{4}\left(\frac{1}{A^2+B^2}\right)(x^2 - A^2)\,(y^2 - B^2). \qquad ..(11)$$

Note:– Approximate values of the torsinal rigidity D and maximum shear stress τ_{max} can also be computed with the help of (7). We recall that

$$D = 2\mu \iint_R \Psi \, dx\, dy, \qquad ..(12)$$

and the maximum shear stress τ_{max} occurs at the mid points of the longer sides of the bar.

If B > A, then

$$\tau_{max} = \tau_{yz}\Big|_{\substack{x=A\\ y=0}}$$

$$= -\mu\,\alpha \left[\frac{\partial \Psi}{\partial x}\right]_{\substack{x=A\\ y=0}}. \qquad ...(13)$$

Inserting the values of $\Psi = \Psi_1$ given in (7) into relations (12) and (13), we find that

$$D_1 = \frac{5}{18}\mu a^2 b\left[\frac{(b/a)^2}{1+(b/a)^2}\right], \qquad ...(14)$$

$$\tau_{max} = \frac{5}{4}\mu\alpha a\left[\frac{(b/a)^2}{1+(b/a)^2}\right], \qquad ...(15)$$

with $a = 2A$, $b = 2B$.

8.11. METHOD OF KANTOROVICH

In 1932, Kantorvich proposed a generalization of the Ritz method. In the present method, the integration of partial differential equation (Euler's equation) reduces to the integration of a system of ordinary differential equations.

In the application of the Ritz method to the problem

$$I(u) = \iint_R F(x, y, u, u_x, u_y)\, dx\, dy = \min, \qquad \ldots(1)$$

we consider approximate solutions in the form

$$u_n = \sum_{k=1}^{\eta} a_k \phi_k (x, y) \quad , \qquad \ldots.(2)$$

where the functions $\varphi_k(x, y)$ satisfy the same boundary conditions as those imposed on the exact solution u (x, y) and a_k are constants. We then determined the coefficients a_k so as to minimize I (u_n).

In the method of Kantorvich, the a_k in (2) are no longer constants but are unknown functions of x such that the product

$$a_k (x)\, \phi_k (x, y)$$

satisfies the same boundary conditions as u(x, y).

This led to minimize

$$I(u_n) = I\left(\sum_{k=1}^{\eta} a_k (x) \phi_k (x, y)\right). \qquad \ldots(3)$$

Since the functions ϕ_k (x, y) are known functions, we can perform integration w. r. t. y in (1) and then obtain a functional of the type

$$I(u_n) = \int_{x_0}^{x_1} f\left[a_k (x), a'_k (x), x\right] dx. \qquad \ldots(4)$$

Kantorvich proposed to determine the function a_k (x) so that they minimize the functional (4). It is clear that a_k (x) can be determined by solving the second order ordinary differential equations (Euler's equations)

$$f_{a_k} - \frac{d}{dx} f_{(a_k)} = 0, \quad \text{for } k = 1,2,\ldots\ldots..,n. \qquad \ldots(5)$$

Once a_k (x) are determined from (5), an approximate solution is known.

8.12. APPLICATION OF KANTORVICH METHOD TO THE TORSINAL PROBLEM

The torsion boundary value problem is

$$\left.\begin{aligned} \nabla^2\Psi &= -2 \quad \text{in } R \\ \Psi &= 0 \quad \text{on } C, \end{aligned}\right\} \qquad \text{....(1)}$$

C being the boundary of R, where R is the rectangle $|x| \le A$, $|y| \le B$.

The variational problem associated with this boundary value problem is (a well known result)

$$I(\Psi) = \iint_R \left(\Psi_x^2 + \Psi_y^2 - 4\Psi\right) dx\, dy = \text{min.} \qquad \text{...(2)}$$

An approximate to its solution is

$$\Psi_1 = a_1(x)\,(y^2 - B^2), \qquad \text{...(3)}$$

with

$$\phi_1(x, y) = y^2 - B^2. \qquad \text{...(4)}$$

Here $\phi_1(x, y)$ vanishes on the part $y = \pm B$ of the boundary C. In order that

$$a_1(x)\,(y^2 - B^2) = 0 \quad \text{on } C, \qquad \text{...(5)}$$

we shall determine $a_1(x)$ such that

$$a_1(A) = a_1(-A) = 0. \qquad \text{...(6)}$$

Inserting the value of Ψ from (3) in (2), we get

$$I(a_1) = \iint \left[\{a_1'(x)\}^2 (y^2 - B^2)^2 + 4y^2 \{a_1(x)\}^2 - 4a_1(x)(y^2 - B^2) \right] dx\, dy$$

$$= \int_{-A}^{A} \left[\{a_1'(x)\}^2 \int_{-B}^{B} (y^4 + B^4 - 2B^2 y^2)\, dy + 4\{a_1(x)\}^2 \right.$$

$$\left. \times \int_{-B}^{B} y^2 dy - 4a_1(x) \int_{-B}^{B} (y^2 - B^2)\, dy \right] dx$$

$$= \int_{-A}^{A} \left[(a_1')^2 \left[\frac{y^5}{5} + B^4 y - \frac{2B^2 y^3}{3} \right]_{-B}^{B} \right.$$

$$\left. + 4a_1^2 \left\{ \frac{y^3}{3} \right\}_{-B}^{B} - 4a_1 \left\{ \frac{y^3}{3} - B^2 y \right\}_{-B}^{B} \right] dx$$

$$= \int_{-A}^{A} \left[2 (a_1')^2 \left(\frac{B^5}{5} + \frac{B^5}{1} - 2\frac{B^5}{3} \right) + 4a_1^2 \left(2\frac{B^3}{3} \right) - 8a_1 \left(\frac{B^3}{3} - B^3 \right) \right] dx$$

$$= \left(\frac{2B^3}{3} \right) \int_{-A}^{A} \left[\frac{8}{5} B^2 (a_1')^2 + 4a_1^2 + 8a_1 \right] dx \ . \qquad ...(7)$$

Here

$$f\,(a_1, a_1', x) = \frac{8}{5} B^2 (a_1')^2 + 4a_1^2 + 8a_1 . \qquad ...(8)$$

$$f_{a_1} = 8a_1 + 8,$$

$$f_{a_1'} = \frac{16}{5} B^2 a_1' . \qquad ...(9)$$

Euler's equation becomes

$$(8a_1 + 8) - \frac{d}{dx} \left(\frac{16}{5} B^2 a_1' \right) = 0$$

$$a_1''(x) - \frac{5}{2B^2} a_1(x) - \frac{5}{2B^2} = 0 . \qquad ...(10)$$

Its solution is

$$a_1(x) = C_1 \cosh . \left(\frac{kx}{B} \right) + C_2 \sinh \left(\frac{kx}{B} \right) - 1, \qquad ...(11)$$

with

$$k = \sqrt{\frac{5}{2}} . \qquad ...(12)$$

The function $a_1(x)$ must be an even function, therefore $C_2 = 0$.

Putting $a_1(A) = 0$, we get,

5 $$C_1 = \frac{1}{\cosh(kx/B)} . \qquad ...(13)$$

Thus, we find

$$\Psi_1 = (y^2 - B^2)\left[\frac{\cosh(kx/B)}{\cosh(kA/B} - 1\right], \qquad ...(14)$$

as an approximate solution obtained by Kantorvich method.

Chapter-9

Waves in Elastic Solids

9.1. WAVES IN AN ISOTROPIC ELASTIC SOLID

In the absence of body force , equations of motion are

$$\tau_{ij,j} = \rho\ \ddot{u}_i \qquad (1)$$

for i , j = 1 , 2 , 3. Here , dot signifies the differentiation with respect to time t and ρ is the density of the solid. τ_{ij} is the stress tensor , u_i is the displacement vector.

The left of (1) is a force in the x_i-direction due to stresses and the right of (1) is the inertia term – mass $\times$ acceleration.

We know that the generalized Hooke's law for an isotropic homogeneous elastic medium is

$$\tau_{ij} = \lambda\, \delta_{ij}\, u_{k,k} + \mu\, (u_{i,j} + u_{j,i}) . \qquad (2)$$

Here , homogeneity implies that ρ , λ and μ are constants throughout the medium , λ and μ being Lame′ constants.

We put

$$\theta = u_{k,k} = \text{div}\ \bar{u} , \qquad (3)$$

$$\bar{\Omega} = \text{curl } \bar{u} . \tag{4}$$

θ is the cubical dilatation and $\bar{\Omega}$ is the rotation vector.

Putting (2) into (1) and using (3) , we obtain the following Navier's equation of motion (exercise)

$$(\lambda + \mu) \text{ grad}\theta + \mu \nabla^2 \bar{u} = \rho \ddot{\bar{u}} , \tag{5}$$

in which ∇^2 is the Laplacian.

Taking divergence of both sides of (5) and using (3) , we obtain

$$(\lambda + \mu) \text{ div}(\text{grad } \theta) + \mu \nabla^2(\theta) = \rho \ddot{\theta}$$

or
$$(\lambda + \mu) \nabla^2 \theta + \mu \nabla^2 \theta = \rho \ddot{\theta}$$

or
$$\nabla^2 \theta = \frac{1}{\alpha^2} \frac{\partial^2 \theta}{\partial t^2} . \tag{6}$$

where

$$\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}} = \sqrt{\frac{k + \frac{4}{3}\mu}{\rho}} . \tag{7}$$

It shows that the changes in the cubical dilation θ propagates through the elastic isotropic solid with speed α .

Here , k is the modulus of compressibility.

Taking curl of (5) both sides , we write

$$(\lambda + \mu) \; curl \; (grad \; \theta) + \mu \nabla^2 \; (curl \; \bar{u}) = \rho \frac{\partial^2}{\partial t^2}(curl \bar{u}) \tag{8}$$

From , vector calculus , we have the identity

$$\text{curl grad } \phi \equiv 0 . \tag{9}$$

Using (4) and (9) , equation (8) gives

$$\nabla^2 \bar{\Omega} = \frac{1}{\beta^2} \frac{\partial^2 \bar{\Omega}}{\partial t^2} , \tag{10}$$

where

$$\beta = \sqrt{\frac{\mu}{\rho}} \; . \tag{11}$$

Equation (10) shows that changes in rotation $\bar{\Omega}$ propagates with speed β.

Remark 1: The speed of both the waves depend upon the elastic parameters λ, μ and the density ρ of the medium. Since $\lambda > 0$ & $\mu > 0$, it can be seen that

$$\alpha > \beta.$$

That is , the dilatational waves propagate faster than the shear waves.

Therefore , dilatational waves **arrive first while** rotational waves arrive after that **on a seismogram.** For this reason , dilatational waves are also called primary waves and rotational waves are called secondary waves.

Remark 2: For a Poisson's solid ($\lambda = \mu$) , we have

$$\alpha = \sqrt{3}\,\beta.$$

For most solids , particularly rocks in Earth , there is a small difference between λ & μ. So, we may take $\lambda = \mu$ and solid is then called a Poisson's solid.

Remark 3: In seismology , the dilatational waves are called **P-waves** and rotational waves are denoted by **S – waves.**

Remark 4: If $\mu = 0$, then $\beta = 0$. That is , there is no S – wave in a medium with zero rigidity.

That is , **in liquids , S – waves cann't exist**. However , P – waves exists in a liquid medium.

Remark 5: Since

$$\text{div } \bar{\Omega} = \text{div}(\text{curl } \bar{u}) \equiv 0 \, ,$$

it follows that a rotational wave is free of expansion/compression of volume.

For this reason , the rotational wave $\bar{\Omega}$ is also called **equivoluminal/dilatationless.**

Remark 6: The dilatational wave ($\theta \neq 0$) causes a change in volume of the material elements in the body. Rotational wave (when $\bar{\Omega} \neq 0$) produces a change in shape of the material element without changes in the volume of material elements.

Rotational waves are also referred as shear waves or a wave of distortion.

Remark 7: For a typical metal like **copper** , the speeds of primary and secondary and waves are estimated **as**

$$\alpha = \mathbf{4.36km/sec} \text{ and } \beta = \mathbf{2.13km/sec} \ ,$$

respectively.

Remark 8: At points far away from initially disturbed region , **the waves are plane waves**.

This suits seismology because the recording station of a disturbance during an earthquake is placed at a great distance in comparison to the dimensions of initial source.

Remark 9: We note that equations (6) and (10) both are forms of wave equations. Equation (6) shows that a disturbance θ , **called dilatation wave/compressive wave** propagates through the elastic medium with velocity α. Similarly , equation (10) shows that a disturbance $\overline{\Omega}$, **called a rotational wave** , propagates through the elastic medium with velocity β.

Thus, we conclude that any disturbance in an infinite homogeneous isotropic elastic medium can be propagated **in the form of two types of** these waves.

The speed α depends upon rigidity μ and modulus of compressibility k. On the other hand , β depends upon rigidity μ only.

Helmholtz's Theorem (P and S wave of Seismology)

Any vector point function $\overline{F}$ which is finite, uniform and continuous and which vanishes at infinity, may be expressed as the sum of a gradient of a scalar function ϕ and curl of a zero divergence vector $\overline{\psi}$.

The function ϕ is called the scalar potential of $\overline{F}$ and $\overline{\psi}$ is called the vector potential of $\overline{F}$.

The equation of motion for an elastic isotropic solid with density ρ (for zero body force) is

$$(\lambda + 2\mu)\ \nabla(\nabla.\ \overline{u}) - \mu\ \nabla \times \nabla \times\ \overline{u} = \rho\ \frac{\partial^2 \overline{u}}{\partial t^2}\ . \quad (1)$$

We write the decomposition for the displacement vector u in the form

$$\mathbf{u} = \nabla\ \phi + \nabla \times\ \overline{\psi}\ , \qquad (2)$$

with

$$\text{div } \bar{\psi} = 0 \tag{3}$$

The scalar point function ϕ and vector point function $\bar{\psi}$ are called Lame′ potentials.

We know the following vector identities.

$$\nabla \times \nabla \phi \equiv \bar{0} \quad , \tag{4}$$

$$\nabla \,.\, (\nabla \times \bar{\psi}) \equiv 0 \,. \tag{5}$$

Now $\boldsymbol{\nabla \,.\, \bar{u} = \nabla.(\nabla \phi + \nabla \times \bar{\psi})}$

$$= \nabla \,.\, \nabla\phi$$

$$= \nabla^2\phi \tag{6}$$

and

$$\nabla \times \bar{u} = \nabla \times(\nabla \phi + \nabla \times \bar{\psi})$$

$$= \nabla \times \nabla \times \bar{\psi}$$

$$= \text{grad (div } \bar{\psi}) - \nabla^2 \bar{\psi}$$

$$= -\nabla^2 \bar{\psi} \,. \tag{7}$$

Using (6) and (7) in (1) , we write

$$(\lambda + 2\mu)\, \nabla\{\nabla^2 \phi\} + \mu\, \nabla \times\{\nabla^2 \bar{\psi}\} = \rho \frac{\partial^2}{\partial t^2}(\nabla \phi) + \rho \frac{\partial^2}{\partial t^2}(\nabla\times \bar{\psi})$$

or

$$(\lambda + 2\mu)\, \nabla \left[\nabla^2\phi - \left(\frac{\rho}{\lambda+2\mu}\right)\frac{\partial^2\phi}{\partial t^2}\right] + \mu\nabla \times\{\left[\nabla^2\bar{\psi} - \left(\frac{\rho}{\mu}\right)\frac{\partial^2\bar{\psi}}{\partial t^2}\right] = \bar{0}\,, \tag{8}$$

which is satisfied if we take ϕ and Ψ to be solutions of wave equations

$$\nabla^2 \phi = \frac{1}{\alpha^2}\;\frac{\partial^2\phi}{\partial t^2}\;, \tag{9}$$

$$\nabla^2 \;\bar{\psi} = \frac{1}{\beta^2}\;\frac{\partial^2\bar{\psi}}{\partial t^2}\;, \tag{10}$$

where

$$\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}} \quad , \qquad (11)$$

$$\beta = \sqrt{\frac{\mu}{\rho}} \ . \qquad (12)$$

Wave equation (9) is a **Scalar wave equation** and equation (10) is a **vector wave equation**.

Waves represented by (9) are P-waves while waves represented by (10) are S-waves.

ϕ and $\bar{\psi}$ are now called the **scalar and vector potentials** associated with P- and S-waves , respectively.

Note : We can write the displacement vector $\bar{u}$ as

$$\bar{u} = \bar{u}_P + \bar{u}_S \ , \qquad (1)$$

where

$$\bar{u}_P = \nabla \phi \quad , \qquad (2)$$

$$\bar{u}_S = \text{curl} \ \bar{\psi} \ . \qquad (3)$$

Hence $\bar{u}_P$ is the displacement due to P-wave alone and $\bar{u}_S$ is the displacement due to S-wave alone.

One – Dimensional Waves

(a) P-waves : Consider the solution where $\bar{\psi} = 0$. Then the one-dimensional P-wave satisfies the equation

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial^2 \phi}{\partial t^2} \quad , \qquad \alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}} \ , \qquad (1)$$

whose solution is

$$\phi = \phi(x \pm \alpha t) \quad , \qquad (2)$$

representing a wave in x –direction. The corresponding displacement vector is

$$\bar{u}_P = \nabla \phi$$

$$= \left(\frac{\partial \phi}{\partial x}\right)\hat{e}_1 \ . \qquad (3)$$

That is , the displacement vector $\bar{u}_P$ is in the direction of propagation of P-wave. Therefore , P-waves are **longitudinal waves**.

Since

$$\text{curl } \bar{u}_P = \text{curl}(\text{grad } \phi)$$

$$= 0 \quad , \qquad (4)$$

So , P-waves are **irrotational/rotationless**. P-waves do not cause any rotation of the material particles of the medium.

Since

$$\text{div } \bar{u}_P = \text{div}(\text{grad } \phi)$$

$$= \nabla^2 \phi$$

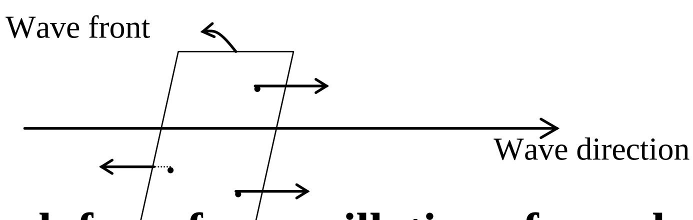
$$\neq 0 \quad , \qquad (5)$$

therefore, P-waves are dilatational/compressional.

In this case , wavefronts are planes ,

$$x = \text{constant} .$$

The particle motion , for P-waves , is perpendicular to the wave fronts(figure).



If we look for a free oscillation of angular velocity ω , then we take

$$\phi(x , t) = e^{i\omega t} \phi(x) , \qquad (6)$$

where

$$\frac{d^2\phi}{dx^2} + h^2 \phi = 0. \qquad (7)$$

Here , $$h = \frac{\omega}{\alpha} , \qquad (8)$$

is the wave number.

(b) S-waves : The one-dimensional wave equation for S-wave is

$$\frac{\partial^2 \overline{\psi}}{\partial x^2} = \frac{1}{\beta^2}\frac{\partial^2 \overline{\psi}}{\partial t^2} \quad , \tag{1}$$

where

$$\beta = \sqrt{\frac{\mu}{\rho}} \; . \tag{2}$$

The corresponding displacement vector is

$$\overline{u}_S = \text{curl } \overline{\psi} \; . \tag{3}$$

Solution of (1) is of the type

$$\overline{\psi}\,(x, t) = \psi_1(x - \beta t)\,\hat{e}_1 + \psi_2(x - \beta t)\,\hat{e}_2 + \psi_3(x - \beta t)\,\hat{e}_3 \quad . \tag{4}$$

From (3) and (4) , we find

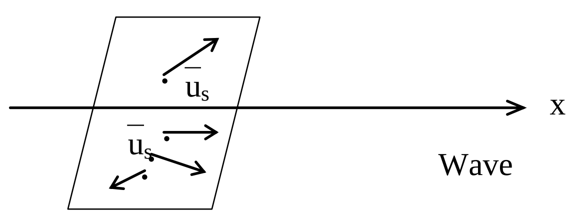
$$\overline{u}_S = -\frac{\partial \psi_3}{\partial x}\,\hat{e}_2 + \frac{\partial \psi_2}{\partial x}\,\hat{e}_3 \, . \tag{5}$$

This shows that the displacement vector $\overline{u}_S$ lies in a plane parallel to yz-plane, which is perpendicular to the x-direction , representing the direction of S-wave propagation.

The wavefronts are planes having x-axis their normals.

Thus, particle motion due to a S-wave is parallel to the wavefronts.

So, S-wave are **transverse waves** (figure).



Since

$$\text{div } \overline{u}_S = \text{div(curl } \overline{\psi}\,)$$

$$= 0 \quad , \tag{6}$$

so, S-waves are **dilationless/equivoluminal**.

Since

$$\text{curl } \bar{u}_s = \text{curl curl } \bar{\psi}_s$$
$$= -\nabla^2 \bar{\psi}$$
$$\neq 0 \quad , \qquad (7)$$

so , S-waves are not irrotational , they are **rotational**.

As strains are not zero , S-waves are **shear/distortional** waves.

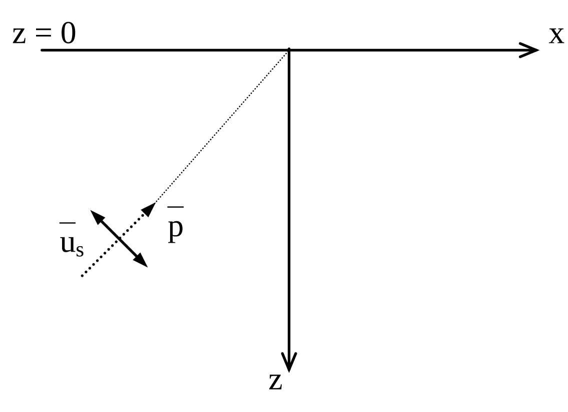
Each component ψ_i satisfy the scalar wave equation for S-waves.

9.2. SV- AND SH-WAVES

We shall now consider the study of waves as related to Earth , for example , waves generated by earthquakes. We consider the surface of the earth (taken as plane , approximately) as horizontal. Let z-axis be taken vertically downwards and xy-plane as horizontal.

To determine x- and y-directions in the horizontal plane, we proceed as below.

Let $\bar{p}$ be the direction of propagation of a S-wave. Let the plane made by the vector $\bar{p}$ and the z-axis be the xz-plane. Then x-axis lies in the horizontal plane (z = 0) bounding the earth and the propagation vector $\bar{p}$ lies in the vertical xz-plane (figure).



(A S-wave)

We choose y-axis as the direction perpendicular to the xz-plane so that x- , y- and z-axis form a right handed system.

The displacement vector $\bar{u}_S$ corresponding to a S-wave propagating in the $\bar{p}$-direction , is perpendicular to $\bar{p}$-direction.

We resolve $\bar{u}_s$ into two components – the first component $\bar{u}_{SV}$ lying in the vertical xz-plane and the second component $\bar{u}_{SH}$ parallel to y-axis(i.e. $\perp$ to xz-plane).

A S-wave representing the motion corresponding to the first component of the displacement vector **is known as a SV-wave**.

For a SV-wave , the particle motion is perpendicular to the direction $\bar{p}$ of propagation of wave and lies in the vertical xz-plane which is normal to the horizontal bounding surface. Let $\bar{u}_{SV}$ denote the corresponding displacement vector. Then , we write

$$\bar{u}_{SV} = (u , 0 , w) ,$$

contains both horizontal component u in the x-direction and vertical component w in the z-direction. A **SV-wave is a vertically polarized shear wave**. SV stands for **vertical shear**.

A S-wave representing the second displacement component , parallel to the y-axis , lying in the horizontal plane , **is known as SH-wave**. Let $\bar{u}_{SH}$ denote the corresponding displacement. Then

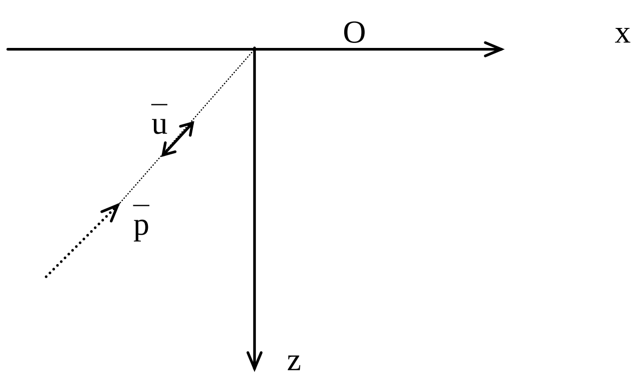
$$\bar{u}_{SH} = (0 , v, 0) ,$$

is parallel to y-axis. The motion due to a SH-wave is perpendicular to $\bar{p}$-direction (i.e., in a transverse direction) and along a horizontal direction. A SH-wave is a horizontally polarized shear wave. SH stands for horizontal shear.

When a P-wave propagates in the $\bar{p}$-direction , then the corresponding displacement , denoted by $\bar{u}_P$, is given by

$$\bar{u}_P = (u , 0 , w) .$$

The displacement $\bar{u}_P$ contains both horizontal component in the x-direction and vertical component in the z-direction(figure)



(A P-wave)

9.3. WAVE PROPAGATION IN TWO – DIMENSIONS

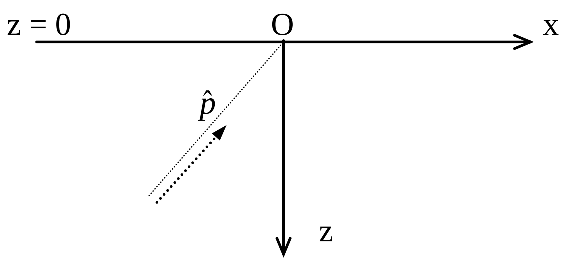
We assume that waves are propagating in planes parallel to the vertical xz-plane containing the wave propagation vector

$$\hat{p} = l\,\hat{e}_1 + n\,\hat{e}_2$$

with

$$l^2 + n^2 = 1.$$

Then the wave motion will be the same in all planes parallel to xz-plane (figure) and independent of y so that $\frac{\partial}{\partial y} \equiv 0$.



Under this assumption , the Navier equation of motion for isotropic elastic materials

$$(\lambda + \mu) \text{ grad div } \bar{u} + \mu \nabla^2 \bar{u} = \rho \ddot{\bar{u}} \qquad (1)$$

gives

$$(\lambda + \mu) \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) + \mu \nabla^2 u = \rho \ddot{u} , \qquad (2)$$

$$(\lambda + \mu) \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \mu \nabla^2 w = \rho \ddot{w} \qquad (3)$$

$$\mu \nabla^2 v = \rho \ddot{v} \quad , \qquad (4)$$

where the displacement vector $\bar{u} = (u, v, w)$ is independent of y ,

$$\text{cubical dilatation} \equiv \theta = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} , \qquad (5)$$

and Laplacian is given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} . \qquad (6)$$

Let z = 0 be the boundary surface. The components of the stress acting on the surface z = 0 are given by

$$\tau_{zx} = 2\mu e_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) , \qquad (7)$$

$$\tau_{zz} = \lambda\,\theta + 2\mu\, e_{zz} = (\lambda + 2\mu)\frac{\partial w}{\partial z} + \lambda \frac{\partial u}{\partial x} , \qquad (8)$$

$$\tau_{zy} = 2\mu\, e_{zy} = \mu \frac{\partial v}{\partial z} . \qquad (9)$$

From above equations , we conclude that the general two – dimensional problem of propagation of plane elastic waves , parallel to the xz-plane, splits into **two independent problems** stated below –

Problem I – Consisting of equations (4) and (9).

Problem II – Consisting of equations (2) , (3) (7) and (8).

Problem I (SH-problem) : In this problem the displacement components are

$$u = w \equiv 0 , \quad , v = v(x , z , t) \qquad (10)$$

and

$$\tau_{zx} = \tau_{zz} = 0 \quad , \tau_{zy} = \tau_{zy}(x , z, t) = \mu \frac{\partial v}{\partial z} . \qquad (11)$$

In this problem , the displacement component v(x , z, t) satisfies the scalar wave equation

$$\nabla^2 v = \frac{1}{\beta^2} \frac{\partial^2 v}{\partial t^2} . \qquad (12)$$

This differential equation is independent of modulus of compressibility k. So the motion due to such waves is equivoluminal. Since the displacement component v is horizontal and is perpendicular to the direction $\hat{p}$ of propagation ($\hat{p}$ lies in xz-plane) , the waves represented by v are **horizontally polarized waves or SH –waves**.

Problem II (P-SV problem) : In this problem

$$v \equiv 0 \ , \ u = u(x, z, t) \ , \ w = w(x, z, t) \ , \qquad (13)$$

$$\tau_{zy} \equiv 0 \ , \ \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) , \ \ \tau_{zz} = (\lambda + 2\mu)\ \frac{\partial w}{\partial z} + \lambda \frac{\partial u}{\partial x} \ ,$$

$$\tau_{xx} = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) + 2\mu \ \frac{\partial u}{\partial x} \ . \qquad (14)$$

In this problem , the in-plane displacement components u and w satisfying the two simultaneous partial differential equations (2) and (3) , and the in-plane stress components τ_{xx} , τ_{zx} and τ_{zz} also contain these two displacement components. The displacement component v plays no role in the solution of this problem and hence taken as identically zero.

The displacement components *u* and *w* can be expressed in terms of two scalar potentials

$$\phi = \phi(x , z , t) \text{ and } \psi = \psi(x, z, t)$$

through the relations

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial z} \ , \qquad (15)$$

$$w = \frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial x} \ , \qquad (16)$$

with help of Helmholtz's theorem (on taking $\bar{\psi} = -\psi \ \hat{e}_2$).

Using (15) and (16) in the equations of motion (2) and (3) , we obtain

$$\frac{\partial}{\partial x}\left[\alpha^2 \nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2}\right] + \frac{\partial}{\partial z}\left[\beta^2 \nabla^2 \psi - \frac{\partial^2 \psi}{\partial t^2}\right] = 0 \ , \qquad (17)$$

$$\frac{\partial}{\partial z}\left[\alpha^2 \nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2}\right] - \frac{\partial}{\partial x}\left[\beta^2 \nabla^2 \psi - \frac{\partial^2 \psi}{\partial t^2}\right] = 0 \ . \qquad (18)$$

These equations are identically satisfied when the scalar potentials ϕ and ψ are solutions of following scalar wave equations

$$\nabla^2 \phi = \frac{1}{\alpha^2} \frac{\partial^2 \phi}{\partial t^2} \ , \qquad (19)$$

$$\nabla^2 \psi = \frac{1}{\beta^2} \frac{\partial^2 \psi}{\partial t^2} \ , \qquad (20)$$

where $$\alpha = \sqrt{\frac{(\lambda+2\mu)}{\rho}} \quad , \quad \beta = \sqrt{\frac{\mu}{\rho}}\,. \tag{21}$$

The plane wave solutions of (19) represents P-waves and those of (20) represent S-waves.

The potentials ϕ and ψ are **called displacement potentials**.

In the displacements (15) and (16) , the contribution from ϕ is due to P-waves and that from ψ is due S waves.

P-SV wave is a combination of P-wave and the SV-wave. The displacement vector (u , 0, w) lies in a vertical plane and S-waves represented by the potential ψ are also propagated in a vertical plane , so these S-waves are vertically polarized shear waves or simple SV-waves.

The stress τ_{zx} and τ_{zz} in terms of potentials ϕ and ψ are

$$\tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$= \mu\left(2\frac{\partial^2\phi}{\partial x\,\partial z} + \frac{\partial^2\psi}{\partial z^2} - \frac{\partial^2\psi}{\partial x^2}\right), \tag{22}$$

$$\tau_{zz} = \lambda \operatorname{div} \bar{u} + 2\mu \,.\, \frac{\partial w}{\partial z}$$

$$= \lambda\, \nabla^2 \phi + 2\mu\left(\frac{\partial^2\phi}{\partial z^2} - \frac{\partial^2\psi}{\partial x\,\partial z}\right)$$

$$= \left(\lambda\nabla^2 + 2\mu\frac{\partial^2}{\partial z^2}\right)\phi - 2\mu\, \frac{\partial^2\psi}{\partial x\,\partial z}\;. \tag{23}$$

Since ,

$$\frac{\lambda}{\mu} = \frac{\lambda+2\mu-2\mu}{\mu} = \frac{\lambda+2\mu}{\mu} - 2 = \frac{\alpha^2}{\beta^2} - 2\,, \tag{24}$$

therefore , we write

$$\tau_{zz} = \mu\left[\left(\frac{\alpha^2}{\beta^2} - 2\right)\nabla^2\phi + 2\frac{\partial^2\phi}{\partial z^2} - 2\frac{\partial^2\psi}{\partial x\,\partial z}\right]. \tag{25}$$

Note 1: In the SH-problem , other stresses are

$$\tau_{xx} = \tau_{yy} = 0\ , \quad \tau_{xy} = \mu \frac{\partial v}{\partial x}\ . \tag{26}$$

Note 2: In the P-SV problem , other stresses are (in terms of ϕ an ψ)

$$\tau_{xx} = \lambda \nabla^2 \phi + 2\mu\, e_{xx}$$

$$= \lambda\, \nabla^2 \phi + 2\mu \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x\, \partial z} \right), \tag{27}$$

$$\tau_{yy} = \lambda\, \nabla^2 \phi + 2\mu\, e_{yy}$$

$$= \lambda\, \nabla^2 \phi \quad , \tag{28}$$

$$\tau_{xy} = 2\mu\, e_{xy} = 0\ . \tag{29}$$

Note 3: In the case of two – dimensional wave propagation , the SH motion is decoupled from the P-SV motion. The displacement vector due to P-SV type motion is

$$\bar{u} = \nabla \phi + \nabla \times (\psi\, \hat{e}_2)$$

$$= (u, o, w)\ .$$

9.4. PLANE WAVES

A geometric surface of all points in space over which the phase of a wave is constant is called a wavefronts.

Wavefronts can have many shapes. For example , wavefronts can be planes or spheres or cylinders.

A line normal to the wave fronts , indicating the direction of motion of wave, is called a ray.

If the waves are propagated in a single direction , the waves are called **plane waves**, and the wavefronts for plane waves are parallel planes with normal along the direction of propagation of the wave.

Thus, a plane wave is a solution of the wave equation in which the disturbance/displacement varies only in the direction of wave propagation and is constant in all the directions orthogonal to propagation direction.

The rays for plane waves are parallel straight lines.

For spherical waves , the disturbance is propagated out in all directions from a point source of waves. The wavefronts are concentric spheres and the rays are radial lines leaving the point source in all directions.

Since seismic energy is usually radiated from localized sources , seismic wavefronts are always curved to some extent. However , at sufficiently large distances from the source , the wavefronts become flat enough that a plane wave approximation become locally valid.

Illustration (1) : For one –dimensional wave equation

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} , \qquad (1)$$

a progressive wave travelling with speed c in the positive x-direction is represented by

$$\phi(x, t) = A\, e^{ik(x-ct)} = A\, e^{i(kx-wt)} , \qquad (2)$$

where

k = wave number ,

ω = c k = angular frequency ,

λ = wavelength = $2\pi/k$,

A = amplitude of the wave.

Let

$$\bar{x} = x\ \hat{e}_1 \quad , \quad \bar{k} = k\ \hat{e}_1 . \qquad (3)$$

Then $\bar{k}$ is called the propagation vector and (2) can be written in the form

$$\phi(x, t) = A\, e^{i(\bar{k}.\bar{x} - \omega t)} . \qquad (4)$$

In this type of waves , wavefronts are planes

$$x = \text{constant} , \qquad (5)$$

which are parallel to yz – plane.

Illustration 2: A two – dimensional wave equation with speed c is

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} . \qquad (1)$$

If we take

$$u = l\,x + m\,y - c\,t\,, \qquad (2)$$

$$v = l\,x + m\,y + c\,t\,, \qquad (3)$$

where l , m are constants , then equation (1) is reduced to (exercise)

$$\frac{\partial^2 \phi}{\partial u\, \partial v} = 0 \quad , \qquad (4)$$

provided

$$l^2 + m^2 = 1. \qquad (5)$$

Integrating (4) , we get

$$\phi = f(u) + g(v)$$

$$= f(lx + my - ct) + g(lx + my + ct) \quad , \qquad (6)$$

where f and g are arbitrary functions.

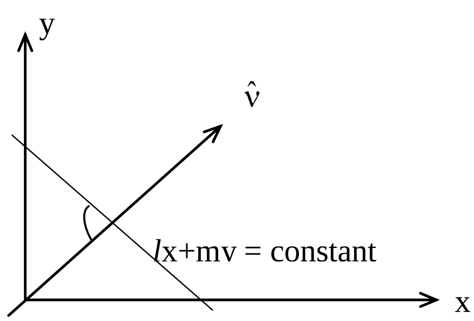
Let

$$\hat{v} = l\,\hat{e}_1 + m\,\hat{e}_2 \quad . \qquad (7)$$

Then $\hat{v}$ is a unit vector perpendicular to the system of straight lines

$$lx + my = \text{constant}\,, \qquad (8)$$

in two-dimensional xy-plane.



At any instant , say t = t_0 , the disturbance ϕ is constant for all points (x, y) lying on the line (8). Therefore , ϕ represents a plane propagating with speed c in the direction $\hat{v}$. The wavefronts are straight lines given by equation (8).

Illustration 3: Three-dimensional wave equation is

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} , \qquad (1)$$

which propagates with speed c.

As discussed earlier , solution of (1) is

$$\phi(x, y, z, t) = f(lx + my + nz - ct) + g(lx + my + nz + ct) \qquad (2)$$

where the constants l, m , n satisfy the relation

$$l^2 + m^2 + n^2 = 1 . \qquad (3)$$

Let

$$\hat{v} = l\hat{e}_1 + m\ \hat{e}_2 + n\ \hat{e}_3 . \qquad (4)$$

Then $\hat{v}$ is a unit vector which is normal to the system of parallel planes

$$lx + my + nz = \text{constant}. \qquad (5)$$

The wavefronts are parallel planes given by equation (5) , which travel with speed c in the direction $\hat{v}$. So, the wave (2) is a three-dimensional progressive plane wave.

Question : Determine the wavelength and velocity of a system of plane waves given by

$$\phi = a \sin (A x + B y + C z - D t) ,$$

where a , A, B , C , D are constants.

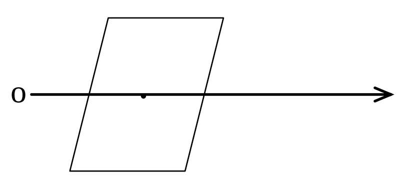
Solution : Let < l , m , n> be the direction cosines of the direction of wave propagation with speed c. Then

$$l^2 + m^2 + n^2 = 1 \qquad (1)$$

Equation of a wave front is

$$l x + m y + n z = p \qquad (2)$$

where p is the length of perpendicular from the origin to the wave front.



< l , m , n >

The plane wave ϕ, therefore , must be of the type

$$\phi = a \sin k\,(l\,x + m\,y + n\,z - c\,t)\,, \qquad (3)$$

where k = wave number and ,

c = wave velocity.

Comparing (3) with the given form

$$\phi = a \sin (A\,x + B\,y + C\,z - D\,t)\,, \qquad (4)$$

we find

$$l = \frac{A}{\sqrt{A^2+B^2+C^2}}\,, \quad m = \frac{B}{\sqrt{A^2+B^2+C^2}}\,, \quad n = \frac{C}{\sqrt{A^2+B^2+C^2}}\,, \qquad (5)$$

and

$$k = \sqrt{A^2+B^2+C^2}\,, \qquad (6)$$

$$c = \frac{D}{\sqrt{A^2+B^2+C^2}}\,. \qquad (7)$$

Therefore ,

$$\text{wave length} = \frac{2\pi}{k} = \frac{2\pi}{\sqrt{A^2+B^2+C^2}}\,, \qquad (8)$$

and

$$\text{wave velocity} = \frac{D}{\sqrt{A^2+B^2+C^2}}\,. \qquad (9).$$

Chapter-10

Surface Waves

10.1. INTRODUCTION

In an elastic body, it is possible to have another type of waves (other than body waves) which are propagated over the surface and penetrate only a little into the interior of the body.

Such waves are similar to waves produced on a **smooth surface of water when** a stone is thrown into it.

These type of waves are **called SURFACE WAVES.**

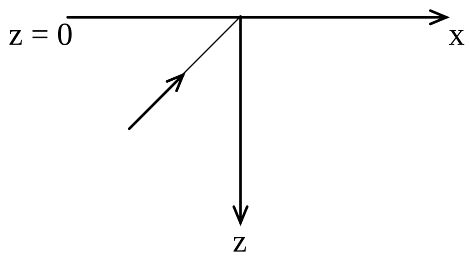
Surface waves are "tied" to the surface and diminish exponentially as they get farther from the surface.

The criterion for surface waves is that the **amplitude** of the displacement in the medium **dies exponentially with the increasing distance from the surface.**

In seismology , the interfaces are , in the ideal case , horizontal and so the plane of incidence is **vertical**. Activity of surface waves is restricted to the neighbourhood of the interface(s) or surface of the medium.

Under certain conditions , such waves can propagated independently along the surface/interface. For surface waves , the disturbance is confined to a depth **equal to a few wavelengths**.

Let us take xz – plane as the plane of incidence with z – axis vertically downwards. Let z = 0 be the surface of a semi-infinite elastic medium (Figure) .



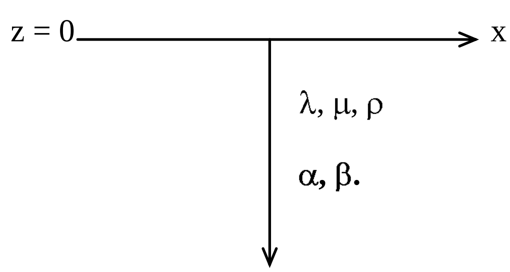
For a surface wave , its amplitude will be a function of z (rather than an exponential function) which tends to zero as $z \to \infty$. For such surface – waves , the motion will be two – dimensional **,** parallel to xz – plane , so that $\frac{\partial}{\partial y} = 0$.

The existence of surface waves raises the question of whether they might (under certain conditions) be able to travel freely along the plane (horizontal) as a guided wave.

10.2. RAYLEIGH WAVES

Rayleigh (1885) discussed the existence of a simplest surface wave propagating on the free – surface of a homogeneous isotropic elastic half – space.

Let the half – space occupies the region $z \geq 0$ with z – axis taken as vertically downwards. Let ρ be the density and λ , μ be the Lame′ elastic moduli and z = 0 be the stress – free boundary of the half – space (figure).



z

Suppose that a train of plane waves is propagating in the media in the positive x – direction such that

(i) the plane of incidence is the vertical plane (xz – plane) so that the motion , i.e. , disturbance is independent of y and hence $\frac{\partial}{\partial y} \equiv 0$.

(ii) the amplitude of the surface wave decreases exponentially as we move away from the surface in the z – direction.

This problem is a **plane strain problem** and the displacement vector $\bar{u}$ is of the type

$$\bar{u} = (u , 0 , w) , \quad u = u(x , z, t) , \quad w = w(x , z , t) . \quad (1)$$

The displacement components u and w are given in term of potential ϕ and ψ through the relations

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial z} , \quad (2)$$

$$w = \frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial x} , \quad (3)$$

where potentials ϕ and ψ satisfy the scalar wave equations

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{\alpha^2} \frac{\partial^2 \phi}{\partial t^2} , \quad (4)$$

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{\beta^2} \frac{\partial^2 \psi}{\partial t^2} , \quad (5)$$

and wave velocities α and β are given by

$$\alpha = \sqrt{(\lambda + 2\mu)/\rho} , \quad (6)$$

$$\beta = \sqrt{\mu/\rho} . \quad (7)$$

This wave motion is of a P-SV type wave travelling along the stress – free surface of the half space and such a motion takes place in the xz – plane.

Now , we seek solutions of wave equations (4) and (5) of the form

$$\phi(x , z , t) = f(z)\, e^{ik(ct-x)} , \quad (8)$$

$$\psi(x , z , t) = g(z)\, e^{ik(ct-x)} \quad , \qquad (9)$$

where k is the wave number , c is the speed of surface wave travelling in the +ve x-direction and $\omega = ck$ is the angular frequency.

The substitution of (8) and (9) into wave equations (4) and (5) leads to two ordinary differential equations (exercise)

$$\frac{d^2 f}{dz^2} - k^2 a^2 f = 0 \quad , \qquad (10)$$

and

$$\frac{d^2 g}{dz^2} - k^2 b^2 g = 0 \quad , \qquad (11)$$

with

$$a = \sqrt{1 - c^2/\alpha^2} > 0 \quad , \qquad (12)$$

and

$$b = \sqrt{1 - c^2/\beta^2} > 0 \quad . \qquad (13)$$

From equations (8) to (11) ; we find (exercise)

$$\phi (x , z , t) = (A\, e^{-akz} + A_1\, e^{akz})\, e^{ik(ct-x)} \quad , \qquad (14)$$

$$\psi(x , z , t) = (B\, e^{-bkz} + B_1\, e^{bkz})\, e^{ik(ct-x)} \quad , \qquad (15)$$

where A , A_1 B and B_1 are constants.

Since the disturbance due to surface waves must die rapidly as $z \to \infty$, we must have

$$A_1 = B_1 = 0 \, . \qquad (16)$$

Thus , for Rayleigh waves , we get

$$\phi(x , z , t) = A\, e^{-akz}\, e^{i(\omega t - kx)} , \qquad (17)$$

and

$$\psi(x , z , t) = B\, e^{-bkz}\, e^{i(\omega t - kx)} , \qquad (18)$$

with

$$c < \beta < \alpha \, . \qquad (19)$$

The equation (19) gives the condition for the existence of surface waves on the surface of a semi – infinite isotropic elastic media with velocity c in the positive x – direction.

To find frequency equation for the velocity c and ratio of A and B ; we use the stress – free boundary conditions. This conditions are

$$\tau_{zx} = \tau_{zy} = \tau_{zz} = 0 \qquad \text{at } z = 0. \tag{20}$$

From relation in (2) and (3) , we find

$$\text{div } \bar{u} = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$= \nabla^2 \phi \,. \tag{21}$$

The stresses in terms of potentials ϕ and ψ are given by

$$\tau_{zx} = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)$$

$$= \mu \left[2\frac{\partial^2 \phi}{\partial x \partial z} + \frac{\partial \psi}{\partial z^2} - \frac{\partial^2 \psi}{\partial x^2} \right], \tag{22}$$

$$\tau_{zz} = \lambda \text{ div } \bar{u} + \mu \frac{\partial w}{\partial z}$$

$$= \mu \left[\left(\frac{\alpha^2}{\beta^2} - 2 \right) \frac{\partial^2 \phi}{\partial x^2} + \; + \frac{\alpha^2}{\beta^2} \frac{\partial^2 \phi}{\partial z^2} - 2\frac{\partial^2 \psi}{\partial x \partial z} \right], \tag{23}$$

$$\tau_{zy} = \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)$$

$$= 0 \,. \tag{24}$$

The boundary condition $\tau_{zx} = 0$ at z = 0 in (20), gives (exercise)

$$2A(-ik)\,(-ak) - B(-ik)^2 + B(-bk)^2 = 0$$

or $$2\, i\, a\, k^2 A + B(1 + b^2)\, k^2 = 0$$

or $$2\, i\, a\, A = -B(1 + b^2)$$

or $$\frac{A}{B} = \frac{i(1+b^2)}{2a} \,. \tag{25}$$

The boundary condition $\tau_{zz} = 0$ at z = 0 , gives (exercise)

$$\left(\frac{\alpha^2}{\beta^2} - 2\right) . \text{ A } . (-k^2) + \frac{\alpha^2}{\beta^2} . \text{ A } . (\alpha^2 k^2) - 2\text{B } (-ik)\ (-bk) = 0$$

or
$$\text{A}\left[-\frac{\alpha^2}{\beta^2} + 2 + \frac{\alpha^2 a^2}{\beta^2}\right]k^2 - 2 \text{ i b B } k^2 = 0$$

or
$$\text{A}\left(2 - \frac{\alpha^2}{\beta^2} + \frac{\alpha^2 a^2}{\beta^2}\right) = 2 \text{ i b B}$$

or
$$\frac{A}{B} = \left(\frac{2ib}{2 - \frac{\alpha^2}{\beta^2} + \frac{\alpha^2 a^2}{\beta^2}}\right). \qquad (26)$$

Eliminating A/B from equations (25) and (26) and substituting the value of a and b from equation (12) and (13) , we obtain

$$(1 + b^2)\left(2 - \frac{\alpha^2}{\beta^2} + \frac{\alpha^2 a^2}{\beta^2}\right) = 4 \text{ a b}$$

or
$$\left(2 - \frac{c^2}{\beta^2}\right)\left[2 - \frac{\alpha^2}{\beta^2} + \frac{\alpha^2}{\beta^2}\left(1 - \frac{c^2}{\alpha^2}\right)\right]$$

$$= 4\sqrt{1 - \frac{c^2}{\alpha^2}}\ \sqrt{1 - \frac{c^2}{\beta^2}}$$

or
$$\left(2 - \frac{c^2}{\beta^2}\right)\left[2 - \frac{\alpha^2}{\beta^2} + \frac{\alpha^2}{\beta^2} - \frac{c^2}{\beta^2}\right] = 4\sqrt{1 - \frac{c^2}{\alpha^2}}\ \sqrt{1 - \frac{c^2}{\beta^2}}$$

or
$$\left(2 - \frac{c^2}{\beta^2}\right)^2 - 4\left(1 - \frac{c^2}{\alpha^2}\right)^{\frac{1}{2}}\left(1 - \frac{c^2}{\beta^2}\right)^{\frac{1}{2}} = 0 . \qquad (27)$$

This equation contains only one unknown c.

This equation determines the speed c for Rayleigh surface waves in an uniform half – space.

Equation (27) is called the **RAYLEIGH EQUATION** for Rayleigh waves. It is also called the **Rayleigh frequency equation**. It is also called the **Rayleigh wave – velocity equation**.

This equation is the **period equation** for Rayleigh waves.

In order to prove that Rayleigh waves really exists , we must show that frequency equation (27) has at least one real root for c. To show this , we proceed as follows :

From equation (27) , we write

$$\left(2-\frac{c^2}{\beta^2}\right)^4 = 16\left(1-\frac{c^2}{\alpha^2}\right)\left(1-\frac{c^2}{\beta^2}\right)$$

or

$$16 - 32\frac{c^2}{\beta^2} + 24\frac{c^4}{\beta^4} - 8\frac{c^6}{\beta^6} + \frac{c^8}{\beta^8}$$

$$= 16\left[1-\frac{c^2}{\alpha^2}-\frac{c^2}{\beta^2}+\frac{c^4}{\alpha^2\beta^2}\right]$$

$$= 16\left[1-\frac{c^2}{\beta^2}\frac{\beta^2}{\alpha^2}-\frac{c^2}{\beta^2}+\frac{c^4}{\beta^4}.\frac{\beta^2}{\alpha^2}\right]$$

or

$$-32 + 24\frac{c^2}{\beta^2} - 8\frac{c^4}{\beta^4} + \frac{c^6}{\beta^6} + 16\frac{\beta^2}{\alpha^2} + 16 - 16\frac{c^2}{\beta^2}.\frac{\beta^2}{\alpha^2} = 0 . \qquad (28)$$

Put

$$s = c^2/\beta^2 , \qquad (29)$$

then equation (28) gives

$$f(s) = s^3 - 8s^2 + \left(24-16\frac{\beta^2}{\alpha^2}\right)s - 16\left(1-\frac{\beta^2}{\alpha^2}\right) = 0 . \qquad (30)$$

It is polynomial **equation of degree 3 in s** with real coefficients.

Hence the frequency equation (30) has at least one real root since complex roots occur in conjugate pairs.

Moreover

$$f(0) = -16\left(1-\frac{\beta^2}{\alpha^2}\right) < 0 , \qquad (31)$$

and

$$f(1) = 1 - 8 + 24 - \frac{16\beta^2}{\alpha^2} + \frac{16\beta^2}{\alpha^2} = 1 > 0 \,, \quad (32)$$

since $\beta < \alpha$.

Hence , by the intermediate theorem of calculus , the frequency equation (30) has at least one root in the range (0 , 1).

That is , there exists s $\in$ (0 , 1) which is a root of the Rayleigh frequency equation.

Further , $0 < s < 1$ implies

$$0 < \frac{c^2}{\beta^2} < 1$$

or $$\mathbf{0 < c < \beta} \,. \quad (33)$$

This establishes that Rayleigh surface waves propagating with speed $c = c_R$, where

$$0 < c_R < \beta < \alpha \,,$$

always exists.

This proves the existence of Rayleigh surface waves.

Special Case (Poisson's solid) : When the semi-infinite elastic medium is a Poisson's solid ,

then $\lambda = \mu$ and $\sigma = \frac{1}{4}$ and

$$\frac{\alpha^2}{\beta^2} = \frac{\lambda + 2\mu}{\mu} = 3. \quad (34)$$

For this type of elastic medium , frequency equation (30) becomes

$$3s^3 - 24\,s^2 + 56\,s - 32 = 0$$

or $$(s - 4)\,(3s^2 - 12\,s + 8) = 0$$

giving

$$s = 4 \,,\; 2 + \frac{2}{\sqrt{3}} \,,\; 2 - \frac{2}{\sqrt{3}}$$

or
$$\frac{c^2}{\beta^2} = 4\,,\ 2+\frac{2}{\sqrt{3}}\,,\ 2-\frac{2}{\sqrt{3}}\,. \tag{35}$$

But c must satisfy the inequality $0 < c < \beta$, therefore , the only possible value for $c = c_R$ is

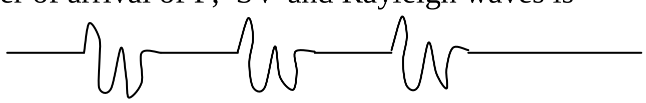
$$\frac{{c_R}^2}{\beta^2} = 2-\frac{2}{\sqrt{3}}\,,$$

or

$$c_R = \left[\sqrt{2-\frac{2}{\sqrt{3}}}\right].\,\beta \cong 0.9195\,\beta\,, \tag{36}$$

giving the speed of propagation of Rayleigh surface waves along the stress free boundary of the Poissonian half-space in the x-direction.

Thus , the order of arrival of P, SV-and Rayleigh waves is



P SV Rayleigh

From equations (12) and (13) ; we write

$$a^2 = 1 - \frac{{c_R}^2}{\alpha^2}$$

$$= 1 - \frac{{c_R}^2}{\beta^2}\,.\,\frac{\beta^2}{\alpha^2}$$

$$= \frac{1}{9}(3+2\sqrt{3})\,,$$

or

$$a = \frac{1}{3}\sqrt{3+2\sqrt{3}} \cong 0.8475 \approx 0.84\,8\,, \tag{37}$$

and

$$b^2 = 1 - \frac{{c_R}^2}{\beta^2}$$

$$= \frac{1}{3}[\,2\sqrt{3} - 3]\,,$$

or $$b \cong 0.3933 \approx 0.393\,. \tag{38}$$

Further , from equations (25) , (37) and (38) , we find

$$\frac{A}{B} = 1.468\, i\quad . \tag{39}$$

Remark 1: Displacements due to Rayleigh waves (Particle motion)

From equations (2) , (3) , (17) , (18) and (26) ; we find

$$u(x\,,z\,,t) = [A(-ik)\,e^{-akz} + B(-bk)\,e^{-bkz}]\,e^{i(\omega t-kx)}$$

$$= k[-i\,A\,e^{-akz} - A\left(\frac{2-c^2/\beta^2}{2i}\right)e^{-bkz}]\,e^{i(\omega t-kx)}$$

$$= -\,i\,k\,A\,[e^{-akz} - \left(1-\frac{c^2}{2\beta^2}\right)e^{-bkz}]\,e^{i(\omega t-kx)}\,. \tag{40}$$

Similarly , we shall find

$$w(x\,,z\,,t) = [(-ak)\,A\,e^{-akz} - (-ik)B\,e^{-bkz}]\,e^{i(\omega t-kx)}$$

$$= k[-\,a\,A\,e^{-akz} + \frac{ie^{bkz}(-2iAa)}{(2-c^2/\beta^2)}]\,e^{i(\omega t-kx)}$$

$$= k\,A\,[\,-a\,e^{-akz} + \left(\frac{a}{1-c^2/2\beta^2}\right)e^{-bkz}\,]\,e^{i(\omega t-kx)}\,. \tag{41}$$

Let $$\theta = \omega t - kx\quad , \tag{42}$$

$$U(z) = e^{-akz} - \left(1-\frac{c^2}{2\beta^2}\right)e^{-bzk}\,, \tag{43}$$

$$W(z) = -a\,e^{-akz} + \left(\frac{a}{1-c^2/2\beta^2}\right)e^{-bkz}\,. \tag{44}$$

Then , taking the real parts of equations (40) and (41) and using equation (42) to (44) ; we find

$$u(x\,,z\,,t) = A\,k\,U(z)\sin\theta\quad , \tag{45}$$

$$w(x\,,z\,,t) = A\,k\,W(z)\cos\theta\quad , \tag{46}$$

we remember that A is the potential amplitude. Eliminating θ from equations (45) and (46) , we obtain

$$\frac{u^2}{[Ak\,U(z)]^2} + \frac{w^2}{[Ak\,W(z)]^2} = 1 \qquad , \qquad (47)$$

which is an equation of an ellipse in the vertical xz-plane.

Equation (47) shows that particles , during the propagation of Rayleigh Surface Waves, describe ellipses.

Remark 2: Particle Motion at the surface (z = 0)

On the surface z = 0 ,we find, at z = 0 ,

$$U(0) = \frac{c^2}{2\beta^2} \quad , \qquad W(0) = a\left(\frac{\frac{c^2}{2\beta^2}}{1-c^2/2\beta^2}\right) \qquad 48)$$

Since $0 < c < \beta$, so

$$U(0) > 0 \quad , \quad W(0) > 0 \, . \qquad (49)$$

Let $a_1 = A\,k\,W(0)$, $b_1 = A\,k\,U(0)$. (50)

Then $a_1 > 0$, $b_1 > 0$, (51)

and equation (47) reduces to

$$\frac{u^2}{b_1^{\,2}} + \frac{w^2}{a_1^{\,2}} = 1 \, . \qquad (52)$$

Remark 3: Particular case (Poisson’s Solid) :

In this case , we find, at z = 0 ,

$$\frac{a_1}{b_1} = \frac{W(0)}{U(0)} = \sqrt{3}\,a \cong 1.5 \quad , \qquad (53)$$

using (37). So

$$a_1 > b_1 \, . \qquad (54)$$

Thus , the surface particle motion (on z = 0) is an ellipse with a vertical major axis.

The horizontal and vertical displacement components are out of phase by $\frac{\pi}{2}$.

The resulting surface particle motion is **Retrograde** (opposite to that of wave propagation).

Remark 4: The dependencies of the displacement components u and w on depth(z) are given by equation (43) to (46) .

There is a value of z for which u = 0 (For Poisson's solid , at z = + 0.19λ , $\lambda = \frac{2\pi}{k}$, u = 0) , whereas w is never zero.

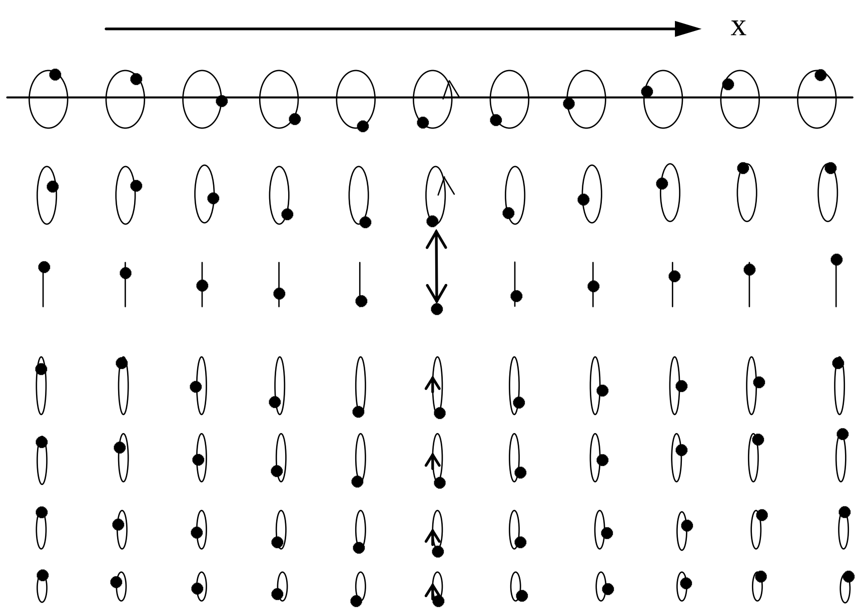
At the depth , where u is zero , its amplitude changes sign.

For greater depths , the particle motion is **Prograde.**

With increasing depth , the amplitudes of u and w decrease exponentially , with w always larger than u.

Thus , the elliptic motion changes from retrograde at the surface to prograde at depth , passing through a node at which there is no horizontal motion.

So , for the propagation of Rayleigh surface waves , a surface particle **describes an ellipse** , about its mean position , in the retrograde sense.



Particle motion for the fundamental Rayleigh mode in a uniform half-space , propagating from left to right. One horizontal wavelength (Λ) is shown ; the dots are plotted at a fixed time point. Motion is counter clockwise (retrograde) at the surface , changing to purely vertical motion at a depth of about $\Lambda/5$, and becoming clockwise (prograde) at greater depths. Note that the time behavior at a fixed distance is given by looking from right to left in this plot.

Remark 5: We see that frequency equation (27) for Rayleigh surface waves is independent of ω. Therefore , the velocity c_R of Rayleigh surface waves is constant and fixed.

This phenomenon is called nondispessive.

That is , **Rayleigh waves are undispresed**.

Remark 6: Maximum displacement parallel to the direction of Rayleigh waves

$$= (u)_{max.}$$

$$= b_1$$

$$= \frac{2}{3} a_1 \quad , \quad \text{for a Poisson solid.}$$

= **two-third of** the maximum displacement in the vertical direction

for a Poisson solid.

Note (1) : Rayleigh waves are important because the largest disturbances caused by an earthquake recorded on a distant seismogram are usually those of Rayleigh waves.

GROUND ROLL is the term commonly used for Rayleigh waves.

Note (2) : Although a “a free surface” means contact with a vacuum , the elastic constants and density of air are so lows in comparison with values for rocks that the surface of the earth is approximately a free surface.

Note (3) : The boundary conditions $\tau_{zx} = \tau_{zz} = 0$ at $z = 0$ require that these two conditions must be satisfied , and so we require two parameters than can be adjusted. Therefore , we assume that both P-and SV-components exist and adjust their amplitude to satisfy the boundary conditions.

Exercise : Show that the displacement components at the surface of an elastic Poisson solid due to Rayleigh waves are

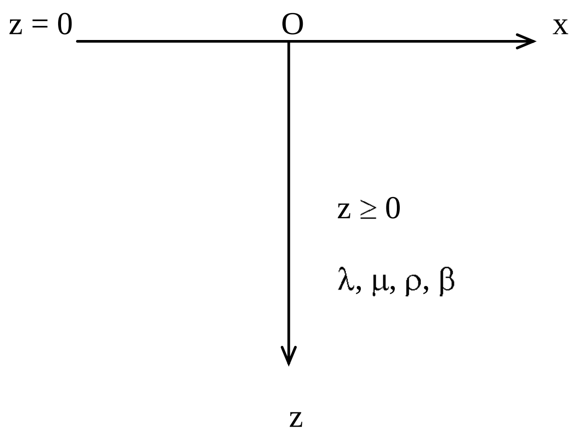
$$u(x, t) = -0.423 \, kA \sin k(x - c_R t)$$

$$w(x, t) = 0.620 \, kA \cos k(x - c_R t) \, , \quad v \equiv 0 \, ,$$

with usual notation.

10.3. SURFACE WAVES OF SH-TYPE IN A HALF-SPACE

We consider first the possibility of the propagation of SH type surface waves (called Love waves) in a homogeneous semi-infinite isotropic elastic medium occupying the half-space $z \geq 0$. The horizontal boundary $z = 0$ of the medium is assumed to be stress free. Let ρ be the density of the **medium and λ , μ Lame′ constants**(figure) .



(Elastic isotropic half-space)

Let the two – dimensional SH-wave motion takes place in the xz-plane. The basic equations for SH- wave motion are

$$u = w = 0 , \qquad v = v(x , z , t) , \tag{1}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2} = \frac{1}{\beta^2} \frac{\partial^2 v}{\partial t^2} , \tag{2}$$

$$\beta^2 = \frac{\mu}{\rho} . \tag{3}$$

We try a plane wave solution of wave equation (2) of the form

$$v(x, z, t) = B . e^{-bkz} . e^{i(\omega t - kx)} , \tag{4}$$

where ω is the angular frequency of wave , k = wave number and $c = \omega/k$ is the speed with which surface waves are travelling in the x-direction on the surface $z = 0$; $b > 0$ and B is an arbitrary constant.

The amplitude of surface wave is B e^{-bkz} which die exponentially as z increases.

Substituting the value of v(x, z, t) from equation (4) into equation (2) , we find

$$b^2 = 1 - \frac{c^2}{\beta^2} . \tag{5}$$

Since b > 0 ,so

$$c < \beta . \tag{6}$$

Using the stress-displacement relations , we find

$$\tau_{31} = \tau_{33} = 0 ,$$

$$\tau_{32} = \mu \frac{\partial v}{\partial z} = - \mu\, b\, k . B . e^{i(\omega t - kx)} . e^{-bkz} \tag{7}$$

Hence , the stress-free condition of the boundary z = 0 implies that , using (7) ,

$$\mu\, b\, k\, B\, e^{i(\omega t - kx)} = 0$$

or

$$B = 0 \quad , \tag{8}$$

as $\mu \neq 0$, $b \neq 0$, $k \neq 0$.

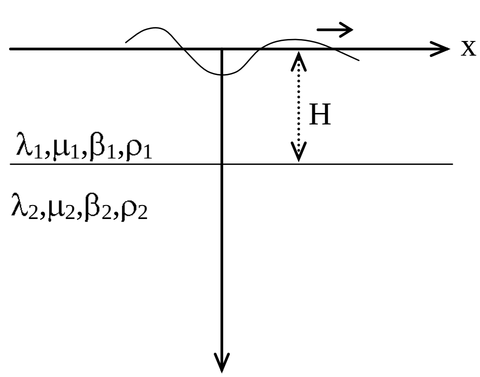
This implies that in the case of a homogeneous isotropic elastic half – space, Love waves do not exist at all.

10.4. PROPAGATION OF LOVE WAVES

Surface waves of the SH-type are observed to occur on the earth's surface. Love (1911) showed that if the earth is modelled as an isotropic elastic layer of finite thickness lying over a homogeneous elastic isotropic half-space (rather than considering earth as a purely uniform half-space) then SH type waves occur in the stress-free surface of a layered half-space.

Now , we consider the possibility of propagation of surface waves of SH-type (Love waves) in a semi-infinite elastic isotropic medium consisting of a horizontal elastic layer of uniform thickness H lying over a half-space.

It is assumed that two elastic isotropic media are welded together and the horizontal boundary z = 0 of the semi-infinite medium is stress – free (see , figure).



o c

$\beta_1 < c_L < \beta_2$

z

Let the layer and the half-space have different densities ρ_1, ρ_2 and different shear moduli μ_1 , μ_2 respectively. Let two-dimensional SH-motion takes place parallel to xz-plane. The basic equation for SH-wave motion are

$$u_1 = w_1 \equiv 0 \,, \quad v_1 = v_1(x\,, z, t) \,, \tag{1}$$

$$\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial z^2} = \frac{1}{\beta_1^2}\frac{\partial^2 v_1}{\partial t^2} \,, \tag{2}$$

$$\beta_1^2 = \frac{\mu_1}{\rho_1} \,, \tag{3}$$

in the layer $0 \le z \le H$, and

$$u_2 = w_2 = 0 \,, \quad v_2 = v_2(x\,, z\,, t) \,, \tag{4}$$

$$\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial z^2} = \frac{1}{\beta_2^2}\frac{\partial^2 v_2}{\partial t^2} \,, \tag{5}$$

$$\beta_2^2 = \frac{\mu_2}{\rho_2} \,, \tag{6}$$

in the half-space ($z \ge H$).

Suitable plane wave solutions of wave equations (2) and (5) are (exercise) , as discussed in detail already ,

$$v_1(x\,, z\,, t) = (A_1\, e^{-b_1kz} + B_1\, e^{b_1kz})\, e^{i(\omega t - kx)} \,, \tag{7}$$

in the layer $0 \le z \le H$, and

$$v_2\,(x\,, z\,, t) = A_2\, e^{-b_2kz} e^{i(\omega t - kx)} \,, \tag{8}$$

in the half-space ($z \ge H$). ω is the angular wave frequency and k is the wave number , and $c = \omega/k$ is the speed of propagation of surface wave (if it exists) in the positive x-direction. A_1, B_1 are constants, b_1 and b_2 are real numbers with $b_2 > 0$. However , b_1 is unrestricted because z is finite in the layer.

Substituting for v_2 from (8) into (5) yields the relation

$$b_2^2 = \left(1 - \frac{c^2}{\beta_2^2}\right) , \qquad (9)$$

and , therefore ,

$$c < \beta_2 , \qquad (10)$$

otherwise

$$v_2 \to \infty \text{ as } z \to \infty.$$

From equations (7) and (2), we find

$$b_1^2 = \left(1 - \frac{c^2}{\beta_1^2}\right) . \qquad (11)$$

The stress-displacement relations imply

$$\tau_{31} = \tau_{33} \equiv 0 \quad , \qquad (12)$$

in the layer as well as in the half space. Also

$$\tau_{32} = \mu_1\, k(-b_1 A_1\, e^{-b_1 kz} + b_1 B_1\, e^{b_1 kz})\, e^{i(\omega t - kx)} , \qquad (13)$$

in the layer $0 \le z \le H$, and

$$\tau_{32} = k\, \mu_2\, A_2\, (-b_2)\, e^{-b_2 kz} .\, e^{i(\omega t - kx)} \qquad (14)$$

in the half space $z \ge H$.

The stress-free boundary $z = 0$ implies that

$$\tau_{32} = 0 \quad , \qquad (15)$$

at $z = 0$. This gives

$$B_1 = A_1 . \qquad (16)$$

Since , there is a welded contact between the layer and the half-space at the interface $z = H$, so the displacement and the tractions must be continuous across the interface $z = H$.

Thus , the boundary conditions at $z = H$ are

$$v_1 = v_2 , \qquad (17)$$

$$\tau_{32}\big|_{\text{layer}} = \tau_{32}\big|_{\text{half-space}} . \qquad (18)$$

From equations (13) , (14) , (17) and (18) , we find

$$A_1\ e^{-kb_1H} + B_1\ e^{kb_1H} = A_2\ e^{-kb_2H}\ , \tag{19}$$

and

$$\mu_1\ k\ b_1[-A_1\ e^{-b_1kH} + B_1\ e^{b_1kH}] = -\ \mu_2\ b_2\ k\ e^{-b_2kH}\ . \tag{20}$$

Equation (16) , (19) and (20) are three homogeneous in A_2 , A_1 , B_1 and A_2.

We shall now eliminate them. From equations (19) & (20) , we write (after putting $B_1 = A_1$)

$$\frac{A_1[e^{-kb_1H} + e^{kb_1H}]}{A_1.\mu_1 b_1(e^{-b_1kH} - e^{b_1kH})k} = \frac{1}{\mu_2 b_2 k}$$

or

$$\frac{e^{-b_1kH} - e^{b_1kH}}{e^{-kb_1H} + e^{kb_1H}} = \frac{\mu_2 b_2}{\mu_1 b_1}$$

or

$$\tan h(b_1\ k\ H) + \frac{\mu_2 b_2}{\mu_1 b_1} = 0. \tag{21}$$

Equation (21) is known as period equation/frequency equation/Dispersion equation for surface Love waves.

Equation (21) can also be written as

$$\tan h\left[kh.\sqrt{1-\frac{c^2}{\beta_1^2}}\right] = -\frac{\mu_2}{\mu_1}\ .\ \frac{\sqrt{1-\frac{c^2}{\beta_2{}^2}}}{\sqrt{1-\frac{c^2}{\beta_1{}^2}}}\ .$$

or

$$\tan h\left[\frac{\omega H}{c}\sqrt{1-\frac{c^2}{\beta_1{}^2}}\right] = -\frac{\mu_2}{\mu_1}\ .\ \frac{\sqrt{1-\frac{c^2}{\beta_2{}^2}}}{\sqrt{1-\frac{c^2}{\beta_1{}^2}}}\ . \tag{22}$$

Equation (22) is a transcendental equation.

For given ω**,** we can find the speed c for surface Love waves. We note **the value of c depends upon** ω **.** This means that **waves of different frequencies will**, in **general , propagate with different phase velocity.**

This phenomenon is known as dispersion.

It is caused by the **inhomogeneity of the medium** (layered medium) due to some abrupt discontinuities within the medium (or due to continuous change of the elastic parameters which is not the present case).

Thus, **Love waves are dispressed.**

We consider now the following two possibilities between c and β_1 .

(i) Either $c \leq \beta_1$, (ii) or $c > \beta_1$. (23)

When $c \leq \beta_1$ **:** In this case b_1 is real (see , equation (11)) and left side of (22) becomes real and positive and right side of (22) is real and negative. Therefore , equation (22) can not possess any real solution for c.

Therefore , in this case , Love waves do not exist.

So , for the existence of surface Love waves , we must have

$$\mathbf{c > \beta_1} \ . \qquad (24)$$

In this case (24), b_1 is purely imaginary and we may write

$$b_1 = \ = \sqrt{1-\frac{c^2}{\beta_1^2}} = i\left(\sqrt{\frac{c^2}{\beta_1}-1}\right) . \qquad (25)$$

Then equation (22) becomes

$$\tan\left(kH\sqrt{\frac{c^2}{\beta_1^2}-1}\right) = \frac{\mu_2}{\mu_1}\left(\frac{\sqrt{1-\frac{c^2}{\beta_2^2}}}{\sqrt{\frac{c^2}{\beta_1^2}-1}}\right) . \qquad (26)$$

From equations (10) and (24) ; we find

$$\beta_1 < c < \beta_2 \ . \qquad (27)$$

Equation (26) is a transcendental equation that yields infinitely many roots for c.

Thus, the possible **speeds of the Love waves are precisely** the roots of equations (26) that lie **in the interval (β_1 , β_2).**

This indicates that the **shear velocity in the layer** must be less than the **shear velocity in the half-space** for the possible existence of Love waves.

This gives **the upper and lower bounds for** the speed of Love waves.

Remark 1: If the layer and the half – space are such that $\beta_1 \leq \beta$, then existence of Love waves are not possible

Remark 2: In the limiting case when the layer is absent , we have

$$\mu = \mu_1 \text{ and } \rho = \rho_1$$

and therefore

$$\beta = \beta_1$$

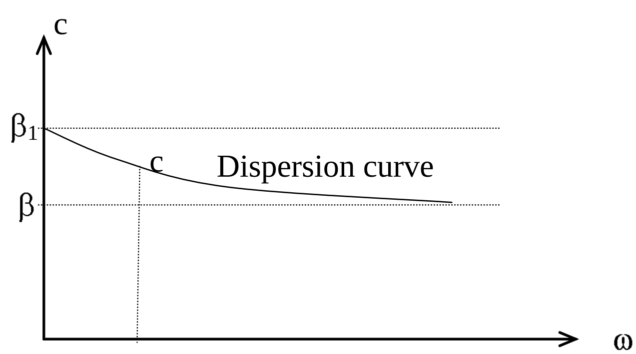
Equation (22) leads to the impossible condition

$$0 = -1 .$$

Hence , in this case , the wave considered can not exist.

Remark 3: When k or $\omega \to 0$, we get $c \to \beta_1$.

The dispersion curve is given in the following figure.



Here , if we assume

$$\mu_1/\mu = 1.8 , \beta = 3.6\text{km/sec} , \beta_1 = 4.6\text{km/sec}$$

then

$$c_L = \text{speed of Love waves} = 4.0\text{km/sec.}$$

10

SURFACE WAVES

10.1. INTRODUCTION

In an elastic body, it is possible to have another type of waves (other than body waves) which are propagated over the surface and penetrate only a little into the interior of the body.

Such waves are similar to waves produced on a **smooth surface of water when** a stone is thrown into it.

These type of waves are **called SURFACE WAVES.**

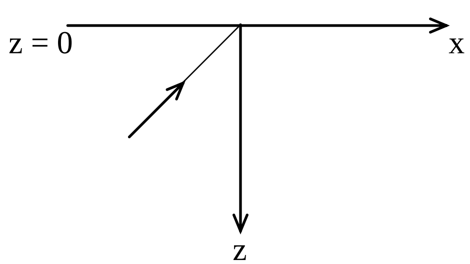
Surface waves are "tied" to the surface and diminish exponentially as they get farther from the surface.

The criterion for surface waves is that the **amplitude** of the displacement in the medium **dies exponentially with the increasing distance from the surface.**

In seismology , the interfaces are , in the ideal case , horizontal and so the plane of incidence is **vertical**. Activity of surface waves is restricted to the neighbourhood of the interface(s) or surface of the medium.

Under certain conditions , such waves can propagated independently along the surface/interface. For surface waves , the disturbance is confined to a depth **equal to a few wavelengths**.

Let us take xz – plane as the plane of incidence with z – axis vertically downwards. Let z = 0 be the surface of a semi-infinite elastic medium (Figure) .



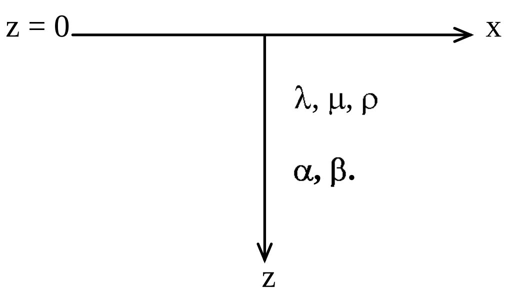
For a surface wave , its amplitude will be a function of z (rather than an exponential function) which tends to zero as $z \to \infty$. For such surface – waves , the motion will be two – dimensional **,** parallel to xz – plane , so that $\frac{\partial}{\partial y} = 0$.

The existence of surface waves raises the question of whether they might (under certain conditions) be able to travel freely along the plane (horizontal) as a guided wave.

10.2. RAYLEIGH WAVES

Rayleigh (1885) discussed the existence of a simplest surface wave propagating on the free – surface of a homogeneous isotropic elastic half – space.

Let the half – space occupies the region $z \geq 0$ with z – axis taken as vertically downwards. Let ρ be the density and λ , μ be the Lame′ elastic moduli and z = 0 be the stress – free boundary of the half – space (figure).



Suppose that a train of plane waves is propagating in the media in the positive x – direction such that

(i) the plane of incidence is the vertical plane (xz – plane) so that the motion , i.e. , disturbance is independent of y and hence $\frac{\partial}{\partial y} \equiv 0$.

(ii) the amplitude of the surface wave decreases exponentially as we move away from the surface in the z – direction.

This problem is a **plane strain problem** and the displacement vector $\bar{u}$ is of the type

$$\bar{u} = (u , 0 , w) , \quad u = u(x , z, t) , \quad w = w(x , z , t) . \quad (1)$$

The displacement components u and w are given in term of potential ϕ and ψ through the relations

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial z} , \tag{2}$$

$$w = \frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial x} , \tag{3}$$

where potentials ϕ and ψ satisfy the scalar wave equations

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{\alpha^2} \frac{\partial^2 \phi}{\partial t^2} , \tag{4}$$

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{\beta^2} \frac{\partial^2 \psi}{\partial t^2} , \tag{5}$$

and wave velocities α and β are given by

$$\alpha = \sqrt{(\lambda + 2\mu)/\rho} , \tag{6}$$

$$\beta = \sqrt{\mu/\rho} . \tag{7}$$

This wave motion is of a P-SV type wave travelling along the stress – free surface of the half space and such a motion takes place in the xz – plane.

Now , we seek solutions of wave equations (4) and (5) of the form

$$\phi(x , z , t) = f(z) e^{ik(ct-x)} , \tag{8}$$

$$\psi(x , z , t) = g(z) e^{ik(ct-x)} , \tag{9}$$

where k is the wave number , c is the speed of surface wave travelling in the +ve x-direction and ω = ck is the angular frequency.

The substitution of (8) and (9) into wave equations (4) and (5) leads to two ordinary differential equations (exercise)

$$\frac{d^2 f}{dz^2} - k^2 a^2 f = 0 , \tag{10}$$

and

$$\frac{d^2 g}{dz^2} - k^2 b^2 g = 0 , \tag{11}$$

with

$$a = \sqrt{1 - c^2/\alpha^2} > 0 , \tag{12}$$

and $$b = \sqrt{1 - c^2/\beta^2} > 0 \quad . \qquad (13)$$

From equations (8) to (11) ; we find (exercise)

$$\phi\,(x\,,z\,,t) = (A\,e^{-akz} + A_1\,e^{akz})\,e^{ik(ct-x)}\,, \qquad (14)$$

$$\psi(x\,,z\,,t) = (B\,e^{-bkz} + B_1\,e^{bkz})\,e^{ik(ct-x)}\,, \qquad (15)$$

where A , A_1 B and B_1 are constants.

Since the disturbance due to surface waves must die rapidly as $z \to \infty$, we must have

$$A_1 = B_1 = 0\,. \qquad (16)$$

Thus , for Rayleigh waves , we get

$$\phi(x\,,z\,,t) = A\,e^{-akz}\,e^{i(\omega t - kx)}\,, \qquad (17)$$

and

$$\psi(x\,,z\,,t) = B\,e^{-bkz}\,e^{i(\omega t - kx)}\,, \qquad (18)$$

with

$$c < \beta < \alpha\,. \qquad (19)$$

The equation (19) gives the condition for the existence of surface waves on the surface of a semi – infinite isotropic elastic media with velocity c in the positive x – direction.

To find frequency equation for the velocity c and ratio of A and B ; we use the stress – free boundary conditions. This conditions are

$$\tau_{zx} = \tau_{zy} = \tau_{zz} = 0 \qquad \text{at } z = 0. \qquad (20)$$

From relation in (2) and (3) , we find

$$\begin{aligned} \operatorname{div}\,\bar{u} &= \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \\ &= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} \\ &= \nabla^2 \phi\,. \qquad (21) \end{aligned}$$

The stresses in terms of potentials ϕ and ψ are given by

$$\tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$= \mu\left[2\frac{\partial^2\phi}{\partial x\,\partial z} + \frac{\partial\psi}{\partial z^2} - \frac{\partial^2\psi}{\partial x^2}\right], \qquad (22)$$

$$\tau_{zz} = \lambda \text{ div } \bar{u} + \mu\frac{\partial w}{\partial z}$$

$$= \mu\left[\left(\frac{\alpha^2}{\beta^2} - 2\right)\frac{\partial^2\phi}{\partial x^2} + \quad + \frac{\alpha^2}{\beta^2}\frac{\partial^2\phi}{\partial z^2} - 2\frac{\partial^2\psi}{\partial x\,\partial z}\right], \qquad (23)$$

$$\tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$= 0\,. \qquad (24)$$

The boundary condition $\tau_{zx} = 0$ at $z = 0$ in (20), gives (exercise)

$$2A(-ik)\,(-ak) - B(-ik)^2 + B(-bk)^2 = 0$$

or $$2\,i\,a\,k^2 A + B(1 + b^2)\,k^2 = 0$$

or $$2\,i\,a\,A = -B(1 + b^2)$$

or $$\frac{A}{B} = \frac{i(1+b^2)}{2a}\,. \qquad (25)$$

The boundary condition $\tau_{zz} = 0$ at $z = 0$, gives (exercise)

$$\left(\frac{\alpha^2}{\beta^2} - 2\right).\ A\ .\ (-k^2) + \frac{\alpha^2}{\beta^2}.\ A\ .\ (\alpha^2\,k^2) - 2B\,(-ik)\,(-bk) = 0$$

or $$A\left[-\frac{\alpha^2}{\beta^2} + 2 + \frac{\alpha^2 a^2}{\beta^2}\right]k^2 - 2\,i\,b\,B\,k^2 = 0$$

or $$A\left(2 - \frac{\alpha^2}{\beta^2} + \frac{\alpha^2 a^2}{\beta^2}\right) = 2\,i\,b\,B$$

or $$\frac{A}{B} = \left(\frac{2ib}{2 - \frac{\alpha^2}{\beta^2} + \frac{\alpha^2 a^2}{\beta^2}} \right). \tag{26}$$

Eliminating A/B from equations (25) and (26) and substituting the value of a and b from equation (12) and (13) , we obtain

$$(1 + b^2) \left(2 - \frac{\alpha^2}{\beta^2} + \frac{\alpha^2 a^2}{\beta^2} \right) = 4\, a\, b$$

or $$\left(2 - \frac{c^2}{\beta^2} \right) \left[2 - \frac{\alpha^2}{\beta^2} + \frac{\alpha^2}{\beta^2} \left(1 - \frac{c^2}{\alpha^2} \right) \right]$$

$$= 4 \sqrt{1 - \frac{c^2}{\alpha^2}}\ \sqrt{1 - \frac{c^2}{\beta^2}}$$

or $$\left(2 - \frac{c^2}{\beta^2} \right) \left[2 - \frac{\alpha^2}{\beta^2} + \frac{\alpha^2}{\beta^2} - \frac{c^2}{\beta^2} \right] = 4 \sqrt{1 - \frac{c^2}{\alpha^2}}\ \sqrt{1 - \frac{c^2}{\beta^2}}$$

or $$\left(2 - \frac{c^2}{\beta^2} \right)^2 - 4 \left(1 - \frac{c^2}{\alpha^2} \right)^{\frac{1}{2}} \left(1 - \frac{c^2}{\beta^2} \right)^{\frac{1}{2}} = 0\,. \tag{27}$$

This equation contains only one unknown c.

This equation determines the speed c for Rayleigh surface waves in an uniform half – space.

Equation (27) is called the **RAYLEIGH EQUATION** for Rayleigh waves. It is also called the **Rayleigh frequency equation**. It is also called the **Rayleigh wave – velocity equation**.

This equation is the **period equation** for Rayleigh waves.

In order to prove that Rayleigh waves really exists , we must show that frequency equation (27) has at least one real root for c. To show this , we proceed as follows :

From equation (27) , we write

$$\left(2-\frac{c^2}{\beta^2}\right)^4 = 16\left(1-\frac{c^2}{\alpha^2}\right)\left(1-\frac{c^2}{\beta^2}\right)$$

or

$$16 - 32\,\frac{c^2}{\beta^2} + 24\,\frac{c^4}{\beta^4} - 8\,\frac{c^6}{\beta^6} + \frac{c^8}{\beta^8}$$

$$= 16\left[1-\frac{c^2}{\alpha^2}-\frac{c^2}{\beta^2}+\frac{c^4}{\alpha^2\beta^2}\right]$$

$$= 16\left[1-\frac{c^2}{\beta^2}\frac{\beta^2}{\alpha^2}-\frac{c^2}{\beta^2}+\frac{c^4}{\beta^4}.\frac{\beta^2}{\alpha^2}\right]$$

or

$$-32 + 24\,\frac{c^2}{\beta^2} - 8\,\frac{c^4}{\beta^4} + \frac{c^6}{\beta^6} + 16\,\frac{\beta^2}{\alpha^2} + 16 - 16\,\frac{c^2}{\beta^2}.\frac{\beta^2}{\alpha^2} = 0\,. \qquad (28)$$

Put

$$s = c^2/\beta^2\,, \qquad (29)$$

then equation (28) gives

$$f(s) = s^3 - 8s^2 + \left(24-16\frac{\beta^2}{\alpha^2}\right)s - 16\left(1-\frac{\beta^2}{\alpha^2}\right) = 0\,. \qquad (30)$$

It is polynomial **equation of degree 3 in s** with real coefficients.

Hence the frequency equation (30) has at least one real root since complex roots occur in conjugate pairs.

Moreover

$$f(0) = -16\left(1-\frac{\beta^2}{\alpha^2}\right) < 0\,, \qquad (31)$$

and

$$f(1) = 1 - 8 + 24 - \frac{16\beta^2}{\alpha^2} + \frac{16\beta^2}{\alpha^2} = 1 > 0\,, \qquad (32)$$

since $\beta < \alpha$.

Hence , by the intermediate theorem of calculus , the frequency equation (30) has at least one root in the range (0 , 1).

That is , there exists $s \in (0, 1)$ which is a root of the Rayleigh frequency equation.

Further , $0 < s < 1$ implies

$$0 < \frac{c^2}{\beta^2} < 1$$

or $$\mathbf{0 < c < \beta} . \tag{33}$$

This establishes that Rayleigh surface waves propagating with speed $c = c_R$, where

$$0 < c_R < \beta < \alpha ,$$

always exists.

This proves the existence of Rayleigh surface waves.

Special Case (Poisson's solid) : When the semi-infinite elastic medium is a Poisson's solid ,

then $\lambda = \mu$ and $\sigma = \frac{1}{4}$ and

$$\frac{\alpha^2}{\beta^2} = \frac{\lambda + 2\mu}{\mu} = 3. \tag{34}$$

For this type of elastic medium , frequency equation (30) becomes

$$3s^3 - 24 s^2 + 56 s - 32 = 0$$

or $$(s - 4)(3s^2 - 12 s + 8) = 0$$

giving

$$s = 4 , 2 + \frac{2}{\sqrt{3}} , 2 - \frac{2}{\sqrt{3}}$$

or $$\frac{c^2}{\beta^2} = 4 , 2 + \frac{2}{\sqrt{3}} , 2 - \frac{2}{\sqrt{3}} . \tag{35}$$

But c must satisfy the inequality $0 < c < \beta$, therefore , the only possible value for $c = c_R$ is

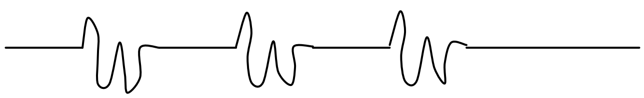
$$\frac{c_R^{\ 2}}{\beta^2} = 2 - \frac{2}{\sqrt{3}} \ ,$$

or

$$c_R = \left[\sqrt{2-\frac{2}{\sqrt{3}}}\right] . \beta \cong 0.9195 \beta \ , \qquad (36)$$

giving the speed of propagation of Rayleigh surface waves along the stress free boundary of the Poissonian half-space in the x-direction.

Thus , the order of arrival of P, SV-and Rayleigh waves is



P SV Rayleigh

From equations (12) and (13) ; we write

$$a^2 = 1 - \frac{c_R^{\ 2}}{\alpha^2}$$

$$= 1 - \frac{c_R^{\ 2}}{\beta^2} . \frac{\beta^2}{\alpha^2}$$

$$= \frac{1}{9}(3 + 2\sqrt{3}) \ ,$$

or

$$a = \frac{1}{3}\sqrt{3+2\sqrt{3}} \cong 0.8475 \approx 0.84\,8 \ , \qquad (37)$$

and

$$b^2 = 1 - \frac{c_R^{\ 2}}{\beta^2}$$

$$= \frac{1}{3}[\,2\sqrt{3} - 3] \ ,$$

or $$b \cong 0.3933 \approx 0.393 . \tag{38}$$

Further , from equations (25) , (37) and (38) , we find

$$\frac{A}{B} = 1.468\, i \quad . \tag{39}$$

Remark 1: Displacements due to Rayleigh waves (Particle motion)

From equations (2) , (3) , (17) , (18) and (26) ; we find

$$u(x , z , t) = [A(-ik)\, e^{-akz} + B(-bk)\, e^{-bkz}]\, e^{i(\omega t-kx)}$$

$$= k[-i\, A\, e^{-akz} - A\left(\frac{2-c^2/\beta^2}{2i}\right) e^{-bkz}]\, e^{i(\omega t-kx)}$$

$$= -\, i\, k\, A\, [e^{-akz} - \left(1-\frac{c^2}{2\beta^2}\right) e^{-bkz}]\, e^{i(\omega t-kx)} . \tag{40}$$

Similarly , we shall find

$$w(x , z , t) = [(-ak)\, A\, e^{-akz} - (-ik)B\, e^{-bkz}]\, e^{i(\omega t-kx)}$$

$$= k[-\, a\, A\, e^{-akz} + \frac{ie^{bkz}(-2iAa)}{(2-c^2/\beta^2)}]\, e^{i(\omega t-kx)}$$

$$= k\, A\, [\,-a\, e^{-akz} + \left(\frac{a}{1-c^2/2\beta^2}\right) e^{-bkz}\,]\, e^{i(\omega t-kx)} . \tag{41}$$

Let $$\theta = \omega t - kx \quad , \tag{42}$$

$$U(z) = e^{-akz} - \left(1-\frac{c^2}{2\beta^2}\right) e^{-bzk} , \tag{43}$$

$$W(z) = -a\, e^{-akz} + \left(\frac{a}{1-c^2/2\beta^2}\right) e^{-bkz} . \tag{44}$$

Then , taking the real parts of equations (40) and (41) and using equation (42) to (44) ; we find

$$u(x , z , t) = A\, k\, U(z) \sin\theta \quad , \tag{45}$$

$$w(x , z , t) = A\, k\, W(z) \cos\theta \quad , \tag{46}$$

we remember that A is the potential amplitude. Eliminating θ from equations (45) and (46) , we obtain

$$\frac{u^2}{[Ak\,U(z)]^2} + \frac{w^2}{[Ak\,W(z)]^2} = 1 \qquad , \qquad (47)$$

which is an equation of an ellipse in the vertical xz-plane.

Equation (47) shows that particles , during the propagation of Rayleigh Surface Waves, describe ellipses.

Remark 2: Particle Motion at the surface (z = 0)

On the surface z = 0 ,we find, at z = 0 ,

$$\mathrm{U}(0) = \frac{c^2}{2\beta^2} \qquad , \qquad \mathrm{W}(0) = \mathrm{a}\left(\frac{\frac{c^2}{2\beta^2}}{1-c^2/2\beta^2}\right) \qquad 48)$$

Since $0 < c < \beta$, so

$$\mathrm{U}(0) > 0 \qquad , \qquad \mathrm{W}(0) > 0\,. \qquad (49)$$

Let $\quad a_1 = A\,k\,W(0)\,, \qquad b_1 = A\,k\,U(0)\,. \qquad (50)$

Then $\quad a_1 > 0 \quad , b_1 > 0 \quad , \qquad (51)$

and equation (47) reduces to

$$\frac{u^2}{b_1^{\,2}} + \frac{w^2}{a_1^{\,2}} = 1\,. \qquad (52)$$

Remark 3: Particular case (Poisson's Solid) :

In this case , we find, at z = 0 ,

$$\frac{a_1}{b_1} = \frac{W(0)}{U(0)} = \sqrt{3}\,\mathrm{a} \cong 1.5 \qquad , \qquad (53)$$

using (37). So

$$a_1 > b_1\,. \qquad (54)$$

Thus , the surface particle motion (on z = 0) is an ellipse with a vertical major axis.

The horizontal and vertical displacement components are out of phase by $\frac{\pi}{2}$.

The resulting surface particle motion is **Retrograde** (opposite to that of wave propagation).

Remark 4: The dependencies of the displacement components u and w on depth(z) are given by equation (43) to (46) .

There is a value of z for which u = 0 (For Poisson's solid , at z = + 0.19λ , λ = $\frac{2\pi}{k}$, u = 0) , whereas w is never zero.

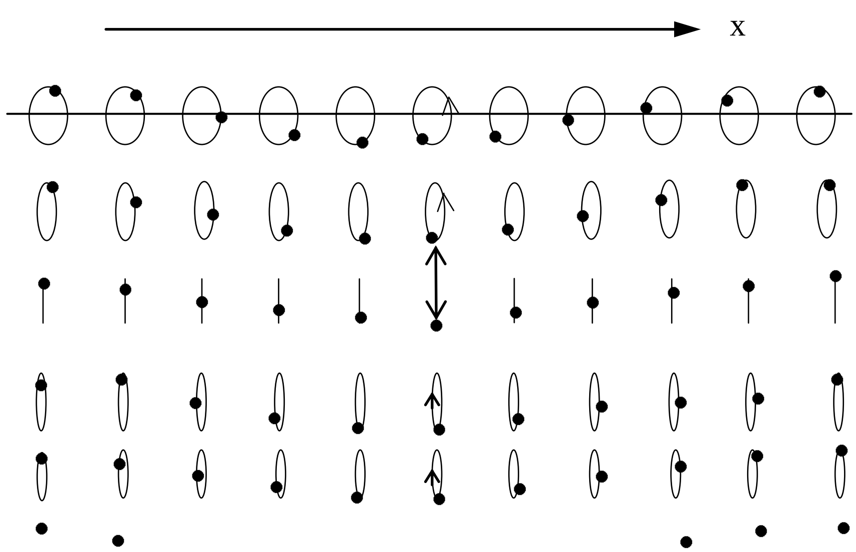
At the depth , where u is zero , its amplitude changes sign.

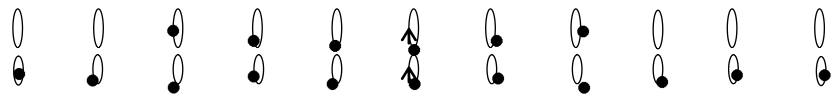
For greater depths , the particle motion is **Prograde.**

With increasing depth , the amplitudes of u and w decrease exponentially , with w always larger than u.

Thus , the elliptic motion changes from retrograde at the surface to prograde at depth , passing through a node at which there is no horizontal motion.

So , for the propagation of Rayleigh surface waves , a surface particle **describes an ellipse** , about its mean position , in the retrograde sense.





Particle motion for the fundamental Rayleigh mode in a uniform half-space , propagating from left to right. One horizontal wavelength ($\wedge$) is shown ; the dots are plotted at a fixed time point. Motion is counter clockwise (retrograde) at the surface , changing to purely vertical motion at a depth of about $\wedge/5$, and becoming clockwise (prograde) at greater depths. Note that the time behavior at a fixed distance is given by looking from right to left in this plot.

Remark 5: We see that frequency equation (27) for Rayleigh surface waves is independent of ω. Therefore , the velocity c_R of Rayleigh surface waves is constant and fixed.

This phenomenon is called nondispessive.

That is , **Rayleigh waves are undispresed**.

Remark 6: Maximum displacement parallel to the direction of Rayleigh waves

$= (u)_{max.}$

$= b_1$

$= \frac{2}{3}\, a_1$, for a Poisson solid.

= **two-third of** the maximum displacement in the vertical direction

for a Poisson solid.

Note (1) : Rayleigh waves are important because the largest disturbances caused by an earthquake recorded on a distant seismogram are usually those of Rayleigh waves.

GROUND ROLL is the term commonly used for Rayleigh waves.

Note (2) : Although a “a free surface” means contact with a vacuum , the elastic constants and density of air are so lows in comparison with values for rocks that the surface of the earth is approximately a free surface.

Note (3) : The boundary conditions $\tau_{zx} = \tau_{zz} = 0$ at $z = 0$ require that these two conditions must be satisfied , and so we require two parameters than can be adjusted. Therefore , we assume that both P-and SV-components exist and adjust their amplitude to satisfy the boundary conditions.

Exercise : Show that the displacement components at the surface of an elastic Poisson solid due to Rayleigh waves are

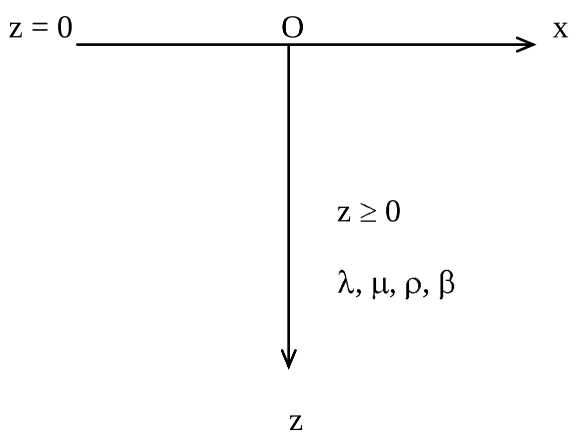
$$u(x, t) = -0.423\ kA \sin k(x - c_R t)$$

$$w(x, t) = 0.620\ kA \cos k(x - c_R t)\ , \qquad v \equiv 0\ ,$$

with usual notation.

10.3. SURFACE WAVES OF SH-TYPE IN A HALF-SPACE

We consider first the possibility of the propagation of SH type surface waves (called Love waves) in a homogeneous semi-infinite isotropic elastic medium occupying the half-space $z \geq 0$. The horizontal boundary $z = 0$ of the medium is assumed to be stress free. Let ρ be the density of the **medium and λ , μ Lame′ constants**(figure) .



(Elastic isotropic half-space)

Let the two – dimensional SH-wave motion takes place in the xz-plane. The basic equations for SH- wave motion are

$$u = w = 0\ , \qquad v = v(x\ , z\ , t)\ , \tag{1}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2} = \frac{1}{\beta^2}\frac{\partial^2 v}{\partial t^2}\ , \tag{2}$$

$$\beta^2 = \frac{\mu}{\rho}\ . \tag{3}$$

We try a plane wave solution of wave equation (2) of the form

$$v(x, z, t) = B\ .\ e^{-bkz}\ .\ e^{i(\omega t - kx)}\ , \tag{4}$$

where ω is the angular frequency of wave , k = wave number and c = ω/k is the speed with which surface waves are travelling in the x-direction on the surface z = 0 ; b > 0 and B is an arbitrary constant.

The amplitude of surface wave is B e^{-bkz} which die exponentially as z increases.

Substituting the value of v(x, z, t) from equation (4) into equation (2) , we find

$$b^2 = 1 - \frac{c^2}{\beta^2} \; . \qquad (5)$$

Since b > 0 ,so

$$c < \beta \; . \qquad (6)$$

Using the stress-displacement relations , we find

$$\tau_{31} = \tau_{33} = 0 \; ,$$
$$\tau_{32} = \mu \frac{\partial v}{\partial z} = -\mu\, b\, k\, .\, B\, .\, e^{i(\omega t - kx)}\, .\, e^{-bkz} \qquad (7)$$

Hence , the stress-free condition of the boundary z = 0 implies that , using (7) ,

$$\mu\, b\, k\, B\, e^{i(\omega t - kx)} = 0$$

or

$$B = 0 \quad , \qquad (8)$$

as $\mu \neq 0$, $b \neq 0$, $k \neq 0$.

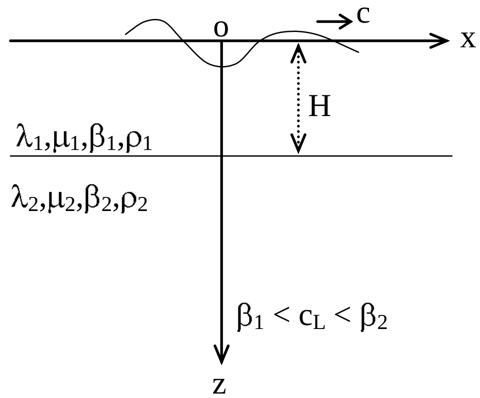
This implies that in the case of a homogeneous isotropic elastic half – space, Love waves do not exist at all.

10.4. PROPAGATION OF LOVE WAVES

Surface waves of the SH-type are observed to occur on the earth's surface. Love (1911) showed that if the earth is modelled as an isotropic elastic layer of finite thickness lying over a homogeneous elastic isotropic half-space (rather than considering earth as a purely uniform half-space) then SH type waves occur in the stress-free surface of a layered half-space.

Now , we consider the possibility of propagation of surface waves of SH-type (Love waves) in a semi-infinite elastic isotropic medium consisting of a horizontal elastic layer of uniform thickness H lying over a half-space.

It is assumed that two elastic isotropic media are welded together and the horizontal boundary z = 0 of the semi-infinite medium is stress – free (see , figure).



Let the layer and the half-space have different densities ρ_1, ρ_2 and different shear moduli μ_1 , μ_2 respectively. Let two-dimensional SH-motion takes place parallel to xz-plane. The basic equation for SH-wave motion are

$$u_1 = w_1 \equiv 0 \,, \;\; v_1 = v_1(x\,, z, t) \;\;, \tag{1}$$

$$\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial z^2} = \frac{1}{\beta_1^2}\frac{\partial^2 v_1}{\partial t^2} \;\;, \tag{2}$$

$$\beta_1^2 = \frac{\mu_1}{\rho_1} \;, \tag{3}$$

in the layer $0 \le z \le H$, and

$$u_2 = w_2 = 0 \,, \quad v_2 = v_2(x\,, z\,, t) \,, \tag{4}$$

$$\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial z^2} = \frac{1}{\beta_2^2}\frac{\partial^2 v_2}{\partial t^2} \;, \tag{5}$$

$$\beta_2^2 = \frac{\mu_2}{\rho_2} \;, \tag{6}$$

in the half-space $(z \ge H)$.

Suitable plane wave solutions of wave equations (2) and (5) are (exercise) , as discussed in detail already ,

$$v_1(x\,, z\,, t) = (A_1\, e^{-b_1 kz} + B_1\, e^{b_1 kz})\, e^{i(\omega t - kx)} \;\;, \tag{7}$$

in the layer $0 \le z \le H$, and

$$v_2(x, z, t) = A_2\, e^{-b_2 kz} e^{i(\omega t - kx)} \quad , \tag{8}$$

in the half-space ($z \ge H$). ω is the angular wave frequency and k is the wave number , and $c = \omega/k$ is the speed of propagation of surface wave (if it exists) in the positive x-direction. A_1, B_1 are constants, b_1 and b_2 are real numbers with $b_2 > 0$. However , b_1 is unrestricted because z is finite in the layer.

Substituting for v_2 from (8) into (5) yields the relation

$$b_2^2 = \left(1 - \frac{c^2}{\beta_2^2}\right) , \tag{9}$$

and , therefore ,

$$c < \beta_2 , \tag{10}$$

otherwise

$$v_2 \to \infty \text{ as } z \to \infty.$$

From equations (7) and (2), we find

$$b_1^2 = \left(1 - \frac{c^2}{\beta_1^2}\right) . \tag{11}$$

The stress-displacement relations imply

$$\tau_{31} = \tau_{33} \equiv 0 \quad , \tag{12}$$

in the layer as well as in the half space. Also

$$\tau_{32} = \mu_1 k(-b_1 A_1\, e^{-b_1 kz} + b_1 B_1\, e^{b_1 kz})\, e^{i(\omega t - kx)} , \tag{13}$$

in the layer $0 \le z \le H$, and

$$\tau_{32} = k\, \mu_2 A_2 (-b_2)\, e^{-b_2 kz} .\, e^{i(\omega t - kx)} \tag{14}$$

in the half space $z \ge H$.

The stress-free boundary z = 0 implies that

$$\tau_{32} = 0 \quad , \tag{15}$$

at z = 0. This gives

$$B_1 = A_1 . \tag{16}$$

Since , there is a welded contact between the layer and the half-space at the interface z = H , so the displacement and the tractions must be continuous across the interface z = H.

Thus , the boundary conditions at z = H are

$$v_1 = v_2 , \tag{17}$$

$$\tau_{32}\big|_{\text{layer}} = \tau_{32}\big|_{\text{half-space}} . \tag{18}$$

From equations (13) , (14) , (17) and (18) , we find

$$A_1\ e^{-kb_1H} + B_1\ e^{kb_1H} = A_2\ e^{-kb_2H} , \tag{19}$$

and

$$\mu_1\ k\ b_1[-A_1\ e^{-b_1kH} + B_1\ e^{b_1kH}] = -\ \mu_2\ b_2\ k\ e^{-b_2kH} . \tag{20}$$

Equation (16) , (19) and (20) are three homogeneous in A_2 , A_1 , B_1 and A_2.

We shall now eliminate them. From equations (19) & (20) , we write (after putting $B_1 = A_1$)

$$\frac{A_1[e^{-kb_1H} + e^{kb_1H}]}{A_1.\mu_1 b_1(e^{-b_1kH} - e^{b_1kH})k} = \frac{1}{\mu_2 b_2 k}$$

or
$$\frac{e^{-b_1kH} -\ e^{b_1kH}}{e^{-kb_1H} + e^{kb_1H}} = \frac{\mu_2 b_2}{\mu_1 b_1}$$

or
$$\tan h(b_1\ k\ H) + \frac{\mu_2 b_2}{\mu_1 b_1} = 0. \tag{21}$$

Equation (21) is known as period equation/frequency equation/Dispersion equation for surface Love waves.

Equation (21) can also be written as

$$\tan h\left[kh.\sqrt{1-\frac{c^2}{\beta_1^{\,2}}}\right] = -\frac{\mu_2}{\mu_1}\ .\ \frac{\sqrt{1-\dfrac{c^2}{\beta_2^{\,2}}}}{\sqrt{1-\dfrac{c^2}{\beta_1^{\,2}}}}\,.$$

or

$$\tan h\left[\frac{\omega H}{c}\sqrt{1-\frac{c^2}{\beta_1^{\,2}}}\right] = -\frac{\mu_2}{\mu_1}\,.\,\frac{\sqrt{1-\dfrac{c^2}{\beta_2^{\,2}}}}{\sqrt{1-\dfrac{c^2}{\beta_1^{\,2}}}}\,. \quad (22)$$

Equation (22) is a transcendental equation.

For given ω, we can find the speed c for surface Love waves. We note **the value of c depends upon ω .** This means that **waves of different frequencies will**, in **general , propagate with different phase velocity.**

This phenomenon is known as dispersion.

It is caused by the **inhomogeneity of the medium** (layered medium) due to some abrupt discontinuities within the medium (or due to continuous change of the elastic parameters which is not the present case).

Thus, **Love waves are dispressed.**

We consider now the following two possibilities between c and β_1 .

(i) Either $c \le \beta_1$, (ii) or $c > \beta_1$. (23)

When $c \le \beta_1$ **:** In this case b_1 is real (see , equation (11)) and left side of (22) becomes real and positive and right side of (22) is real and negative. Therefore , equation (22) can not possess any real solution for c.

Therefore , in this case , Love waves do not exist.

So , for the existence of surface Love waves , we must have

$$\mathbf{c > \beta_1}\ . \quad (24)$$

In this case (24), b_1 is purely imaginary and we may write

$$b_1 = \ = \sqrt{1-\frac{c^2}{\beta_1^{\,2}}} = i\left(\sqrt{\frac{c^2}{\beta_1}-1}\right)\ . \quad (25)$$

Then equation (22) becomes

$$\tan\left(kH\sqrt{\frac{c^2}{\beta_1^2}-1}\right) = \frac{\mu_2}{\mu_1}\left(\frac{\sqrt{1-\frac{c^2}{\beta_2^2}}}{\sqrt{\frac{c^2}{\beta_1^2}-1}}\right). \qquad (26)$$

From equations (10) and (24) ; we find

$$\beta_1 < c < \beta_2 . \qquad (27)$$

Equation (26) is a transcendental equation that yields infinitely many roots for c.

Thus, the possible **speeds of the Love waves are precisely** the roots of equations (26) that lie **in the interval (β_1 , β_2).**

This indicates that the **shear velocity in the layer** must be less than the **shear velocity in the half-space** for the possible existence of Love waves.

This gives **the upper and lower bounds for** the speed of Love waves.

Remark 1: If the layer and the half – space are such that $\beta_1 \leq \beta$, then existence of Love waves are not possible

Remark 2: In the limiting case when the layer is absent , we have

$$\mu = \mu_1 \text{ and } \rho = \rho_1$$

and therefore

$$\beta = \beta_1$$

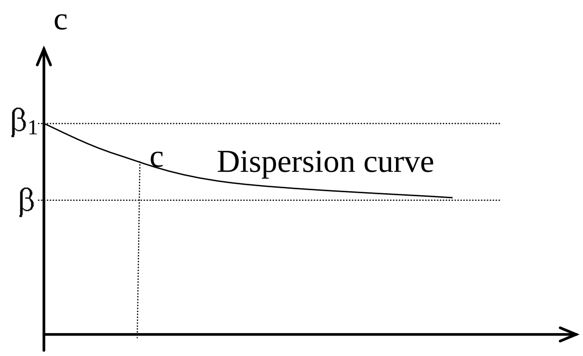
Equation (22) leads to the impossible condition

$$0 = -1 .$$

Hence , in this case , the wave considered can not exist.

Remark 3: When k or $\omega \to 0$, we get $c \to \beta_1$.

The dispersion curve is given in the following figure.



ω

Here , if we assume

$$\mu_1/\mu = 1.8\ ,\ \beta = 3.6\text{km/sec}\ ,\ \beta_1 = 4.6\text{km/sec}$$

then

$$c_L = \text{speed of Love waves} = 4.0\text{km/sec.}$$